Lecture 17, March 25, 2019

TODAY

QUIZ (not)

1) m, G, I
2) Rotational Dynamics

$\boxed{\text{"Let X = X"}}$ Laurie Anderson music

Distribution of mass

How much?: m_{TOT}

Where?: G, $\vec{r}_G$

How spread out?: I^G, (I^O)

155

$M_{TOT} \equiv$ add up, one way or another

m particle

$m_1 + m_2$ 2 particles

$\sum m_i$ system of particles

$\int dm$ continuum

$\int \rho \, dV$ volume

ρ = mass per unit volume

$\int \rho_s \, dA$ surface flat or curved

ρ_s = m per unit area

$\int \rho_\ell \, ds$ curve

ds = arc length

ρ_ℓ = mass per unit length

associative rule of addition

$\sum m_I$ system of systems

$$\underbrace{(m_1 + m_2 \cdots)}_{m_I} + \underbrace{(m_{37} + m_{38} \cdots)}_{m_{II}}$$

Center of mass

"Center of Gravity", G, CoM,

Average pos. of mass in system,
"weighted" by mass.

Analogy: test scores: $\overbrace{x_1, x_2 \ldots}^{\text{scores}}$

$x_{Ave} = ?$

$= x$ one test

$x_{Ave} = \dfrac{x_1 + x_2}{2}$ 2 tests

$x_{Ave} = \dfrac{\sum x_i}{n}$ lots of tests

score

# of stud. w/ score

$$x_{Ave} = \frac{n_1 x_1 + n_2 x_2 + \cdots n_m x_n}{n_{stud\,tot}}$$

if n_i students got score x_i

back to CoM : pos x

↑ analog to but scalar

$$m_{tot} X_G \equiv \begin{cases} X & \text{one part} \\ X_1 m_1 + X_2 m_2 & \text{2 part.} \\ \sum m_i X_i & \text{collection} \\ \int X\, dm & \text{continuum} \\ \int_V X \rho\, dV & \text{Volume} \\ \int X \rho_A\, dA & \text{Surface} \\ \int X \rho_s\, ds & \text{Curve} \\ m_I X_I + m_{II} X_{II} + m_{III} X_{III} \text{ etc} & \text{System of systems} \end{cases}$$

Vector position of G: $\vec{r}_G$

$$m_{TOT}\,\vec{r}_G = \left[\begin{array}{l} \sum m_i \vec{r}_i \\ \\ \int \vec{r}\,dm \\ \\ \underbrace{\sum \vec{r}_I^{\,G}\, m_I}_{\text{system of systems}} \end{array}\right\} \text{main cases}$$

Center of mass rel. to some position A: $\vec{r}_{G/A} = \vec{r}_{AG}$

$$m_{TOT}\,\vec{r}_{G/A} = \sum \vec{r}_{i/G}\, m_i$$

$$\vec{r}_{o/G} = ?$$

$$\underbrace{m\,\vec{r}_{G/G}}_{\text{expect this to be zero}} = \sum \vec{r}_{i/G}\, m_i$$

