Dynamics, MAE 2030

TODAY *Intro. Wed Jan 23, 2019

*1D dynamics

(Please do not sit in last 3 rows)

What is Mechanics?

Given a system (suggests a model)

" some info. about the forces (dir., mag, loc.)

" some inf. about distortion & motion

Find? Other info. about forces, distortion & motion.

How?

Write "true" equations that involve known & unknown quantities

Then: solve eqs for vars or fns. of interest.

Main Eqn:

"Let $X = X$" — Laurie Anderson

think about something 2 different ways

What are these eqns.?

3 general types of Eqns.

"pillars"

I. Constitutive Laws or

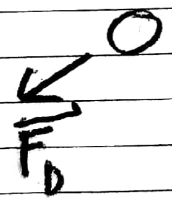
Force Laws or

Material Properties

Ex: $F = k(\Delta l)$ spring

ex) $F_{12} = \frac{G m_1 m_2}{r_{12}^2}$ central force gravity

ex) $F = gm$ near earth gravity

ex a) 

$\vec{F}_D = -c\vec{v}$

↑ linear drag

ex b) $\vec{F}_D = -c|\vec{v}|^2 \frac{\vec{v}}{|\vec{v}|}$

↑ quadratic drag

ex) $F_f = \mu N$ friction

ex) $v^+ = -e v^-$ collision

ex) $\sigma = E\epsilon$ Hook's laws 3

All **bad**; All good enough,

$F = mg$ typical error $1/4$ %

or 2 %

$\Rightarrow (g = 10\ m/s^2)$

↑ Andy likes

best

worst: $F = \mu N$ typical errors

const. laws:

$10\% - 50\%$

Friction: Jaeger's 3rd law: "A friction exp. will make a monkey out of you."

II. Geometry & time

geometry in motion

Kinematics ~~Kinetics~~

Lengths, angles, time

$\boxed{\text{no forces}}$

ex) $v = \dot{x}$

ex $\epsilon = \frac{-y}{\rho}$

ex) $a = \dot{v} = \ddot{x}$

ex) $\epsilon = \frac{\Delta l}{l}$

ex) $c^2 = a^2 + b^2 - 2ab\cos\theta$

ex) $\vec{V} = \vec{\omega} \times \vec{r}$

"5 term acceleration formula"

ex) $\vec{a}_P = \vec{a}_{O'} + \vec{a}_{rel} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'})$

$+ \dot{\vec{\omega}} \times \vec{r}_{rel} \times 2\vec{\omega} \times \vec{v}_{rel}$

III. Laws of Mechanics

Linear Momentum Balance

LMB

$$\boxed{\vec{F}^{ext}_{TOT} = m_{TOT}\vec{a}_G}$$

Angular Momentum Balance

AMB

$$\sum \vec{r}_{i/C} \times \vec{F}_i^{ext} = \dot{\vec{H}}_{/C}$$

$$= \sum \vec{r}_{i/C} \times m_i \vec{a}_i$$

$$\int \vec{r}_{/C} \times \vec{a}\, dm$$

What for?

It describes how world works.

It's useful sometimes.

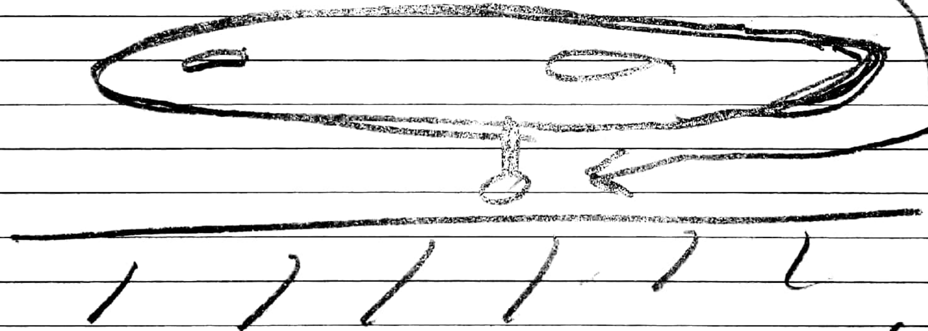
It builds useful skills.

$m\vec{a} \approx \vec{0} \Rightarrow$ statics

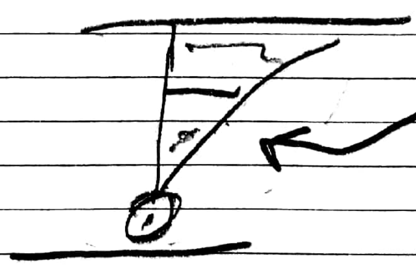
much more useful than dynamics

Airplane landing:

Easy to estimate force on gear from Dynamics



Lots of Statics & Strength of materials to design the gear



Some people really use dynamics. (Like me)

↑ Prof. ANDY

Skills are useful

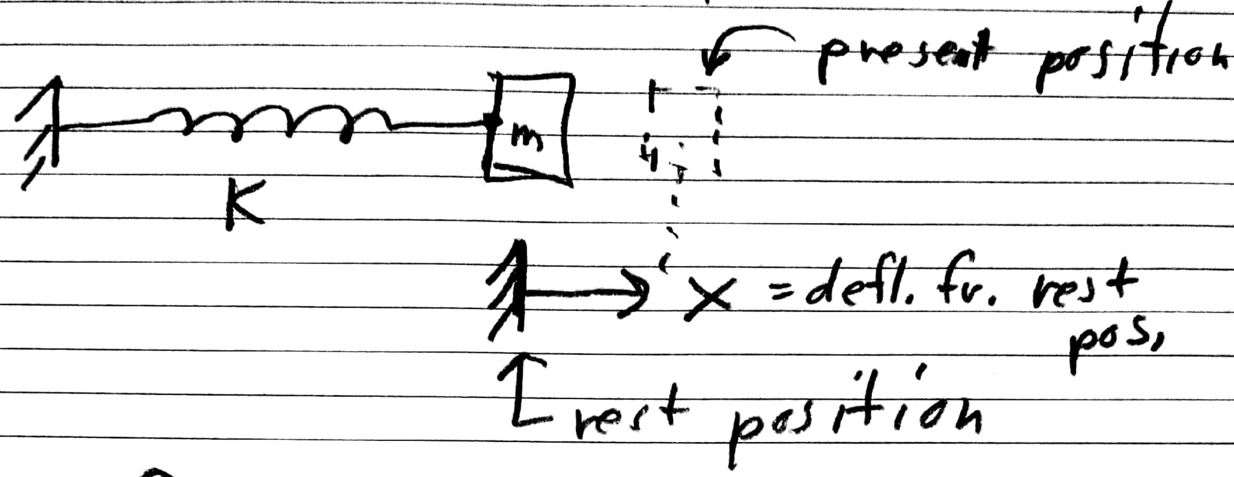
* you will get better at statics
* " " build probl. solvi skills
* " " get good at Matlab
* " " gain an understadig of vibrations

Matlab Question Plot is ?

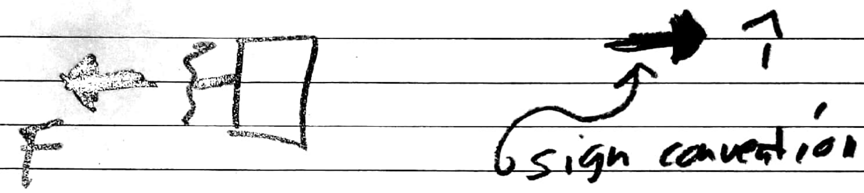
A) Straight line
B) Circle
c) elipse
d) parabola
e) sine wave

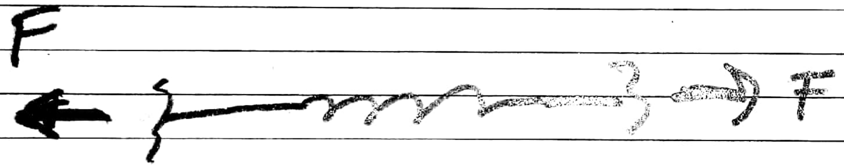
(see matlab code for lecture 1)

Spring & mass: 1D



FBD!





$$F = kx$$

LMB $\left\{ \sum \vec{F} = m\vec{a} \right\} \cdot \hat{\imath}$

$$-F = m\ddot{x} \quad \leftarrow \dot{v}$$

$$\boxed{-kx = m\ddot{x}}$$

or $\quad m\ddot{x} + kx = 0$

or $\quad \ddot{x} + \frac{k}{m}x = 0$

used in matlab code $\rightarrow$ $\left\{ \begin{aligned} \dot{v} &= -\frac{k}{m}x \\ \dot{x} &= v \end{aligned} \right]$ 1st order form (natural)

Soln: Euler's method

$$X_{new} = X_{old} + \dot{X} \cdot \Delta t$$

$$V_{new} = V_{old} + \underbrace{\dot{V}}_{= -\frac{k}{m} X} \cdot \Delta t$$

Solve numerically on computer

(1st lecture code)

TODAY Jan 28, 2019

* Quiz
* Review of harmonic osc.

Generalities

* $F = -kx$ $\Rightarrow$ $f(x, v, t)$

list of 1st order eqns $\Rightarrow$ vector form Z

* Matlab $\Rightarrow$ break into parts

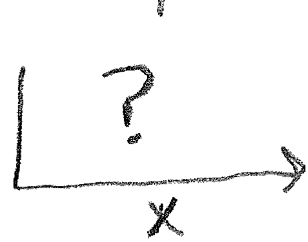
Q: $\dot{x} = v$ } ODEs

Quiz $\dot{v} = -c\,v\,|v|$

$x = 0, \quad v = 1$ } ICs

$c = 1$ parameters

plot $\log(v)$



a) $\underbrace{\text{circle}}_{\text{(partial)}}$ centered at origin

b) Straight line w/ slope $= -1$

c) exponential decay to const. assyptote

d) Upwards parabola

e) Sine wave

SOLUTION of QUIZ

$$\dot{V} = -cV^2$$

$V > 0$

$\uparrow$ if V ever < 0 then soln. does not apply

$$\left[\frac{dV}{V^2} = -c\,dt \right]$$

$$\frac{dV}{dt}\underbrace{\frac{dt}{dx}}_{1/V} = -cV^2 \underbrace{\frac{dt}{dx}}_{1/V}$$

$$\frac{dV}{dx} = -cV$$

1 lin. O.D.E.

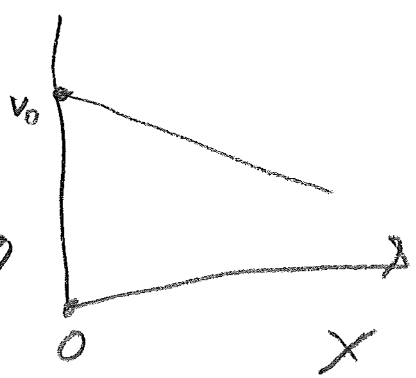
$$\boxed{V = V_0 e^{-cX}}$$

$c = 1$

$V_0 = 1$

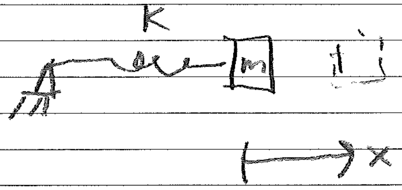
$$\Rightarrow V = e^{-X}$$

$$\log(V) = -X$$



soln (b) from prev. page

Review: Harmonic Oscillator



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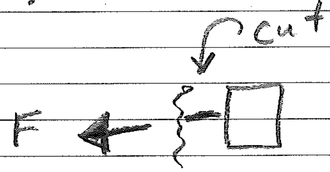


I, Const Law: $F = -Kx$

II, Kinematics: $v = \dot{x}, \Rightarrow \dot{x} = v$

$a = \dot{v}, \Rightarrow \dot{v} = a$

III Laws of Mech: $\underline{LMB} \Rightarrow$

$F = ma$

I, II, III $\Rightarrow$
$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= -Kx/m \end{aligned}}$$

Soln: $x = C_1 \cos(\omega t)$, $\omega = \sqrt{K/m}$

$v = -\omega C_1 \sin(\omega t)$

(There's also an soln. $x = C_2 \sin \omega t$, $v = C_2 \omega \cos \omega t$)

Num, Soln's

Almost all ODES have no practical analytic Soln. $\Rightarrow$ NUMERICS

ODEs in the real world

ex) first order ODE

$$\dot{z} = f(z)$$

$$z(t) = z_0 + \left.\frac{dz}{dt}\right|_0 t + \left.\frac{d^2z}{dt^2}\right|_0 \frac{t^2}{2} + \frac{d^3z}{dt^3}\frac{t^3}{3!} + \ldots$$

Taylor series

Truncate series (approximation)

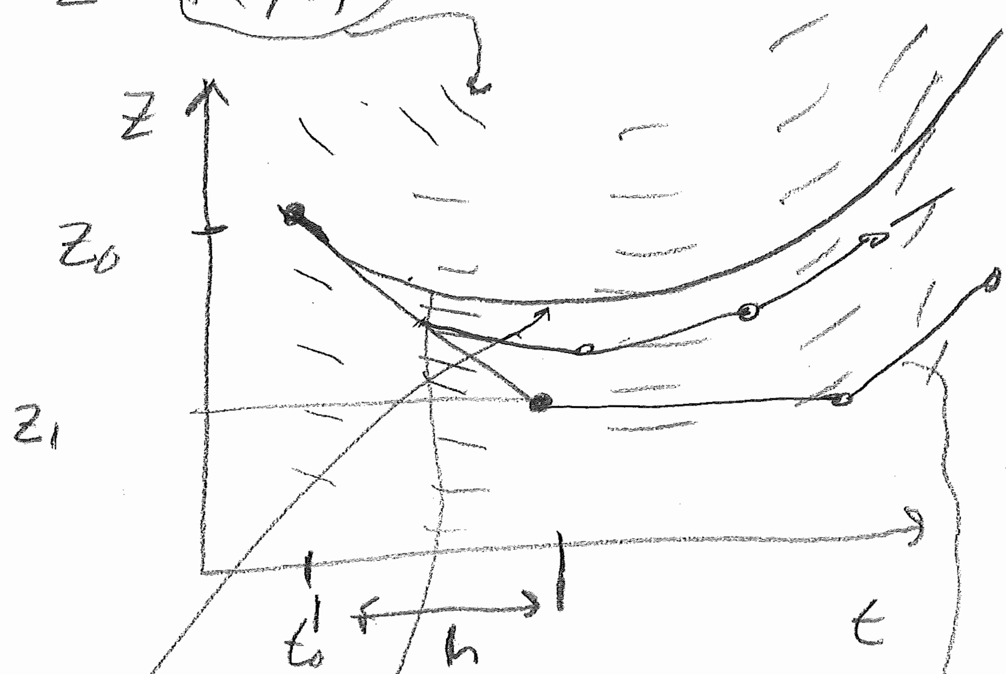
$$z(t) = z_0 + \dot{z}\,\Delta t$$

again & again, Euler's method

for successive z values

ODE $\Rightarrow$ dir field

$\dot{z} = f(t,t)$ = "slope field"



exact soln:
tan to dir. of
dir. field

Euler approx:

$$Z_1 = Z_0 + h \cdot \dot{Z}\Big|_{t_0}$$

approximate Euler Soln.