$$= \sum (\vec{r}_i - \vec{r}_G)\, m_i$$

doesn't vary w/ i

$$= \underbrace{\sum \vec{r}_i\, m_i}_{\parallel} - \sum \vec{r}_G\, m_i$$

$$= m_{TOT}\,\vec{r}_G - \vec{r}_G \underbrace{\sum m_i}_{m_{TOT}}$$

$$= m_{TOT}\,\vec{r}_G - m_{TOT}\,\vec{r}_G$$

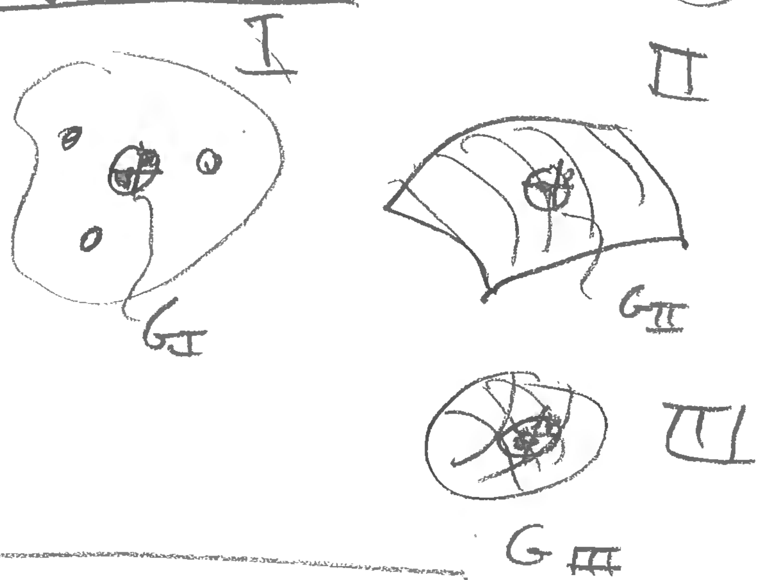
$$= \vec{0}$$ ✓ as expected

useful math for

$\vec{H}$ & E_K

decomposin ang. mom — Koenig's Thms

Apply to subsystems

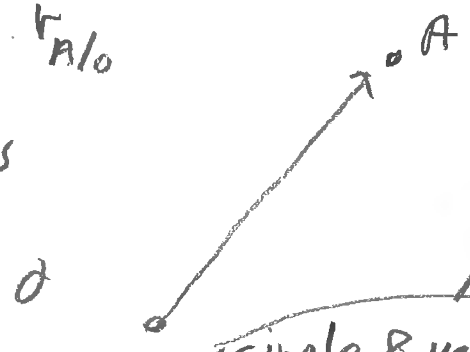


$$\boxed{m_{TOT}\,\vec{r}_{G\,TOT} = \sum m_I\,\vec{r}_{GI}}$$

$\boxed{\text{Very useful}}$

ASIDE

$$\vec{r}_A \underset{\text{means}}{\equiv} \vec{r}_{A/O}$$



$\vec{r}$ vs $\left[\vec{r}_{A/O}\right]_{xyz}$

simple & vague

precise & confusing

Moment of inertia

(How spread out is mass)

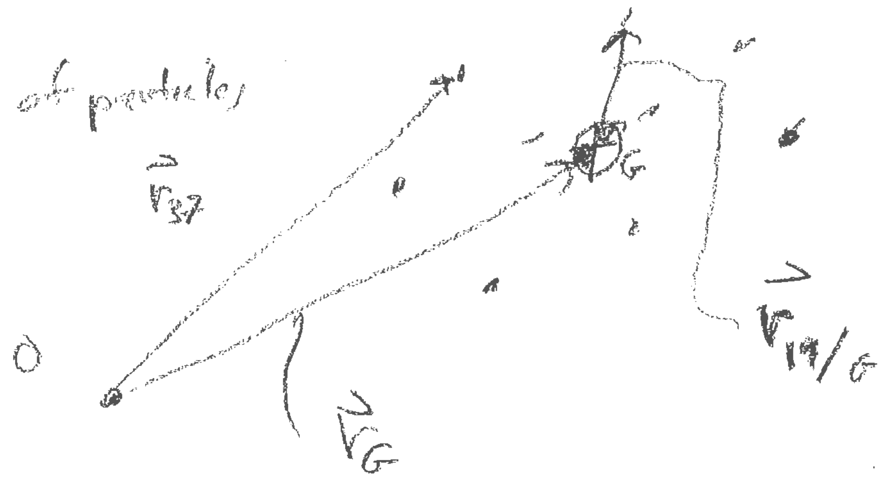
$\underline{\underline{I}}^G$ = moment of inertia rela to center of mass (of object)

Applies to a $\boxed{\text{single rigid object}}$

(in 2D it's a scalar)