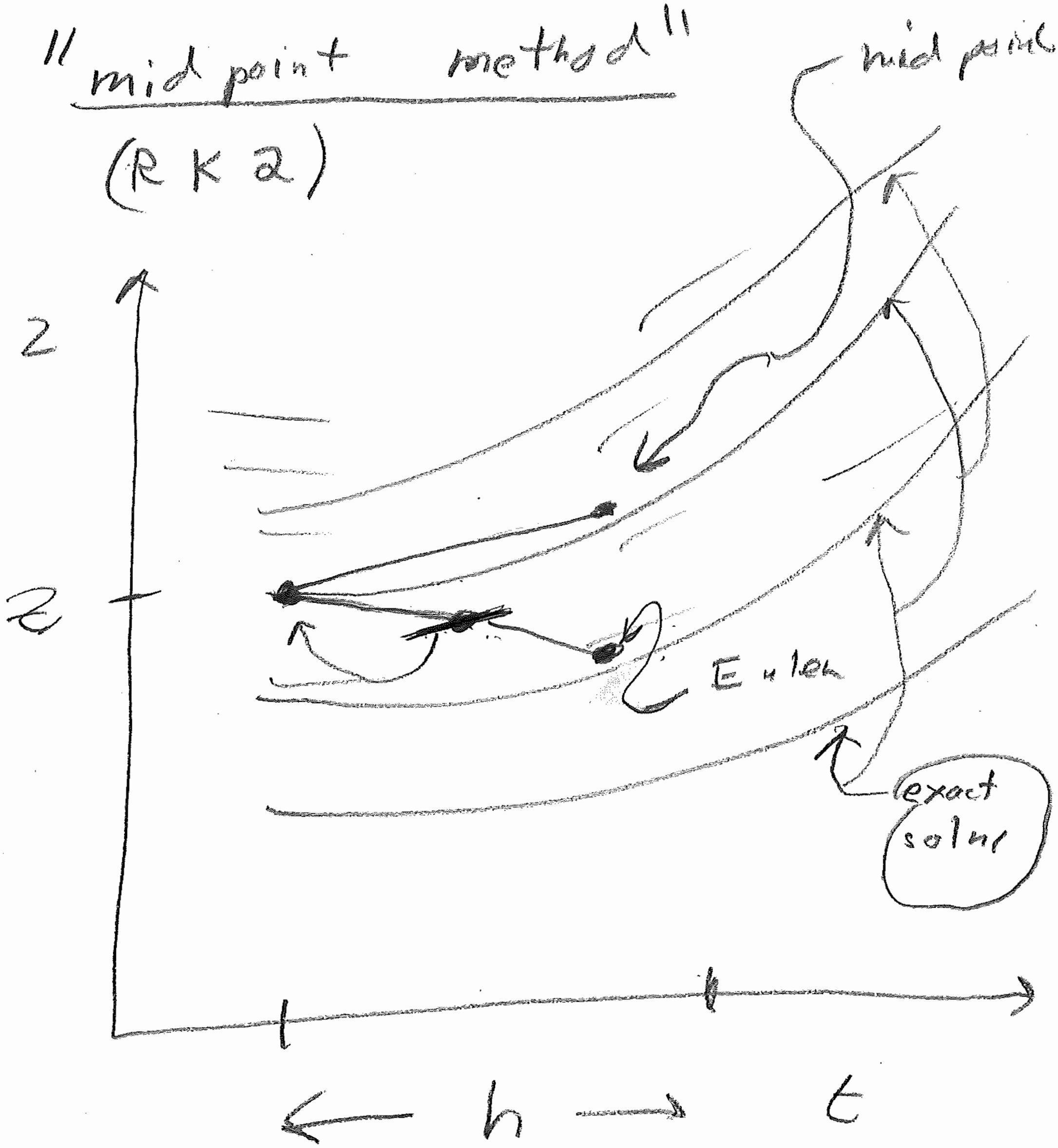
smaller time step
$\Rightarrow$ more accurate

More accurate method

(2nd order method)

"mid point method"

(RK2)



$$\dot{z}_{temp} = z_0 + \dot{z}\big|_{t_0} h/2$$ half step

$$\dot{z}_{new} = f(t_{temp}, z_{temp})$$ new slope calc

$$z_{new} = z_0 + h\,\dot{z}_{new}$$ full step

Crystal Ball interp. of ODEs

Know state of world.
Know rules of world.] $\Rightarrow$ Predict the Future

State: z = a list of numbers

rules: $f(t, z)$ ← "RHS" function (right hand side)

<u>ODEs</u>: $\boxed{\dot{z} = f(t, z)}$

ex) $z = \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ Harmonic Oscillator

$$\dot{z} = \begin{bmatrix} v \\ -kx/m \end{bmatrix} = \begin{pmatrix} z_2 \\ -z_1 k/m \end{pmatrix}$$

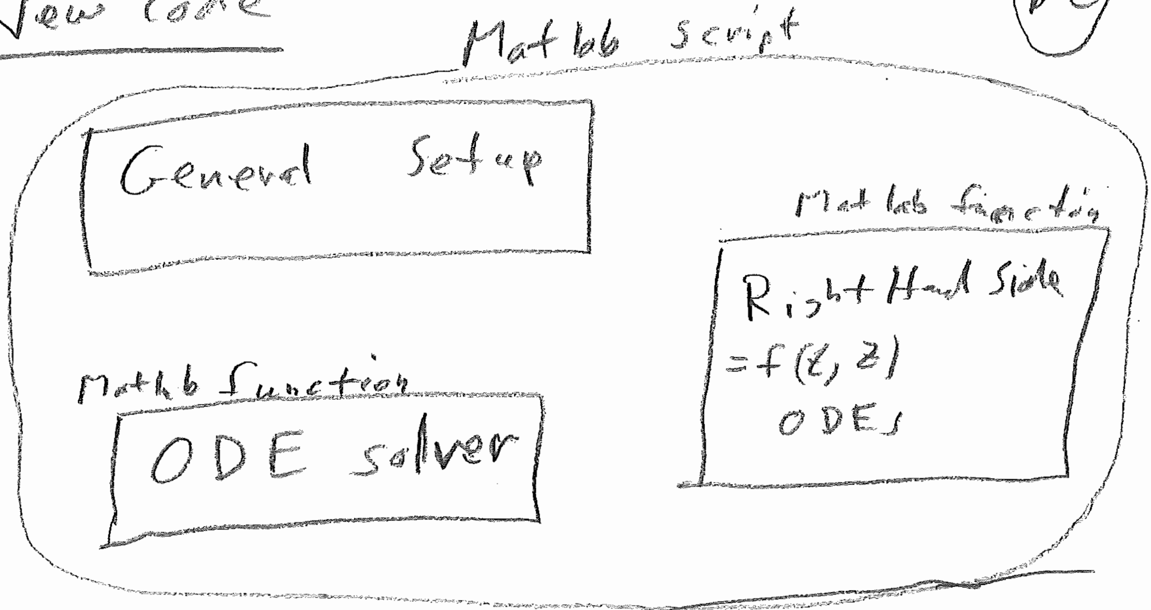
Num. Methods work same way

$z_{new} = z_{old} + \dot{z} \cdot h$ ← RHS fn.

← Euler's method

z is now a list

New code



See code

(½ of lecture was coding)

TODAY Wed Jan 30, 2019

Quiz $F_D = cv$ vs cv^2

3) Thms in 1D dynamics

1) gen. examples of 1D dyn.

~~Num. methods cont'd~~

2) Examples

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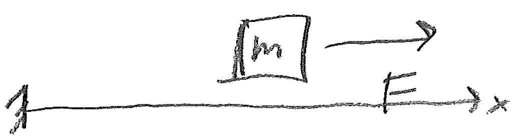
1) Quiz (see next pg)

Gen Ex 1D motion

$\longrightarrow \hat{i}$



fill in

I. Force laws : $F = \text{----}$

II. Kin : $\dot{x} = v$, $\dot{v} = a$

III. Laws of mech. : $F = ma$

ex) $F = 0 \Rightarrow \boxed{\dot{v} = 0}$ or $a = 0$

"An object in motion . . . "

Quiz!

$\rightarrow V_0$ (same for both)

$F_{D1} \leftarrow \boxed{m}$ $\qquad F_{D1} = cV$ (1)

$m = m$

$F_{D2} \leftarrow \boxed{m}$ $\qquad F_{D2} = cV^2$ (2)

V_0 is big

which one goes farther in the end?

a) (1)

b) (2)

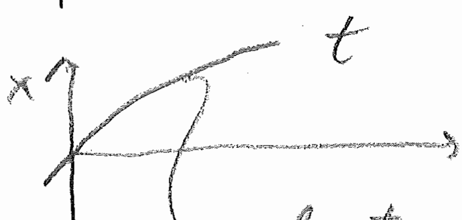
c) depends on c

d) depends on V_0

✓ e) depends on comb. of c, V_0 & m

$c_1 + c_2 e^{-t}$

lin. drag



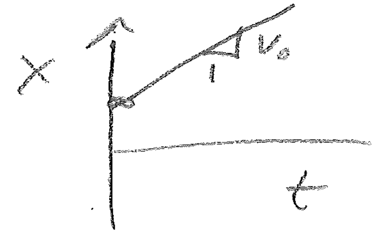
quad drag

$\sim \log t$, goes ∞ distance

$\Rightarrow \dot{V} = 0 \quad \Rightarrow V = V_0$

$\dot{x} = V \quad \Rightarrow \boxed{x = x_0 + V_0 t}$

$\ddot{x} = 0 \quad \Rightarrow$



const.

ex) $F = F_0 \Rightarrow \quad \dot{V} = F_0/m \equiv a_0$

$\Rightarrow \quad V = V_0 + a_0 t$

$x = \frac{1}{2} a_0 t^2 + V_0 t + x_0$

"$d = \frac{1}{2} a t^2$" ANDY 5th grade report

$\boxed{E = \cancel{mc^2}}$ — Hanley Rising

$\boxed{etc}$

ex) $F = f(t)$

$a(t) = \frac{f(t)}{m}$

$\dot{V} = a(t) = f(t)/m$

$V = \int f(t)\,dt/m + C_1$ $\boxed{\text{Math 1910}}$

$x = \int v(t)\,dt/m + C_2$

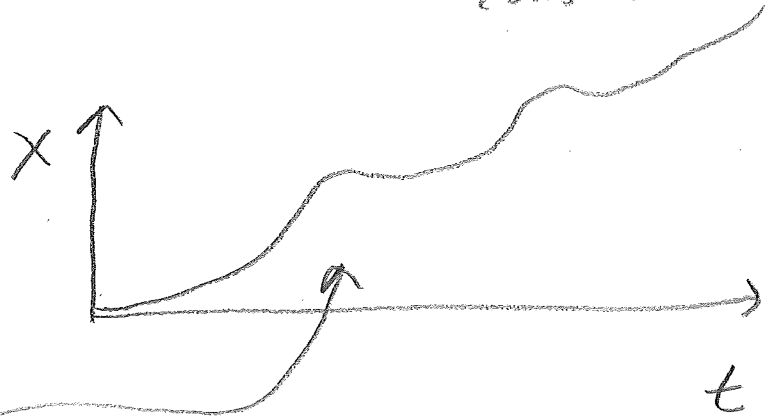
sub ex)

$f(t) = f_0 \sin \omega t$

const.