particle $\quad I^G = 0$

coll. of particles

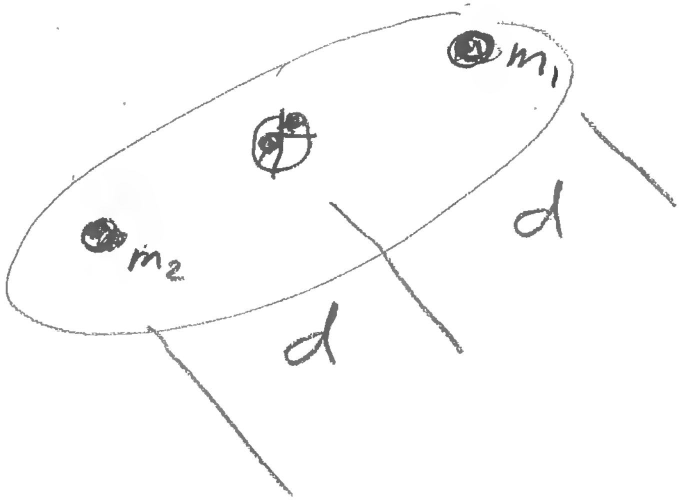


$$\boxed{I^G = \sum m_i \underbrace{r_{i/G}^2}_{|\vec{r}_{i/G}|^2}}$$

Continuum

$$I^G = \int r_{/G}^2 \, dm$$

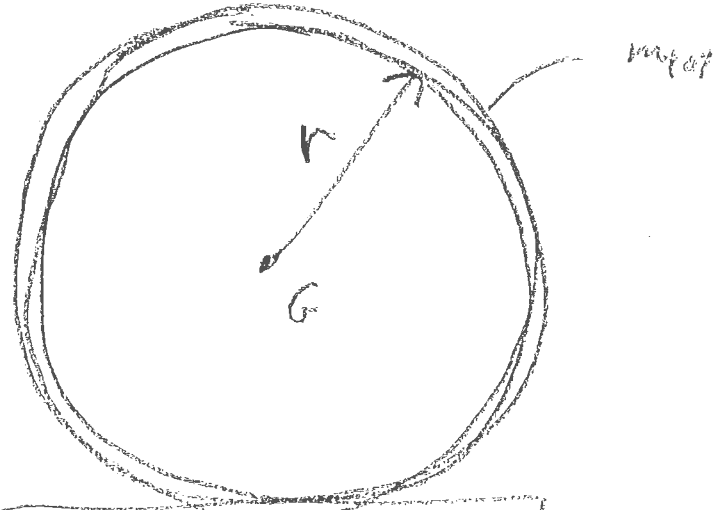
ex) 2 identical masses



$$I^G = m_1 d^2 + m_2 d^2$$

$$\boxed{I^G = m_{tot} \, d^2}$$

ex) ring



$$\boxed{I^G = r^2 m_{TOT}}$$

$$I^G = \int r^2 \, dm$$

$$I^G = \int_0^{2\pi} r^2 \gamma \underbrace{\underbrace{r \, d\theta}_{ds}}_{dm}$$

$$\int_0^{2\pi} d\theta = 2\pi$$

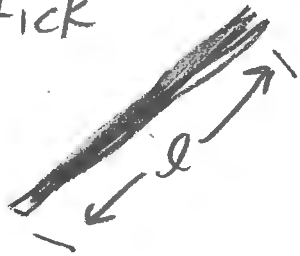
$$I^G = \int_0^{2\pi} r^3 \gamma \, d\theta = 2\pi r^3 \gamma$$