$X_0 = 0$

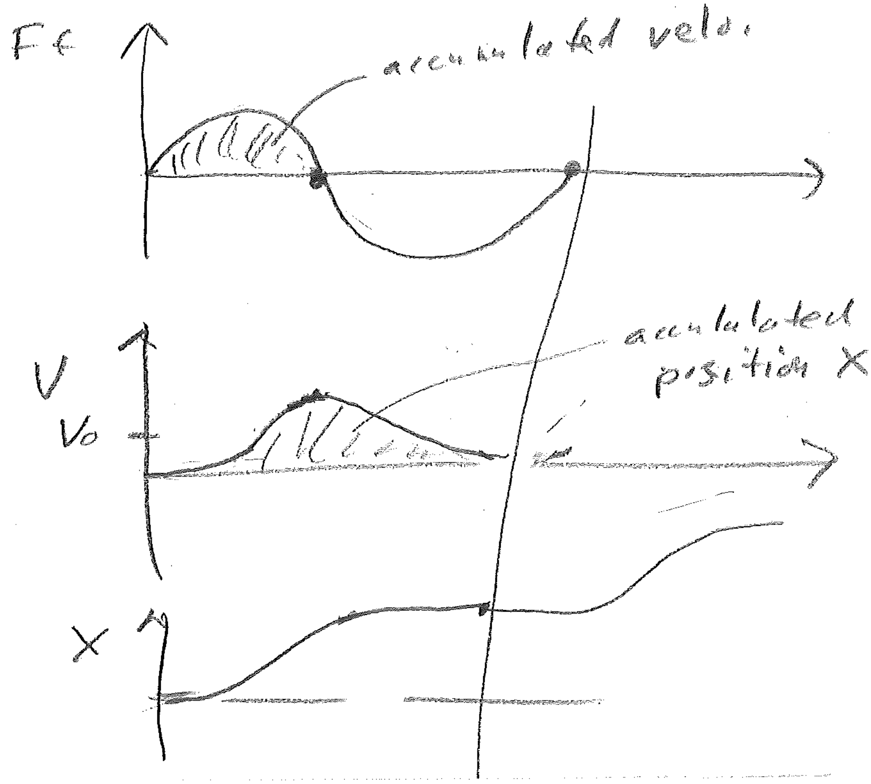
$V_0 = 0$



a) sine wave X

b) " + const X

c) " " const + lin fn. ✓



subex) arb. fcs of t

subsubex) $f(t) = c\dfrac{e^{-bt}}{t}$

$$\dot{V} = \frac{1}{m} c \frac{e^{-bt}}{t}$$

$\Rightarrow$ no analytic soln. (even if $t_0 > 0$)

ex.) $F = g(V)$

$$\Rightarrow a = \frac{1}{m} g(V)$$

$$\boxed{\dot{V} = \frac{1}{m} g(V)} \Rightarrow$$ separable eqn. $\Rightarrow$ might have anal. soln.

$$\dot{x} = V$$

subex) $F = -cV$

$$V = V_0 e^{-ct}$$

$$x = \ldots$$ (Quiz problem)

subex) (last lecture) Quiz problem

$F = -cV|V|$ $\quad$ V^2 if $V>0$

$$\Rightarrow \boxed{V = V_0 e^{-cx}}$$

$$\Rightarrow x \sim \log(t)$$

ex) $F = h(x)$ force is fn. of position

$$\left.\begin{aligned} \dot{v} &= \frac{1}{m} h(x) \\ \dot{x} &= v \end{aligned}\right]$$

[Cons of energy
$\Rightarrow$ $v(x)$]
(not shown yet)

subex) $h(x) = -kx$ restoring ~~~~ spring (linear)

$$\Rightarrow \boxed{m\ddot{x} + kx = 0}$$

or $\begin{bmatrix} \dot{x} = v \\ \dot{v} = -\frac{k}{m} x \end{bmatrix} \Leftrightarrow \dot{z} = f(z)$

rules of change $\leftarrow$ | $\rightarrow$ state

$$x = A\cos(\omega t) + B\sin\omega t$$
$$\omega = \sqrt{k/m}$$

ex) mix & match

subex) Most famous $\omega_f \neq \omega_n = \omega \neq \sqrt{k/m}$

$$\boxed{F = -kx - cv + F_0 \sin \omega_f t}$$

$\Rightarrow$ $$\dot{x} = v$$

$$\dot{v} = \frac{1}{m}\left(-kx - cv + F_0 \sin \omega t\right)$$

$$\boxed{m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)}$$

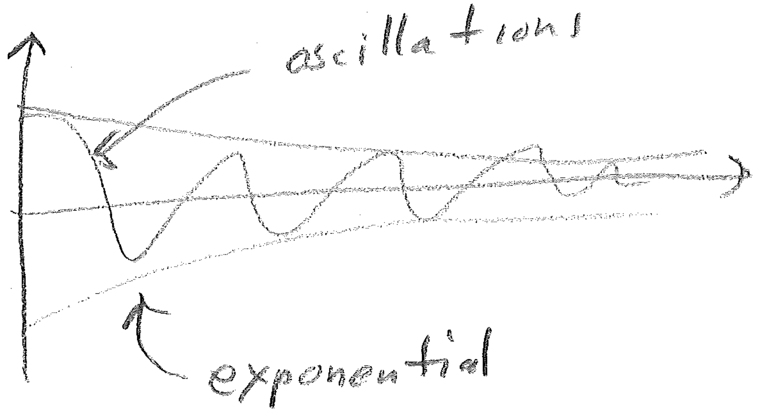
$x(t) = ?$

"Forced, damped harmonic oscillator"

subex) c is small

$F_0 = 0$

underdamped vib. decay



Gen Case of 1D motion

$$F = F(x, v, t)$$

$$\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= \tfrac{1}{m} F(x, v, t) \end{aligned}}$$

gov. eqn. "EoM"

"Equations of motion"

Can solve (at least numerically)

$$\Rightarrow \underbrace{x(t), \quad v(t)}_{\text{solution of EoM}}$$

Can know some things ahead of time: thms / facts. checks ✓✓✓

Thms

$$\{ F = ma \}$$

$\{\}\cdot v$

$$vF = m a \cdot v$$

$$a = \frac{dv}{dt}$$

$$vF = m \frac{d}{dt}\left(\tfrac{1}{2} v^2\right)$$

$$\underbrace{Fv}_{\text{Power} = P} = \frac{d}{dt}\underbrace{\left(\tfrac{1}{2} m v^2\right)}_{\text{Kin. En.} = \text{"}\cancel{KE}\text{"}}$$

E_K, $\cancel{T}$, $\cancel{K}$

$$\boxed{\boxed{P = \dot{E}_K} \quad ① \quad \text{power balance}}$$

<u>Next</u>

$\int ① \, dt$ $\qquad \int P\,dt = \int \dot{E}_K \, dt$

$$\boxed{\int P\,dt = \Delta E_K} \quad ②$$

"Power x time = Energy"

$$\int P\,dt = \int P \frac{dt}{dx}\,dx$$

$$= \int F v \frac{1}{v}\,dx$$

$$= \underbrace{\int F\,dx}_{\text{work}}$$

$$\boxed{\underbrace{\int F\,dx}_{W} = \Delta E_K}$$

Work = change in kin E_y

$\boxed{W = \Delta E_K}$ Work = change in Kin. Energy

Lin. Mom

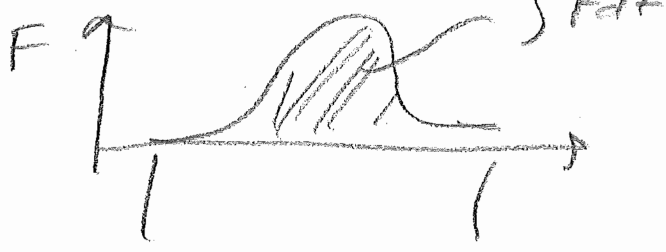
$$\{F = ma\}$$

$\int\{\}dt$ $\quad \int F\,dt = \int m\,a\,dt$ ($a = dv/dt$)

$$\underbrace{\int F\,dt}_{\text{"Impulse"}} = m\,\Delta V$$

$mv = L =$ lin. mom

not. nec. sortly sudden



Summary

* $P = \dot{E}_K$
* $W = \Delta E_K$, $\int P\,dt = \Delta E_K$
* $\int F\,dt = \Delta(mv) = \Delta L$

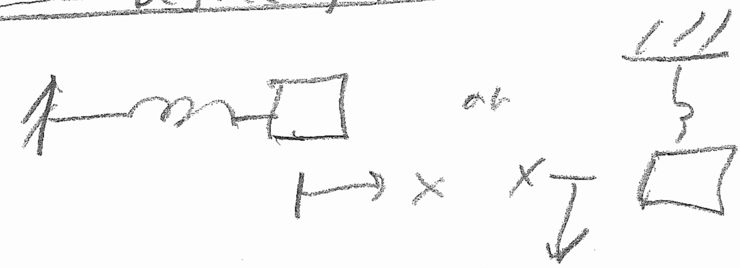
TODAY, Mon Feb. 4, 2019

* Various Quizes (last page)
* spring mass w/ distractions
* multi-D.O.F. oscillation

D. = Degree(s), O. = of, F. = Freedom

Stereotype

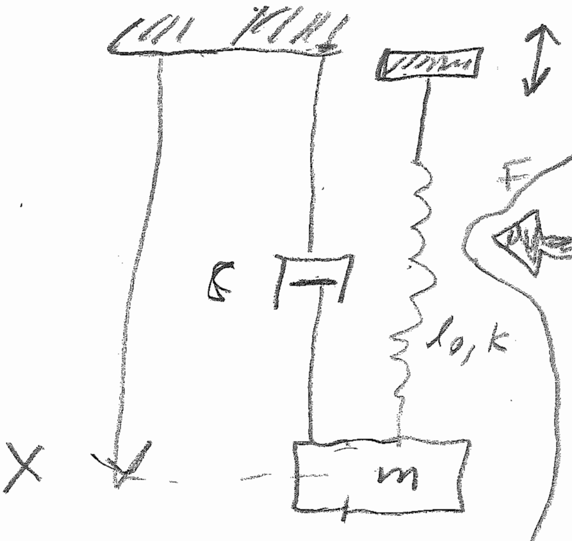
Harmonic Oscillator



real problems!

complexifications, distractions, real life $\rightarrow$ $\ell_0 \neq 0$, $g \neq 0$, dashpot, shaking foundations ("forcing")

ex) $\downarrow g$



"forcing"

$x_0 = \delta \sin(\omega_f t)$

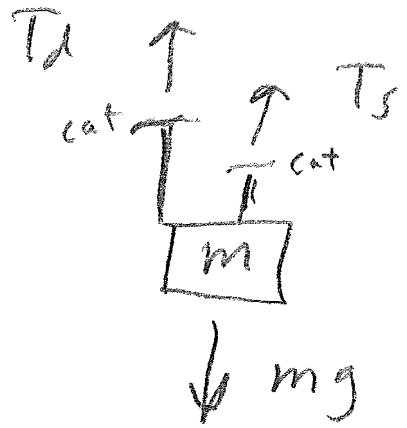
dashpot

$$F = \underset{+}{c}\,\dot{\ell}$$

force $\propto$ vel.

"leaky piston" $\rightarrow$ "linear damping"

FBD



LMB

$$F = m\underbrace{a}_{\ddot{x}}$$

$$mg - T_s - T_d = m\ddot{x}$$

$$T_d = c\dot{x}$$

$$T_s = K(\Delta \ell)$$

✓

$$\Delta \ell = \underbrace{\ell - \ell_0}_{x - x_0}$$

$$x_0 = \delta \sin(\omega_s t)$$

"Eqn of Motion"

$\Rightarrow$ EoM

$$\boxed{m\ddot{x} = -K(x - \delta \sin(\omega_s t) - \ell_0) - c\dot{x} + mg}$$

2930 version

$$\overbrace{m\ddot{x} + c\dot{x} + kx}^{x \text{ stuff}} = \overbrace{\underbrace{k\delta \sin(\omega_f t)}_{\text{sine wave}} + \underbrace{k l_0 + mg}_{\text{const}}}^{\text{other stuff}}$$

$m>0,\ c>,\ k>0$

Soln. Anal.

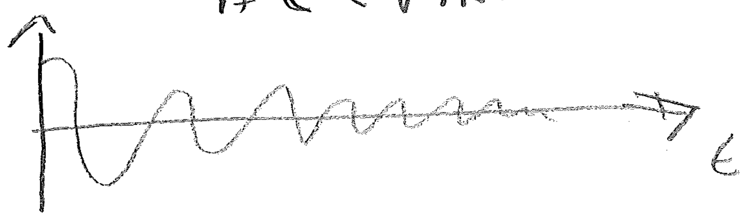
$$x = x_h + x_p$$

x_h: transient, "homogeneous"

x_p: "particular", steady state

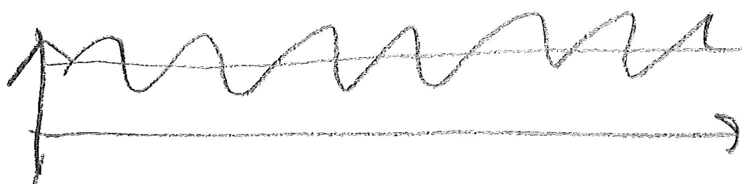
if $c < \sqrt{4km}$

x_h

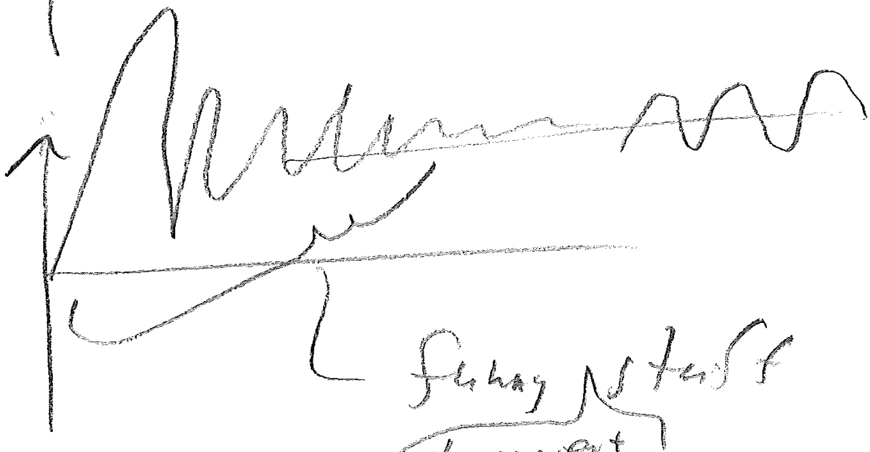


t

x_p



x



fishy stuff

transient

First order form

$$\dot{x} = v$$

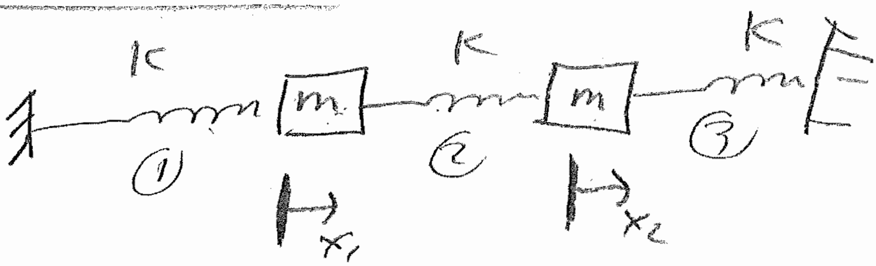
$$\dot{v} = \frac{1}{m}\left[-c\dot{x} - kx + \delta k \sin(\omega_f t) + l_0 k + gm\right]$$

I.Cs. $x(0) = x_0$, $v(0) = v_0$

(see Matlab code)

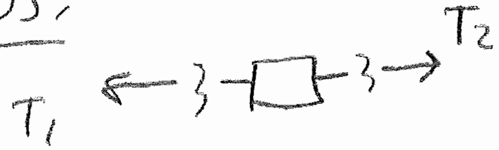
Multi-DoF

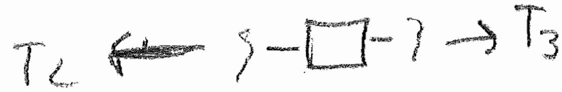
ex)



Assume: tensions $= 0$ in relaxed config.