$$I^G = \underbrace{m_{TOT}}_{2\pi\gamma} r^2$$

as expected

Know these integrals

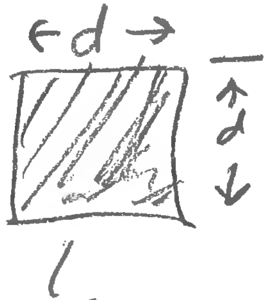
stick



$I^G = ml^2/12$

$I^G = \frac{m}{12}(d^2+h^2)$

$I^G = mr^2/2$

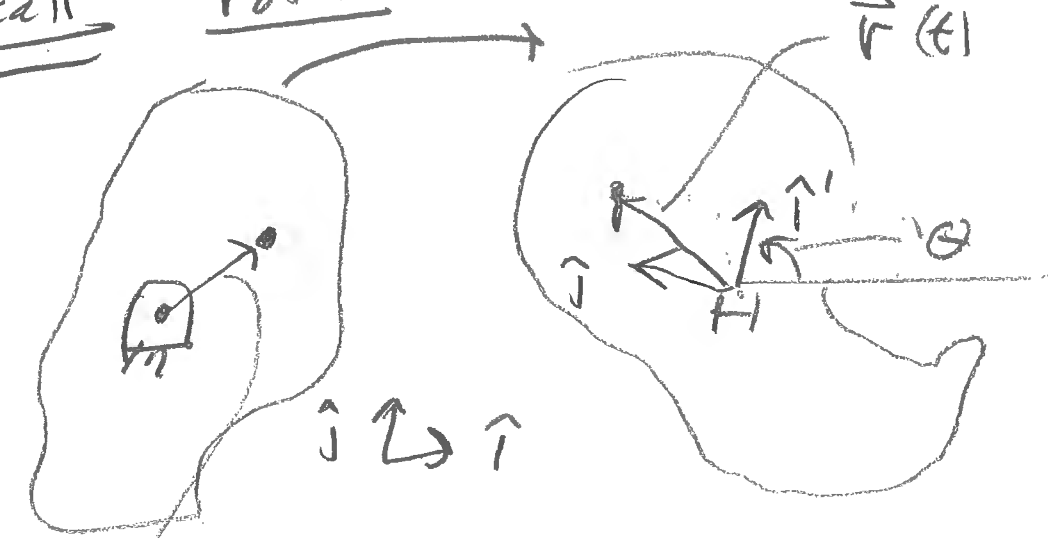


$I^G = md^2/6$

Back to Kinematics

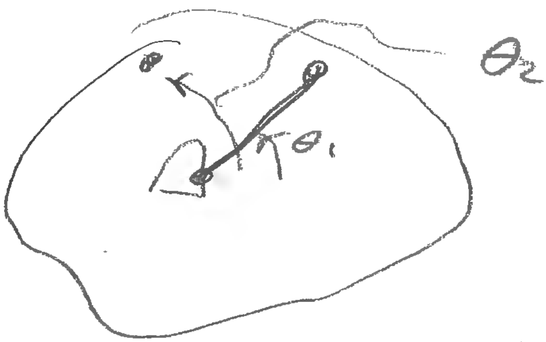
Rotation of rigid objects

recall rotated



$$\vec{r}^{\,0} = x_0 \hat{\imath} + y_0 \hat{\jmath}$$

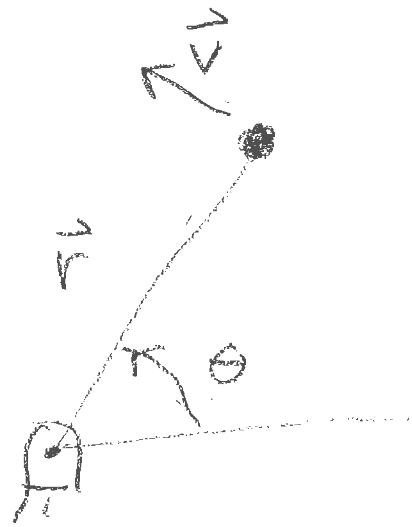
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



$$\theta_1 \neq \theta_2$$

$$\dot{\theta}_1 = \dot{\theta}_2$$

Vel



$\hat{j}$, $\hat{i}$

$\hat{k} = \hat{i} \times \hat{j}$

$$\vec{V} = r\dot{\theta}\hat{e}_\theta$$

θ for that particle

$$= \dot{\theta}(\hat{k} \times \vec{r})$$

same dir, same mag. $\Rightarrow$ same vector "McCarthy"

$$= (\dot{\theta}\hat{k}) \times \vec{r}$$

$$= \vec{\omega} \times \vec{r}$$

$\vec{\omega}$: $\dot{\theta}\hat{k}$

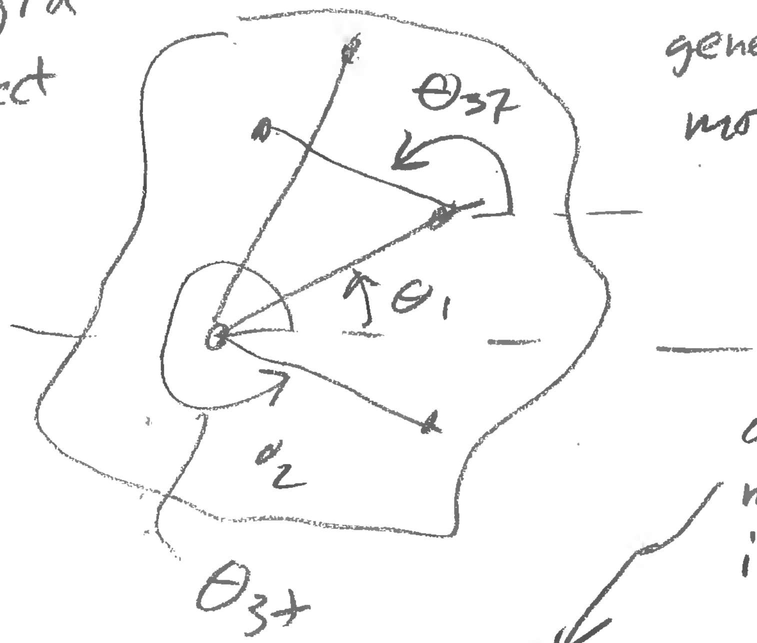
$\boxed{\vec{\omega} = \text{angular velocity of object}}$

$\omega = |\vec{\omega}|$

More specific

rigid object

even for general motion



any line marked in object

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_{37} = \omega$$

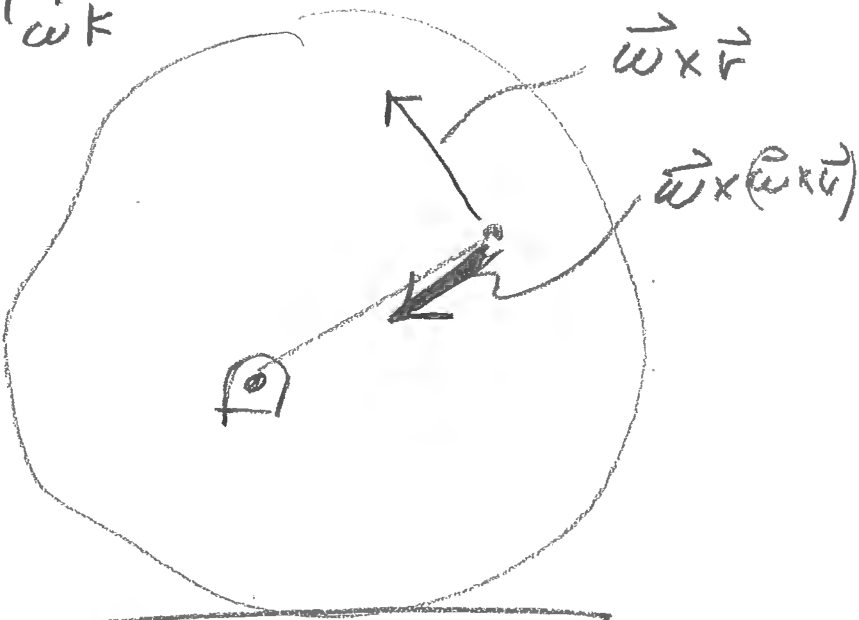
ω = rot. rate of every line on object.

Circ. motion acceleration

$$\vec{V} = \vec{\omega} \times \vec{r}_{/o}$$

$$\dot{\vec{V}} = \vec{a} = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \underbrace{\dot{\vec{r}}}_{(\vec{\omega} \times \vec{r})}$$

$$= \underbrace{\dot{\vec{\omega}}}_{\dot{\omega}\hat{k}} \times \vec{r} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{-\omega^2 \vec{r}}$$



$$\boxed{\vec{a}_{/o} = \dot{\vec{\omega}} \times \vec{r} - \omega^2 \vec{r}}$$

circ. motion

AMB/C

For any FBD

$$\sum \vec{M}^{ext}_{/C} = \begin{cases} \vec{r}_{i/C} \times (m_i \vec{a}_i) = \dot{\vec{H}}_{/C} \\ \int \vec{r}_{/C} \times \vec{a}\, dm = \dot{\vec{H}}_{/C} \\ \sum_{\text{subsystem}} \dot{\vec{H}}_{I/C} \quad \text{of subsystems} \\ \frac{d(\vec{H}_{/C})}{dt} \quad \text{for a fixed pt. at } C \end{cases}$$

$\vec{H}$ = angular momentum ("$\vec{L}$" in Physics)

Key fact:

derived result.
(deriv. in book)

gen. formula for 2D

2D is gen. motion

$$\sum \vec{M}_{/C} = \underbrace{\left(\vec{r}_{G/C} \times m_{tot}\,\vec{a}_G + I^G \dot{\omega}\,\hat{k}\right)}_{\sum \vec{r}_{i/C} \times m_i \vec{a}_i}$$

any 2D motion of rigid object

$$\sum \vec{M}_{/O} = I^O \dot{\omega}\,\hat{k}$$

for rot. about fixed pt. O

BEWARE

Advice: Ignore I^O unless you understand.