FBDs.





$$T_1 = K(\Delta \ell_1) = K x_1$$

$$T_2 = K(\Delta \ell_2) = K(x_2 - x_1)$$

$$T_3 = K(\Delta \ell_3) = K(-x_2)$$

<u>LMB!</u>

$$m\ddot{x}_1 = T_2 - T_1$$

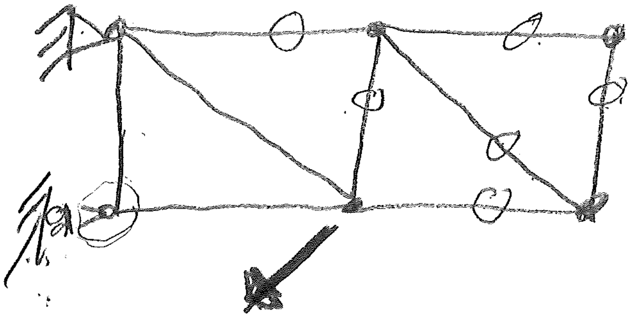
$$m\ddot{x}_2 = T_3 - T_2$$

$$\Rightarrow \boxed{\begin{aligned} m\ddot{x}_1 &= K(-x_1 + x_2) \\ m\ddot{x}_2 &= K(-x_2 + x_1) \end{aligned}}$$

Eqs. of motion

QUIZ 1 Feb. 4

2D, No gravity



How many "zero-force members" (bars w/ T=0)

a) 0

b) 2

c) 4

d) 6 ✓

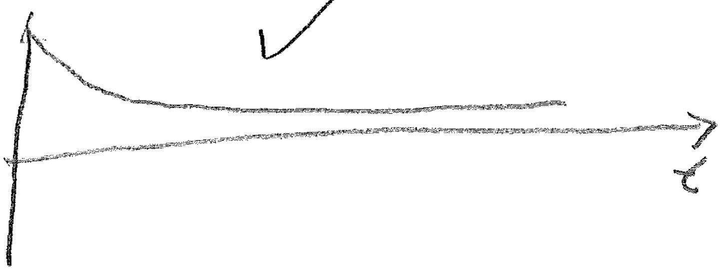
e) None of above

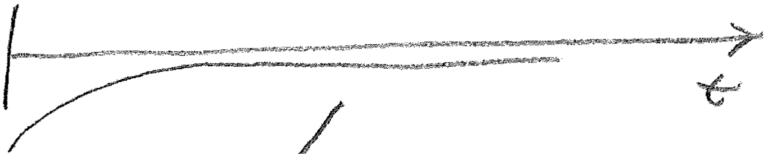
Quiz 2 Feb 4

$$f(t) = a e^{-bt}$$

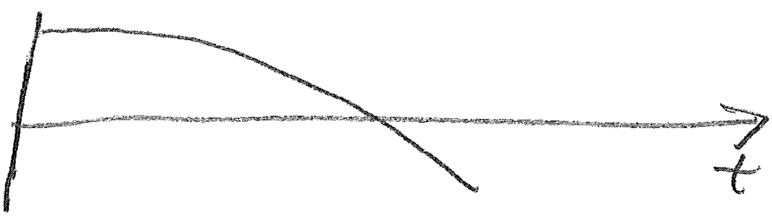
$a > 0$, $b > 0$

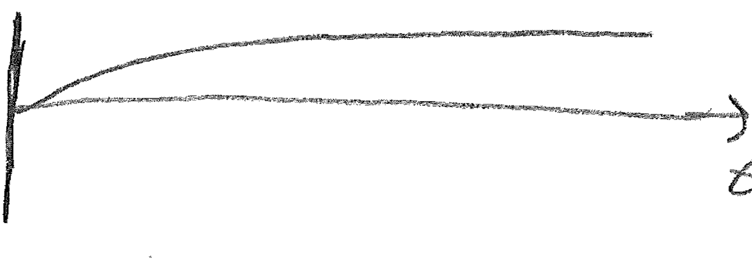
graph?

a) 

b) 

c) 

d) 

e) 

Quiz 3, Feb 4

If $\dot{x} = 2x$ and $h = 0.1$

and $x_0 = 3$ at $t = 0$

For Euler's method

$x(t = 0.1) = ?$

a) 3

b) 3.1

c) 3.2

d) 3.3

e) 3.6 ✓

Soln:

$x(t+h) =$

$x(t) + \underset{.1}{h} \cdot \underset{2 \times 3 = 6}{\dot{x}}$

$= 3 + .6$

$= 3.6$

TODAY, Wed Feb. 6, 2019

1 DoF review

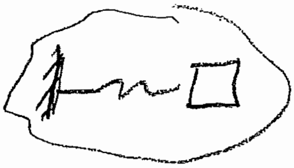
2 DoF (cont'd)

demo

matlab

Normal Modes

See end for quiz questions



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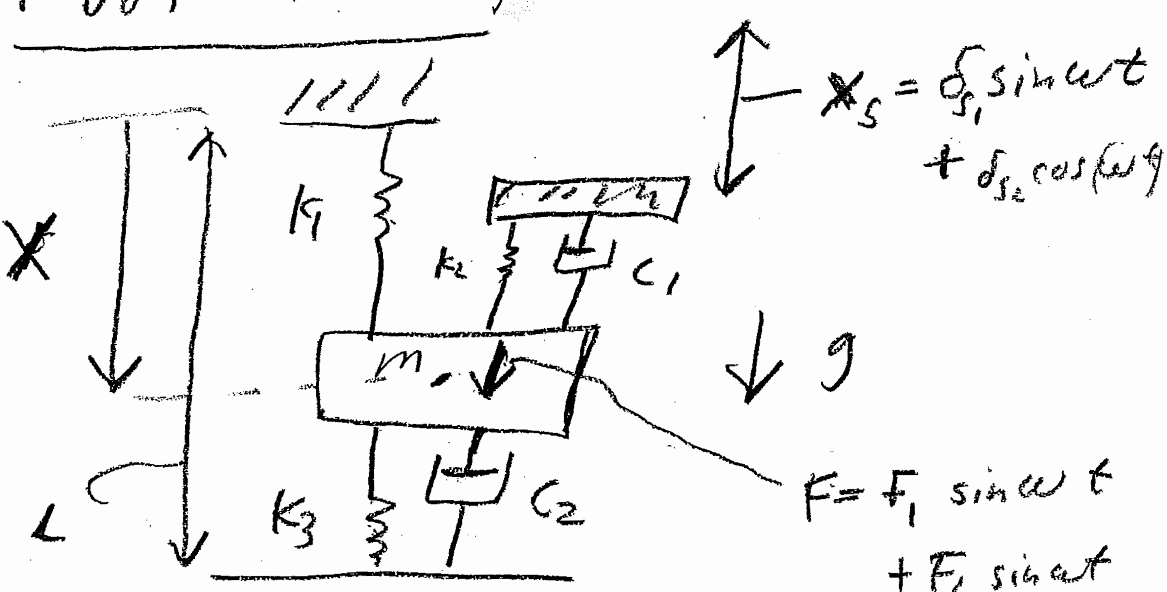
1 DoF review:



mixture of K's

$$\Rightarrow \underline{m}\ddot{x} + c\dot{x} + \underline{K}x = \text{Const} + A \cos \omega t + B \sin \omega t$$

mixture of C's

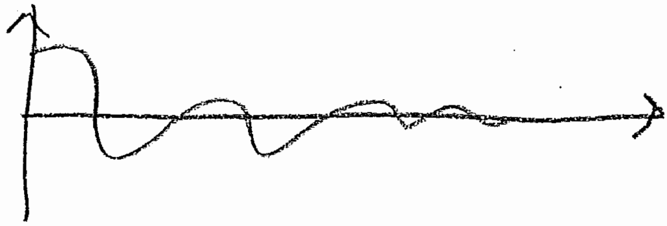
main idea

"forcing"

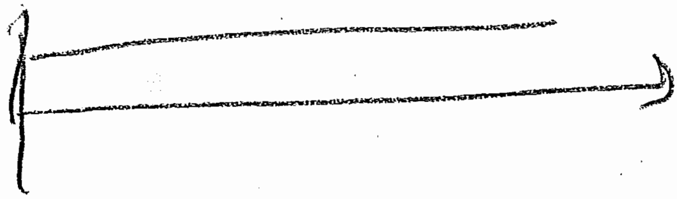
Soln:

sum of terms

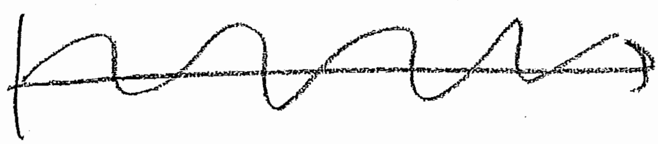
x_h



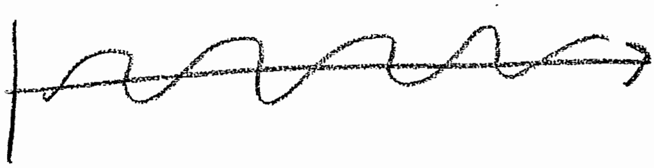
$+ \quad x_{p_1}$



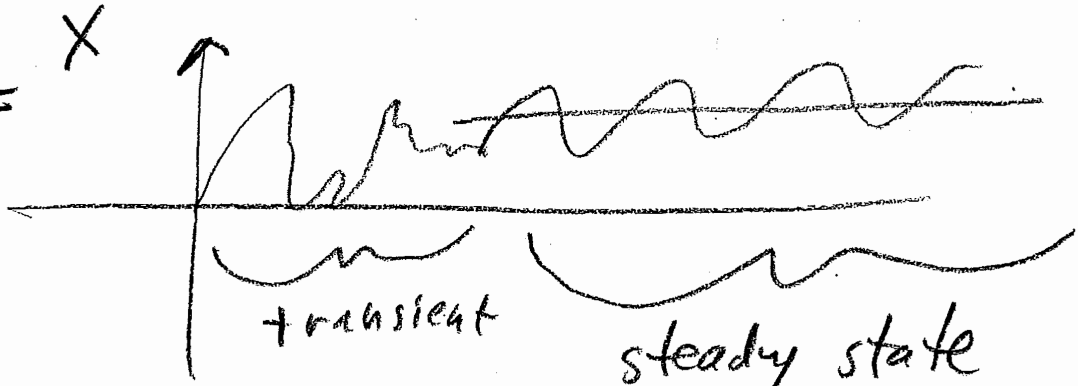
$+ \quad x_{p_2}$



$+ \quad x_{p_3}$



$= \quad x$

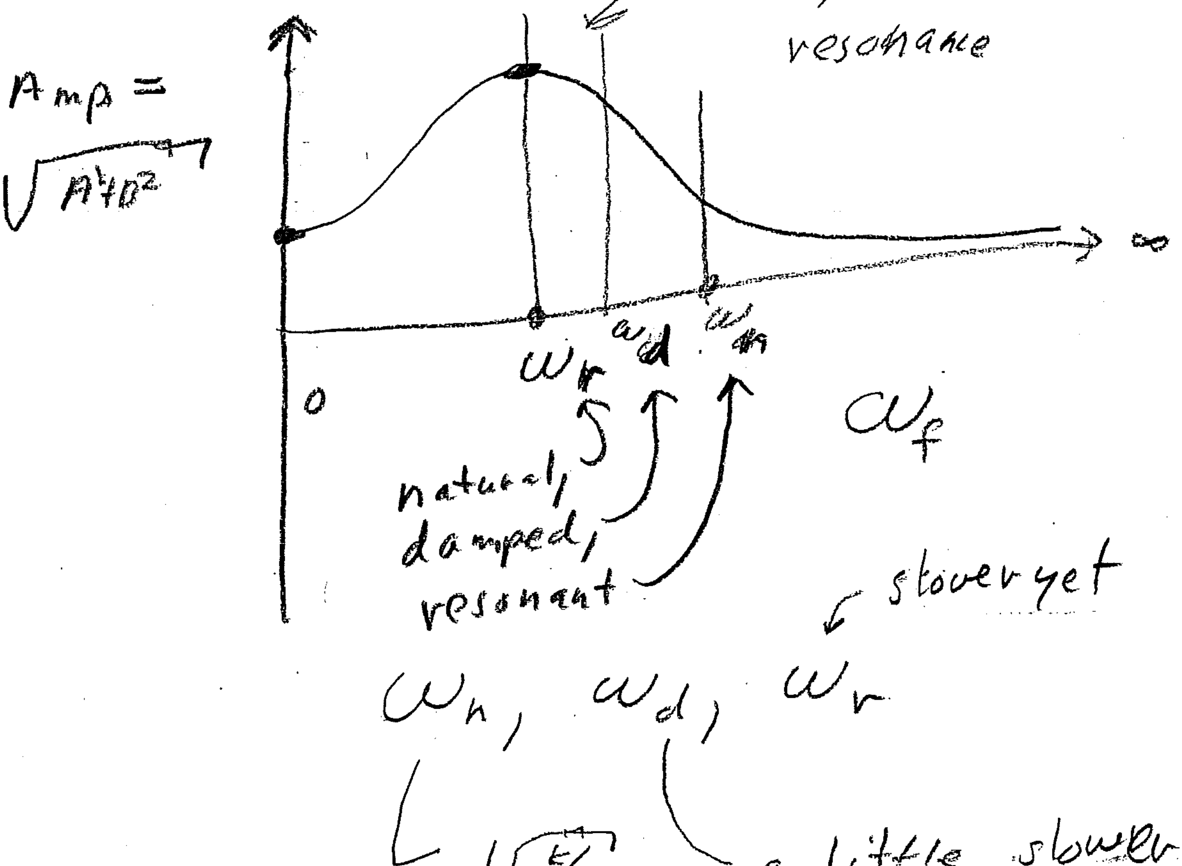


Amplitude of steady state respons

$X_p = A\cos(\omega t) + B\sin(\omega t)$

sine wave forcing

$$\text{Amplitude} = \sqrt{A^2 + B^2}$$



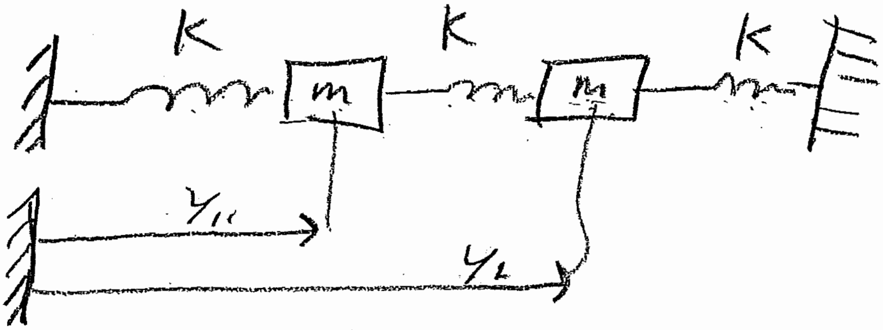
slower yet

ω_n, ω_d, ω_r

$\sqrt{k/m}$ a little slower

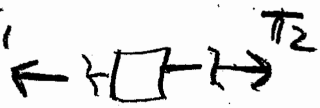
m DoF

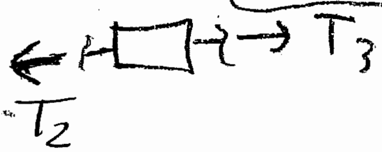
multi-DoF (cont'd from last lecture)



equilib. ≠ springs relaxed
= statics soln,
= dev. from const. partic. soln

FBDs





assume: Spring relaxed at $x_1 = x_2 = 0$

Not really right but gives right answer

$$T_1 = k x_1$$

$$T_2 = k(x_2 - x_1)$$

$$T_3 = -k x_2$$

LMB

mass 1: $\Sigma F = m\ddot{x}_1$

$$m\ddot{x}_1 = -T_1 + T_2$$
$$= -kx_1 + k(x_2 - x_1)$$

mass 2: $\Sigma F = m\ddot{x}_2$

$$m\ddot{x}_2 = -T_2 + T_3$$
$$= -k(x_2 - x_1) + -kx_2$$

---

demo

$$\ddot{x}_1 = (-2k\,x_1 + kx_2)/m$$

$$\ddot{x}_2 = (kx_1 - 2kx_2)/m$$

$$\Rightarrow \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \frac{-1}{m} \underbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[\dot{V}] = -\frac{1}{m} K [X]$$

$$[V] = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

x_1
x_2

$$K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$[\dot{X}] = [V]$$

QUIZ 1

Solution of

$$m\ddot{x} + kx = 0$$

$m > 0, \; k > 0$?

a) Linear fn,

b) Sinusoidal fn. ✓ $\ddot{x} = \frac{-k}{m} x$

c) parabolic fn.

d) exponential fn.

e) log fn.

Quiz 2

$$m\ddot{x} + kx = 0$$

$x(0) = 1$ $\quad$ $m = 4\ kg$

$\dot{x}(0) = 0$ $\quad$ $k = 1\ N/m$

period of oscillation $t_p = ?$

a) $t_p = 1\ s$

b) $t_p = \pi\ s$

c) $t_p = 2\pi\ s$

d) $t_p = 4\pi\ s$ ✓

e) $t_p = \frac{\pi}{2}\ s$

Soln to quiz 2

$X = A\cos(\omega_n t)$

ω_n natural freq.

$= \sqrt{K/m}$

in one period

$(\ \ ) = 2\pi$

$\omega_n t_p = 2\pi$

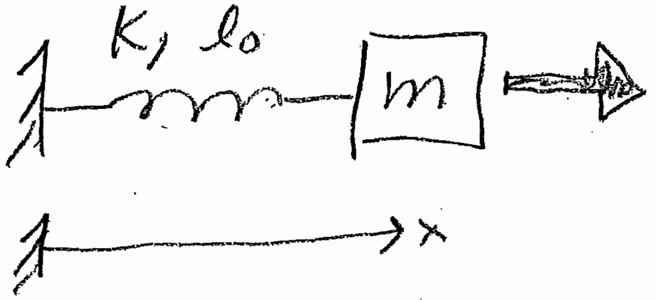
$t_p = \frac{2\pi}{\omega_n}$ $\sqrt{\frac{1\,N/m}{4\,Kg}}$

$= \frac{1}{2}\ s^{-1}$

$t_p = 4\pi\ s$

QUIZ 3

1 D, no gravity



$F = \text{const.}$

Force on wall $= F$?

a) yes, all the time ×

b) sometimes, eventually ✓

c) × No, never so long as $F \neq 0$

d) Depends on initial conditions and F, K, m, l_0 ✓/2

e) none of above ×

TODAY

* Review
* Normal Modes
* "Prof., I was wondering... units"
* Collisions

Review

What is this course about?

Given info. about force, geometry, motion.

Find other info. about force, geom., motion.

So far: 1 D motion of one particle

3) "F = ma"

2) $\dot{x} = v$, $\dot{v} = a$, $d = x_2 - x_1$

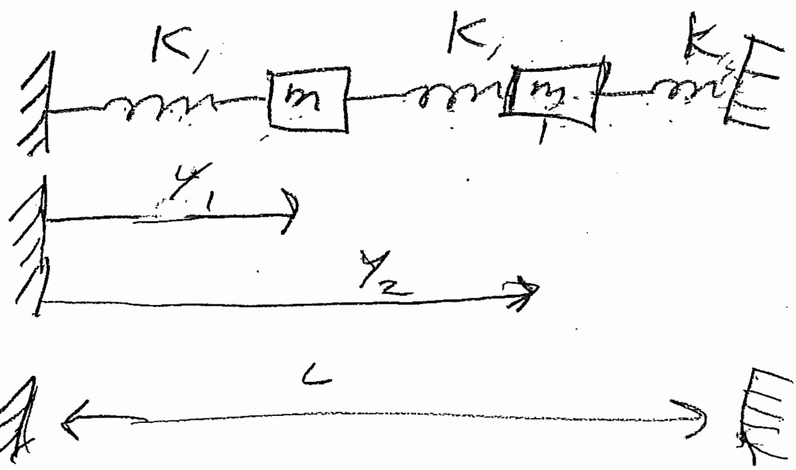
1) $F = \begin{cases} -kx, \text{ or} \\ -c_1 v, \text{ or} \\ -c_2 v^2, \text{ or} \\ mg, \text{ or} \end{cases}$

Tools: Matlab, Euler's ~~Method~~ plotting; ODEs: 1) $a\ddot{x} + b\dot{x} + cx = f(t)$ 2) $\frac{dx}{dt} = f(x)$

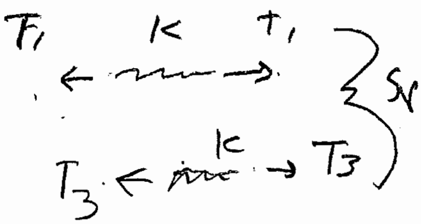
ex) Vibrations: Read text for details ch 9, 10.

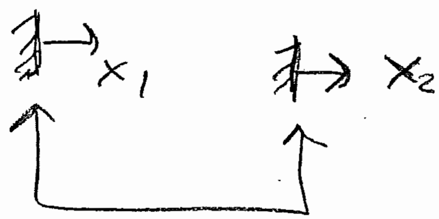
Normal Modes

ex)

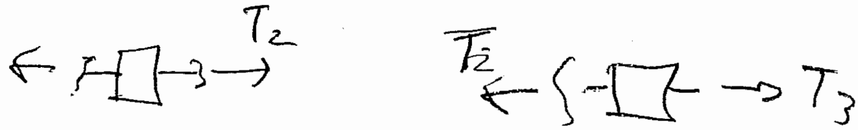


FBDs





Masses



$$\left.\begin{aligned} T_1 &= K x_1 \\ T_2 &= K(x_2 - x_1) \\ T_3 &= K(-x_2) \end{aligned}\right\} \text{Constit. Laws}$$

Mechanics

LMB:

$$\left.\begin{aligned} &\text{block 1}: \quad m\ddot{x}_1 = T_2 - T_1 \\ &\text{block 2}: \quad m\ddot{x}_2 = T_3 - T_2 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} m\ddot{x}_1 &= -2Kx_1 + Kx_2 \\ m\ddot{x}_2 &= -2Kx_2 + Kx_1 \end{aligned}\right]$$

Matrix form

$$\Rightarrow \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = - \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \underbrace{M}_{2\times 2}\ \underbrace{\ddot{X}}_{\ddot{x}_1,\ \ddot{x}_2} = -\underbrace{K}_{2\times 2}\ \underbrace{X}_{x_1,\ x_2}$$

$$\ddot{X} = -\underbrace{M^{-1}}_{\text{inv}(M)} \cdot K \cdot X$$

$$M^{-1} \cdot M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

mech probs.: M^{-1} exists

$$\boxed{\ddot{X} = -A X}$$

$A = M^{-1} K$

Solve w/ Matlab.

A = M^(-1) * K, or

M\K

backslash

ASIDE

$$B x = y$$

vectors

Matlab's backslash

x = B\y

$$\left.\begin{array}{l} \dot{x} = v \\ \\ \dot{v} = -A x \\ \\ z = \begin{bmatrix} x \\ v \end{bmatrix} \\ \\ \dot{z} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} \end{array}\right\} \text{ODEs}$$

<u>ANALYTIC Approach</u> to soln.

$$\ddot{x} = A x$$

$$\boxed{m \ddot{x} + K x = 0}$$

<u>Simple Solns:</u>

ex) $x_1 = n_1 \cos \omega t$

$x_2 = n_2 \cos \omega t$

from 2 mass prob.! $\Rightarrow$ ex) $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>eqn</u>

$$\boxed{M\ddot{x} + K x = 0} \quad \ast$$

<u>guess</u>

$$x = \begin{bmatrix} n_1 \\ n_2 \\ n_{\vdots} \end{bmatrix} \underline{\cos(\omega t)}$$

$$\left.\begin{matrix} n_1, n_2 \ldots \\ \omega \end{matrix}\right\} \text{all unknown}$$

$\uparrow$ const

$\ddot{x} = -\omega^2 n \cos(\omega t)$

$\uparrow$

$\begin{bmatrix} n_1 \\ n_2 \\ \vdots \end{bmatrix}$

($\ast$) plug into $\ast$

matrices

$$\Rightarrow M \cdot (-\omega_n^2 \cancel{\cos(\omega_n t)})\, n + K \cdot n \,\cancel{\cos(\omega_n t)}$$

$$\begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ \vdots \end{bmatrix}$$

$$-\omega_n^2 M n + K n = 0$$

$$\boxed{(-\omega^2 M + K) \cdot n = 0}$$

algebraic problem

$M^{-1} * \{ \}$

$$\Rightarrow \left(-\omega^2 \underbrace{I}_{\begin{bmatrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{bmatrix}} + M^{-1}K\right)\underbrace{n}_{\begin{bmatrix}n_1\\ \vdots\end{bmatrix}} = \underbrace{0}_{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}$$

$$\boxed{\left[\underbrace{A}_{M^{-1}K} - \lambda I\right] n = 0}$$

given A find λ, n

$$\boxed{\boxed{A n = \underbrace{\lambda}_{\omega^2} n}}$$

<u>ask Matlab</u>

$$\mathrm{eig}(\underbrace{M^{-1} * K}_{A})$$

$\Rightarrow$ <u>n vectors &</u>

<u>λ values that solutions</u>

Professor, I was wondering...

UNITS

QUANTITIES

$m_A = 100 kg$

$v = 2 m/s$

$L = m \cdot v = (100 kg) \cdot 2 (m/s)$

$L = \boxed{200 kg \cdot m/s}$

units must match

3 basic units: mass, length, time

MKS: kg, m, s

ex) $m = 7g \cdot (1)$

$= 7\cancel{g} \cdot \left(\frac{1 kg}{1000\cancel{g}}\right) = \frac{7}{1000} kg$

Force

$$[F] = N = 1\,kg \cdot \frac{1 \cdot m}{s^2}$$

$$1 = \left(\frac{1\,N}{1\,kg\,m/s^2}\right)$$

$$[W] = \{E\} = \{F\} \cdot \{d\} -$$

$$J = 1 \cdot N \cdot 1\,m$$

$$1 = \left(\frac{1\,J}{N \cdot m}\right)$$

$$E = \frac{1}{2} m v^2$$

$$[E] = [m]\,[v]^2$$

$$\boxed{J = 1 \cdot kg \left(\frac{m}{s}\right)^2}$$

1 Kg moving at 1 m/s

$$E \neq 1\ kg\ m^2/s^2$$

$m = 100\ kg$

"weighs 100 kg"

"$F = 100 kg$" $\boxed{[F] = kgf}$

$\boxed{kgf = g \cdot kg}$ Note: $1 kgf \neq 1 kgm$

English

$[m] = lbm$

$[t] = s$

$[L] = ft$

$1 = \frac{12 in}{1 ft}$

$1 = \frac{5280 ft}{1 mi}$

$[F] = [m] \cdot [a]$

"Poundal" $= 1 lbm \cdot 1 ft/s^2$

$lbf \equiv g \cdot lbm \neq$ poundal

"pound force = gravity force on pound mass lbm ≠ lbf

---

lbm, ft, s, poundal

$\boxed{\text{slug, ft, s, lbf}}$

consistant units

Quiz

A system of m linear equations in n unknowns $(x_1, x_2, \ldots)$ can be put in the form

$$Ax = b$$

What are the sizes of A, x & b?

Size	A	x	b
a)	$n \times n$	$n = n \times 1$	$n = n \times 1$
b)	$m \times n$	$m = m \times 1$	$n = n \times 1$
c)	$n \times m$	$m = m \times 1$	$n = n \times 1$
d)	$m \times m$	$m = m \times 1$	$m = m \times 1$
✓ e)	$m \times n$	$n = n \times 1$	$m = m \times 1$

Lecture 7, Feb. 13, 2019

TODAY

UNITS
Normal Modes
Collisions

Quiz @ end

Systems of units

[t]	[l]	[mass]	[F] N	[W] = [E] J	[P] W
s	m	kg	1 kg m/s^2	1 kg m^2/s^2	1 kg m^2/s^3
s	ft.	lbm	1 lbm ft/s^2 pdl poundal	lbm ft^2/s^2	lbm ft^2/s^3
s	ft.	slug	lbf slug ft/s^2	– –	– –
s	m	mug	Kgf = mug m/s^2	– –	– –

2 Rules:

1: Carry units

2: Mult. by 1 as you like

UNITS (cont'd)

Some versions of 1

$$1\,ft = 12\,in \implies 1 = \frac{12\,in}{1\,ft} = \frac{1\,ft}{12\,in}$$

12 in = 1 ft
3 ft = 1 yd
5280 ft = 1 mi

1760 yds = 1 mi

2.2 lbm = 1 kgm

lbm · g = lbf

$$L\begin{cases} 9.8\,m/s^2 \\ 32.2\,ft/s^2 \end{cases}\Bigg] \quad 1 = \frac{32\,ft/s^2}{9.8\,m/s^2}$$

Kg · g = 9.8 N

$$g = \frac{1\,lbf}{1\,lbm} = \frac{1\,Kgf}{1\,Kgm} = \frac{9.8\,N}{Kgm}$$

1 lbf ≈ 5.5 N

39.37 in = 1 m

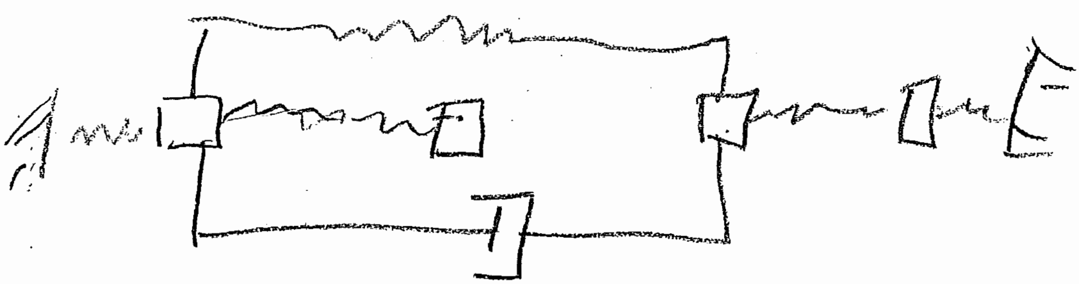
1 W = J/s

1 hp = 550 ft lbs/s
= 746 watts

Normal Modes / Vibes etc

"General" Case

line of springs, mass & dashpots
w/ forcing $\rightarrow F$



LMB for each mass; set of eqs.

$$m_1 \ddot{X}_1 = K_{12}(X_2 - X_1) + K_{13}(X_3 - X_1)$$
$$c_{14}\ \dot{X}_2 - \dot{X}_1 + \text{const. stuff}$$
$$+ \text{forcing}$$

$$m_2 \ddot{X}_2 = \ldots$$

$\vdots$

$\Rightarrow$ Linear Eqs $\Rightarrow$ Matrix form

ignore for vibes

$$\left\{ M\ddot{x} + C\dot{x} + Kx = C_1 t \right\}$$

$F_0 \sin \omega t$

$M = n \times n$

$C = n \times n$

$K = n \times n$

$x = n \times 1$

$\llcorner x(t)$

$C_1 = n \times 1$

$F_0 = n \times 1$

$\omega =$ scalar

$M^{-1}\{\ \} \Rightarrow$

$$\ddot{x} = -M^{-1} \cdot \left[C\dot{x} + Kx - \cancel{C_1} F_0 \sin \omega t \right]$$

$$\left[\begin{array}{l} \dot{V} = -M^{-1} \left[\quad " \quad " \quad " \quad - \quad \right] \\ \\ \dot{x} = V \end{array} \right.$$

2n first order ODE's

2 Key ideas

I: You can solve in matlab

II. Normal Modes

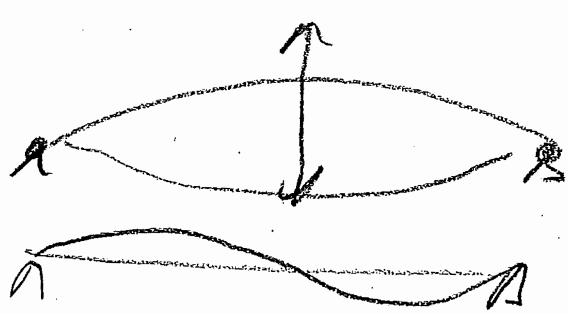
Normal Modes: $C = 0,\ C_1 = 0$
$F_0 = 0$

$$\{ M\ddot{x} + Kx = 0 \}$$

$$M^{-1}\cdot\{\} \Rightarrow \ddot{x} + M^{-1}Kx = 0$$

$$x = \underset{\substack{\uparrow \\ \text{const}}}{W} \underset{\substack{\uparrow \\ \text{scalar}}}{\sin\omega t} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \end{pmatrix} \sin\omega t$$

(violin string)



59

$\omega \neq w$
double-ω
omega

<u>eqn:</u> $\ddot{x} + M^{-1} K x = 0$

<u>plug in guess</u>

$$\frac{d^2}{dt^2}\left(w \sin \omega t\right) + M^{-1}K\left[w \sin \omega t\right] = 0$$

$$\left[-\omega^2 + M^{-1}K\right] w = 0$$

$$\left[\underbrace{-\omega^2}_{\lambda} I + M^{-1}K\right] w = 0$$

$$\left[A - \lambda I\right] w = 0$$

$M^{-1}K$

$$\boxed{A w = \underbrace{\lambda}_{\omega^2} w}$$

$[V, D] = \text{eig}(A)$

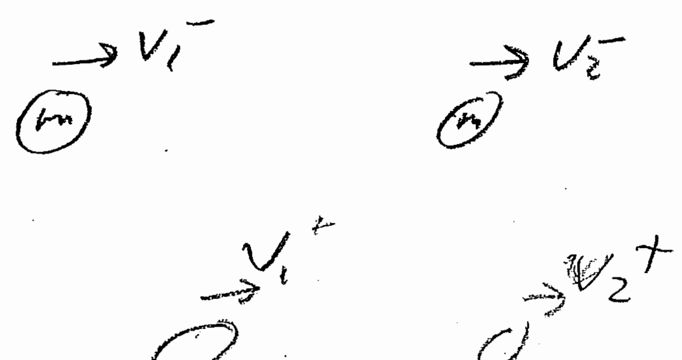
$w = I \cdot w$

e-vector $\equiv V$; $\quad A\,v = \underbrace{\lambda}_{\text{e-value.}}\,v$

$$\underbrace{A\cdot(7v) = \lambda\cdot(7v)}_{\text{const * e-vector} \Rightarrow \text{another e-vector}}$$

$$\lambda = \omega^2$$

Collisions 1D:



$\underbrace{v_2 < v_1}_{\text{collide}}$

collision law :

$$(v_2 - v_1)^+ = -\underbrace{e}_{0 \le e \le 1}\,(v_2 - v_1)$$

$\pm 2\%$ 6

Quiz

1 lbm starts from rest.
A const 1 lbf is applied
to it for 1 s.

a) Distance $= X \approx 16$ ft

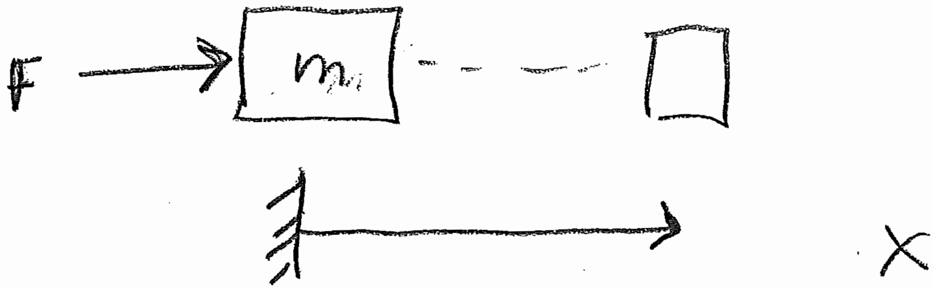
b) $E_K \approx 500$ lbm ft^2/s^2

c) Work done ≈ 500 lbm ft^2/s^2

d) all of above (a, b & c) ✓

e) Any of these:
none of a, b, c
a & b, a & c, b & c

Soln to Quiz ①



LMB

$$F = ma$$

$$\Rightarrow a = F/m$$

$$= 1\,lbf / 1\,lbm$$

$$\boxed{1\,lbf = g \cdot 1\,lbm}$$

$$= g$$

$$a = 32\,ft./s^2$$

at $t = 1s$ $$V = at = \left(32\,\frac{ft}{s^2}\right) \cdot (1\,s)$$

$$= 32\,ft./s$$

$$x = \frac{1}{2}at^2 = (32\,ft/s^2)\,(1\,s)^2$$

$$= 16\,ft.$$

$$L = m\,V = (32\,ft/s) \cdot (1\,lbm) = 32\,lbm\,ft./s$$

$$E_k = \frac{1}{2}\,mV^2 \approx 500\,lbm\,ft/s^2 \;(= F \cdot x)$$

LECTURE 8

MON, Feb. 18, 2019

Quiz questions at end

TODAY

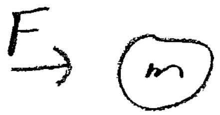
1) Collisions (cont'd)

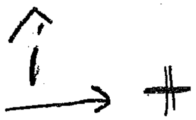
2) Particles in space (vectors)

Collisions (cont'd)

Recall

1-D FBD





$$F = ma$$

$$a = \frac{dV}{dt}$$

$$\Rightarrow \int_{t_1}^{t_2} F\,dt = m\int_{t_1}^{t_2} a\,dt$$

$$'' = m(V_2 - V_1)$$

$$'' = L_2 - L_1$$

$$\boxed{\text{Impulse} = \Delta L}$$

change in linear momentum

In collisions

1) Collision force so big that other forces are negligible, ⟹ collisional FBD neglects "finite" forces (e.g., mg, springs, etc)

2) We ignore details of $F_{collision}$ & instead only note net impulse,

⟹ From just before coll. to just after;

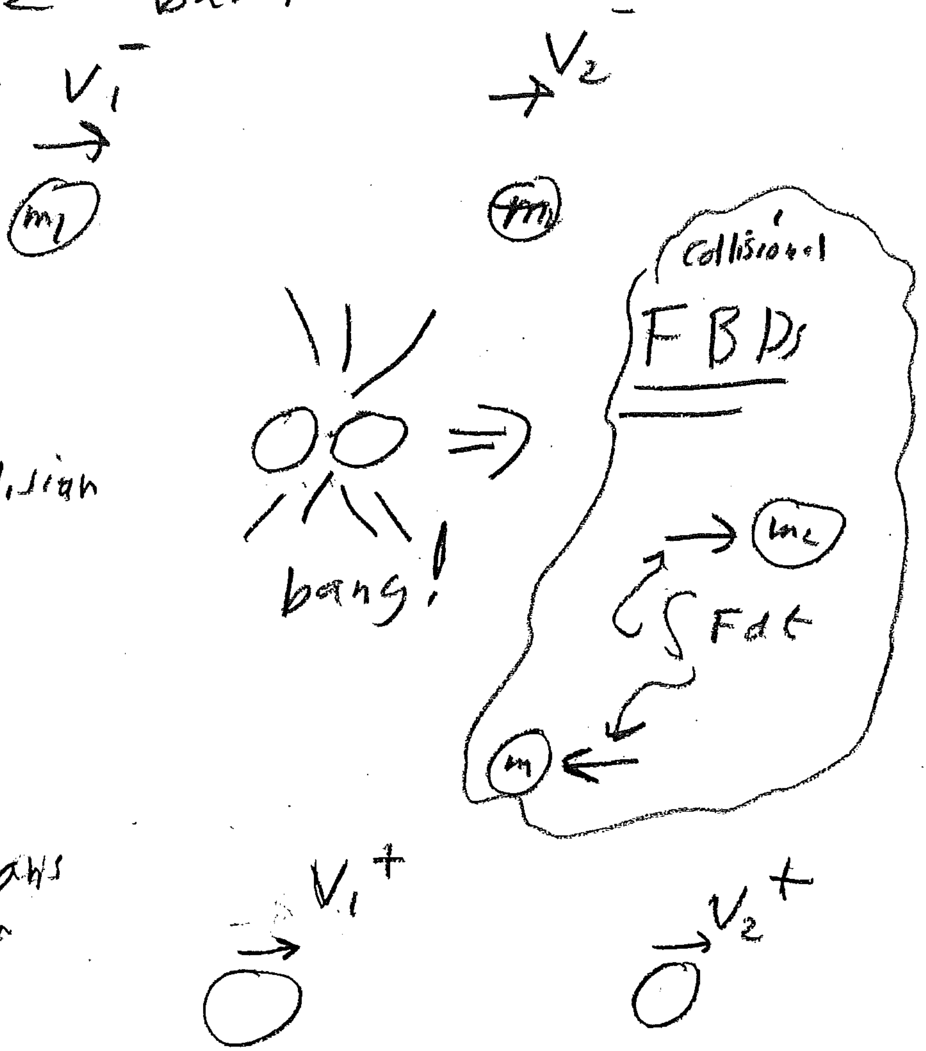
ΔL only due to coll. forces (coll. impulses) ex) 2 balls

— means before coll.

then

during collision

+ means after



Coll. Eqs

LMB:

$$\boxed{\begin{aligned} m_2 V_2^+ - m_2 V_2^- &= \int F dt \\ m_1 V_1^+ - m_1 V_1^- &= -\int F dt \end{aligned}}$$

Note

$$L_{tot} = L_1 + L_2$$

$$\Rightarrow \text{const.}$$

$$\boxed{L_{tot}^{+} = L_{tot}^{-}}$$

Cons. of Lin. mom. for system

Constitutive Law

Coeff. of restitution : e

$$\underbrace{(\text{separ. speed})}_{\text{after}} = e \cdot \underbrace{\begin{pmatrix}\text{approach} \\ \text{speed}\end{pmatrix}}_{\text{before}}$$

e = coeff. of restitution

Usually

$$0 \le e \le 1$$

What about energy?

Given V_1^-, V_2^-, e (m_1, m_2)

can calc. V_1^+, V_2^+ $\Big\}$ 2 unknowns

How?

1) $\Delta L_{TOT} = 0$

2) $(V_2^+ - V_1^+) = -e(V_2^- - V_1^-)$

1) & 2) are 2 eqs in 2 unknowns,

$\Rightarrow V_1^+ \;\&\; V_2^+$

See Matlab Sample from Lect. 7

$\Rightarrow$ cal de cal,

$$\Delta E_k = \left(\frac{1}{2} m_1 V_1^{-2} + \frac{1}{2} m_2 V_2^{-2}\right) - \left(\frac{1}{2} m_1 V_1^{+2} + \frac{1}{2} m_2 V_2^{+2}\right)$$

a bunch of algebra

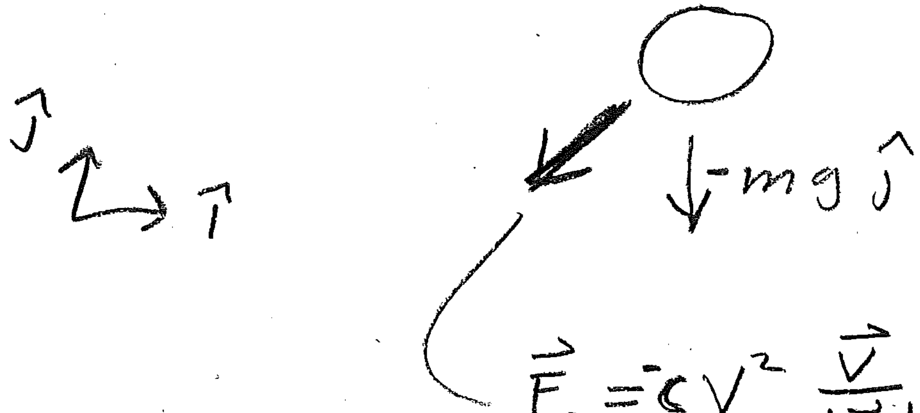
$\Rightarrow$ $e = 1 \Rightarrow$ const E_k

$e = 0 \Rightarrow$ maximizes dissipation

Motion in 2D & 3D

ex) quadratic drag $\vec{V}$

FBD



$$\vec{F}_D = -cV^2 \frac{\vec{V}}{|\vec{V}|}$$

$$= -c\,\vec{V}\,|\vec{V}|$$

ALTERNATIVE!

Linear Drag

$$\vec{F}_D = -c\vec{V}$$

$\Rightarrow$ Linear ODEs, but not continued in this example quadratic drag ballistics (70)

not solvable "in closed form"

$\Rightarrow$ Num. Methods

LMB

$$\vec{F}_{TOT} = m\vec{a}$$

$$\boxed{-mg\hat{j} - c\vec{V}|\vec{V}| = m\vec{a}} \quad (*)$$

"EoM"
Eqs. of Motion

$$\dot{z} = f(t, z)$$

$$z = \begin{bmatrix} x \\ y \\ V_x \\ V_y \end{bmatrix}$$

$$\{*\}\cdot\hat{\imath} \Rightarrow -cV_x\sqrt{V_x^2+V_y^2} = m\dot{V}_x$$

$$\{*\}\cdot\hat{j} \Rightarrow -mg - cV_y\sqrt{V_x^2+V_y^2} = m\dot{V}_y$$

$$\left.\begin{aligned}\dot{x} &= V_x \\ \dot{y} &= V_y\end{aligned}\right] \text{Kinematics}$$

"EOM"

$$\dot{V}_x = \frac{1}{m}\left(-c\, V_x \sqrt{V_x^2 + V_y^2}\right)$$

$$\dot{V}_y = \frac{1}{m}\left[\left(-c\, V_y \sqrt{V_x^2 + V_y^2}\right)\right] - g$$

$$\dot{z} = f(t, z)$$

ICs: $x_0 = 0$, $y_0 = 0$

$V_{x0} =$ ___ , $V_{y0} =$ ___

Solve on Computer to get

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

Quiz

$$0 \xrightarrow{V_A^- = 1} \qquad \qquad V_B^- = -1 \quad \leftarrow 0$$

$$\boxed{e = -1}$$

action on B

a) Impulse > 0

b) Impulse < 0

c) Impulse $= 0$ ✓

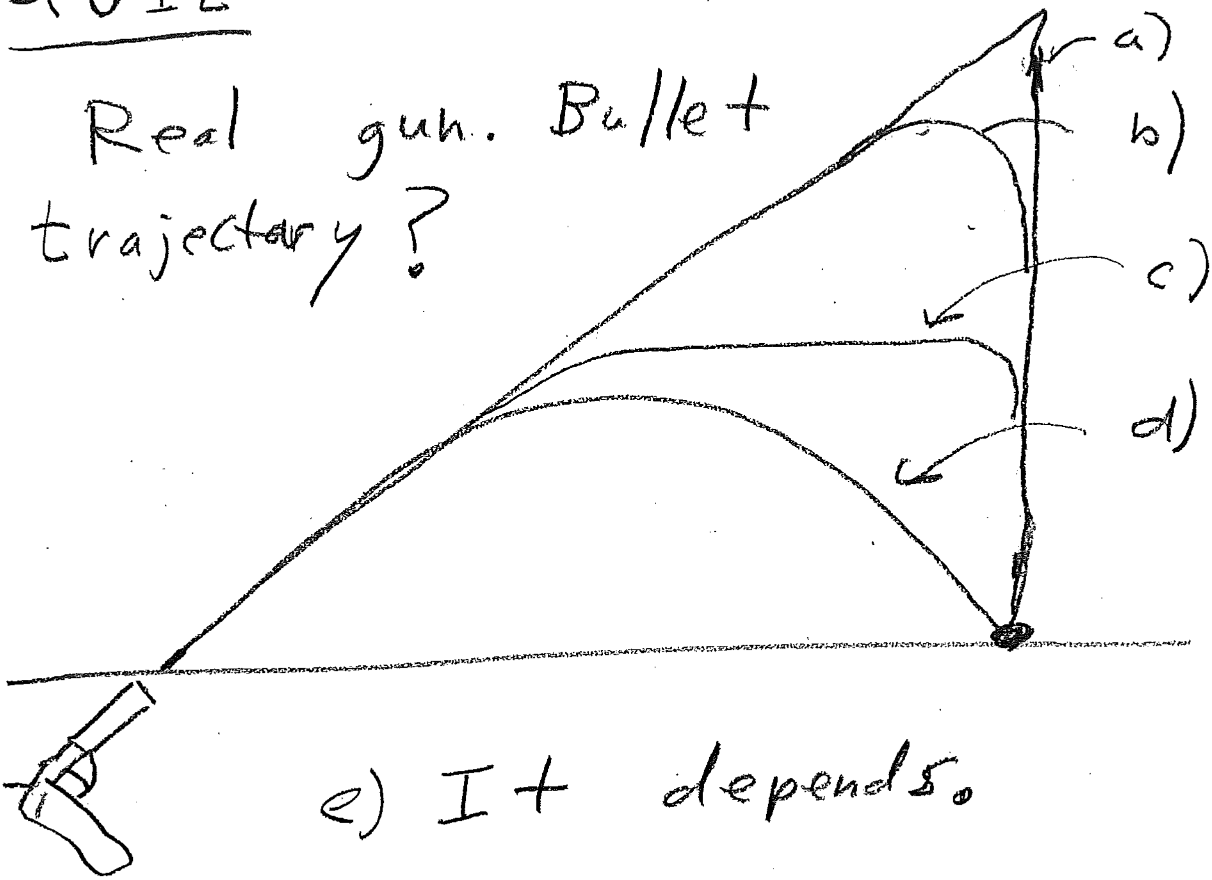
d)

e)

See Matlab code
to check

QUIZ

Real gun. Bullet trajectary?



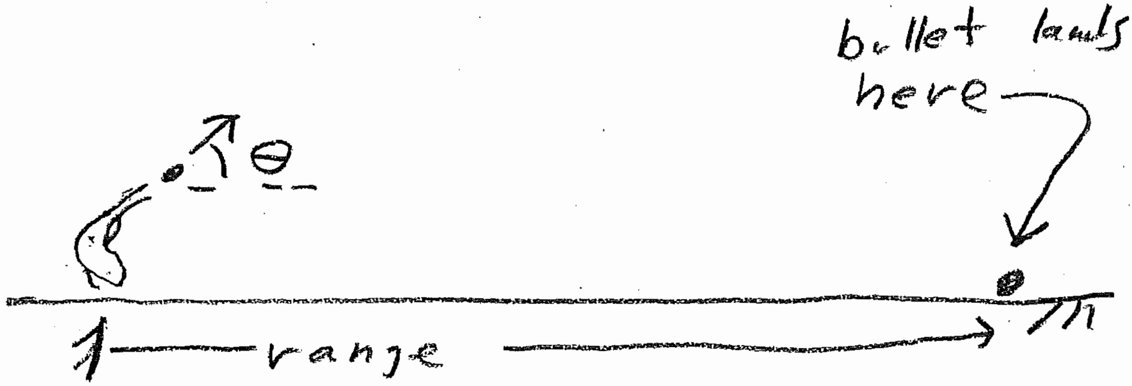
e) It depends.

b ✓ Try it in matlab

Quiz

Shooting some real gun on flat ground. Calm day.

Angle for max range?



$\theta_{max} = ?$

a) 0°

b) $\theta_m < 45^\circ$ ✓ (try it in Matlab)

c) $\theta_m = 45^\circ$

d) $\theta_m > 45^\circ$

e) depends on m, v, etc.

QUIZ oups 83

TODAY

* Misc review
* 2 D dyn. of Particle (cont'd)
* theorems

Normal Modes review

ANY springs & masses;
Rel. to equilib.;
X_i are displ. rel. to equilib.;

$$M \ddot{X} + K X = 0 \qquad *$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Guess $X(t) = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \sin(\omega t)$

double-u

omega

Plug into *
$\Rightarrow$ Good guess if ω^2 is an e-value of A with e-vector W

$A = M^{-1} \cdot K$

$M^{-1} \cdot M = I$

E-vector eqn.

$$\boxed{A w = \underset{\omega^2}{\lambda} w}$$

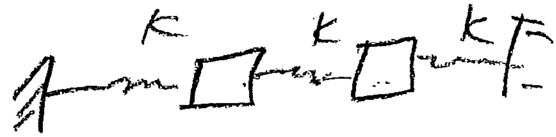
$\omega \neq w$

Vibes probs $\Rightarrow$ allways get A w/ n Lin. Ind. e-vectors

$$\boxed{\omega_i = \sqrt{\lambda_i}}$$

<u>Some special probs</u> $\Rightarrow$ can guess/ see soln.

$$\omega = \sqrt{\frac{K_{eff}}{m}}$$

ex) 

e-vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$K_{eff} = 3k$

$\Rightarrow \omega_n = \sqrt{3k/m}$

$$\underline{M\ddot{x} + Kx = 0}$$

$$\Rightarrow \quad \ddot{x} = -\underbrace{M^{-1}K}_{A}\, x$$

$$\left[\begin{array}{l} \dot{v} = -\underset{3\times 3}{A}\, x \\ \hline \dot{x} = v \end{array}\right]$$

1st order form

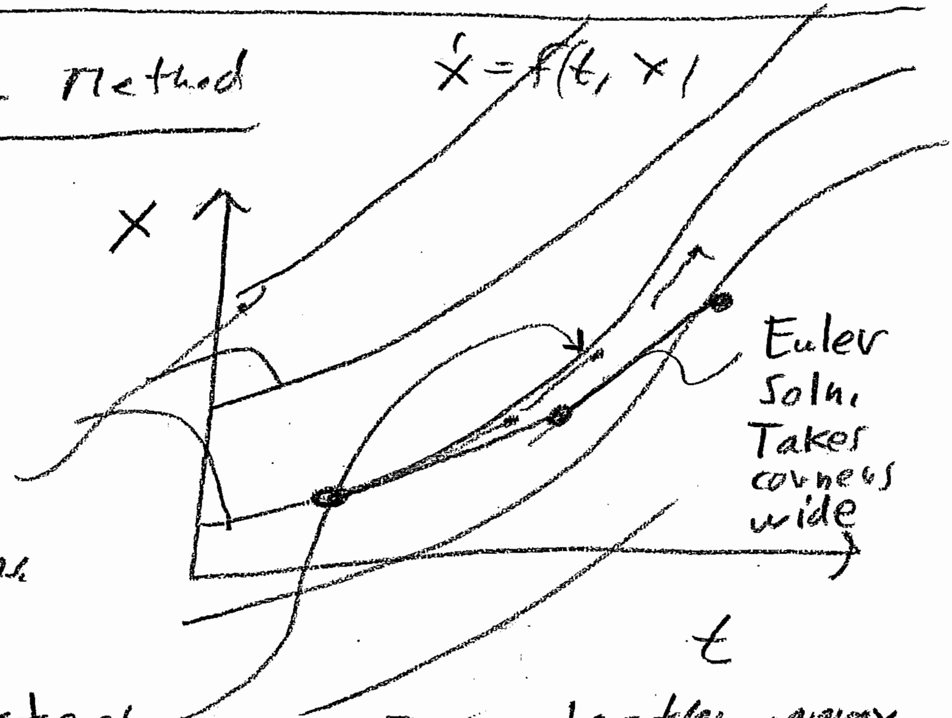
$$\dot{z} = \underset{6\times 6}{B}\, z \qquad B = \left[\begin{array}{ccc|ccc} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ \hline & -A & & 0&0&0 \\ &&&0&0&0 \\ &&&0&0&0 \end{array}\right]$$

$$\dot{z} = f(t, z)$$

Euler Method

$$\dot{x} = f(t, x)$$

ex)



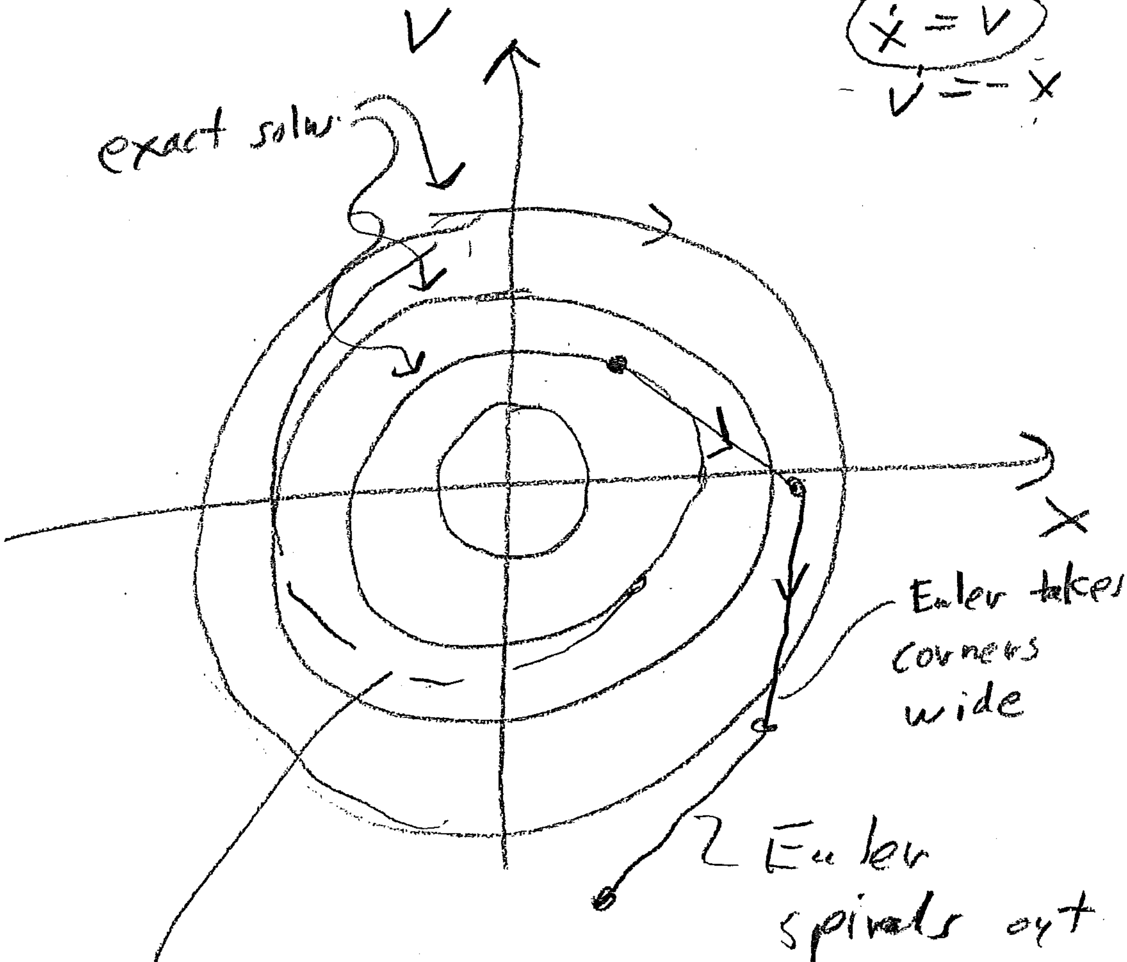
smaller steps $\Rightarrow$ better approx.

ex) Harmonic Osci

$$\ddot{x} + x = 0$$

$$\dot{x} = v$$

$$\dot{v} = -x$$



smaller Δt
smaller h } $\Rightarrow$ less spiral

$$\boxed{\dot{z} = f(t, z)}$$

MAIN DYN. SYST. EQN

Gen. 1D Vibe prob.

$$m\ddot{x} + cx' + Kx = C_1 + C_2 \sin(\omega t)$$

Ignore forcing & damping $\Rightarrow$ $\boxed{m\ddot{x} + Kx = 0}$

Lots of masses $\Rightarrow$ $\underset{n\times n}{M}\,\underset{n\times 1}{\ddot{x}} + \underset{n\times n}{K}\,\underset{n\times 1}{x} = \underset{n\times 1}{0}$

guess: $x(t) = W \sin(\omega t)$

const

$$A = [\;\;] = M^{-1}K$$

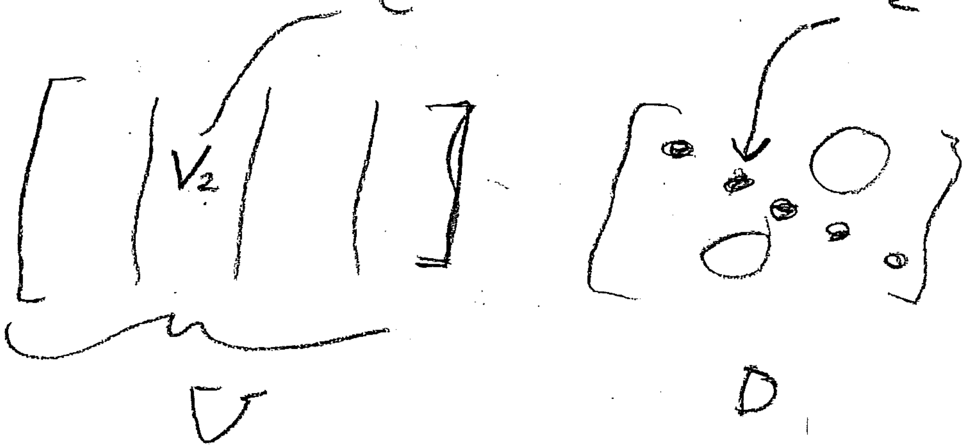
MATLAB soln

$[V, D] = \text{eig}(A)$

corresponding

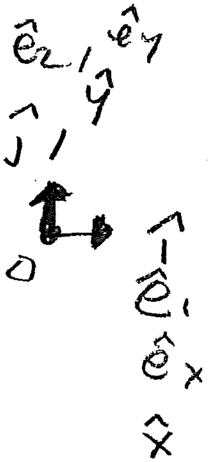
e-vector 2

e-value 2

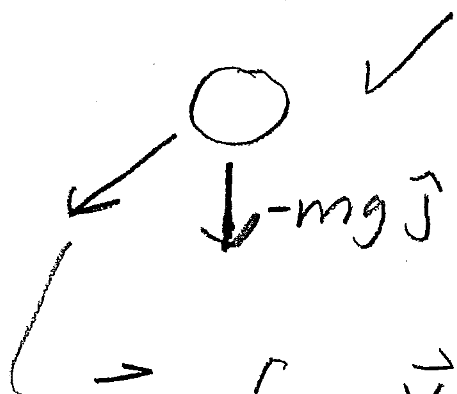


2D motion of particle

Review trajectory



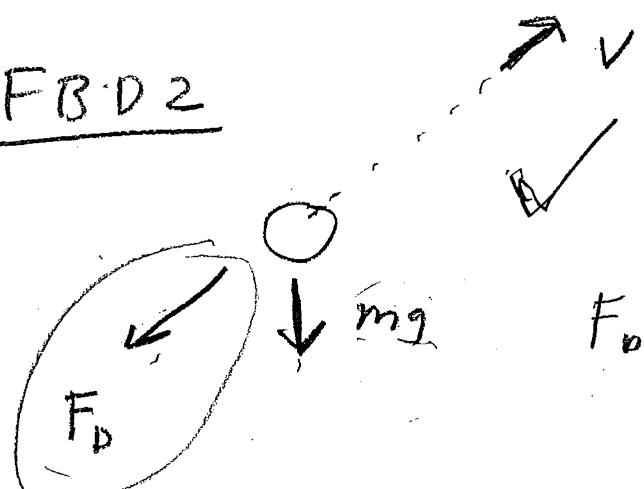
FBD 1 !!



$$\vec{F}_D = \begin{cases} -c_1 \vec{V} \\ -c_2 |\vec{V}| \vec{V} \end{cases}$$

FBD should have enough info to write EOM

FBD 2



$$F_D = \begin{cases} cV \\ cV|V| \end{cases}$$

scalar * unit vector drawn

LMB

$$\sum \vec{F}^{ext} = m\vec{a}$$

$$\Rightarrow \quad m\vec{a} = -mg\hat{j} + \vec{F}_D$$

$$\vec{a} = -g\hat{j} + \vec{F}_D/m$$

linear

$$\vec{F}_D = \begin{cases} -c\vec{V} \\ -c\vec{V}|\vec{V}| \end{cases}$$

quadratic

1st order form

$$\begin{cases} \dot{\vec{r}} = \vec{V} \\ \dot{\vec{V}} = -g\hat{j} + \vec{F}_D/m \end{cases}$$

4 1st order ODEs

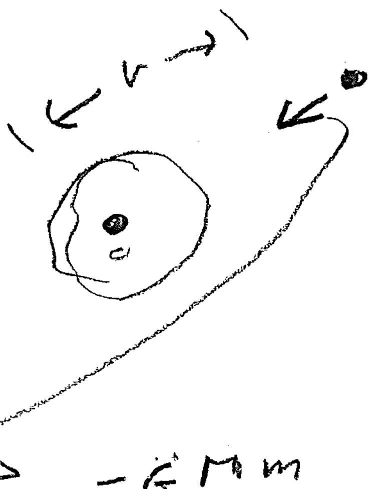
Various forces

drag forces : $\vec{F}_D = \begin{cases} -c\vec{v}\,|\vec{v}| \\ -c\vec{v} \end{cases}$

(see prev. pages)

near earth gravity : $-mg\hat{j}$

"Real" gravity : inverse square



$\left.\frac{mMG}{r^2}\right|_{R_e=r} = mg$ at surf. of earth

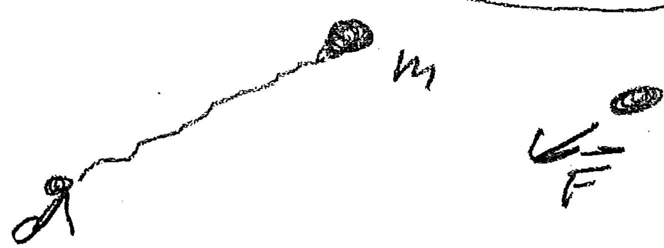
$\Rightarrow \boxed{MG = gR_e^2} \Rightarrow$

$$\vec{F}_g = -\frac{GMm}{r^2}\frac{\vec{r}}{|\vec{r}|}$$

$$= -g\frac{\vec{r}}{(r/R_e)^2}\frac{m}{r}$$

$$= -gm\frac{R_e^2}{r^3}\vec{r}$$

Spring

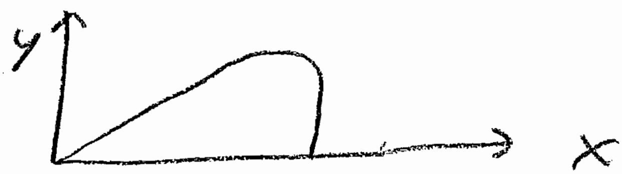


$$\vec{F}_s = -K(|\vec{r}| - l_0)\,\vec{r}/|\vec{r}|$$

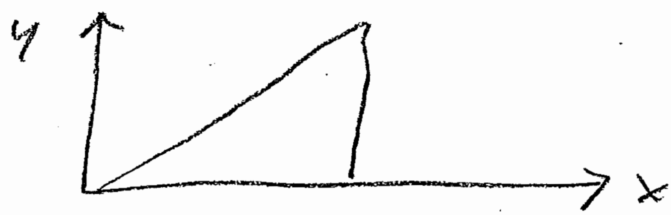
QUIZ

Trajectory on Matlab ex?

(a)



(b)



(c)

None of above (d)

Ans: huge drag, huge V_0 $\Rightarrow$ (c)

Feb. 27, 2019 WED

TODAY

2D & 3D dynamics

Review (Math)

Lots of probs. reduce to

soln. of * Lin. Alg Eqs ⇒ \ (backslash)

* ODEs ⇒ midpoint solver

$\dot{z} = f(t, z)$

1D Mechanics

$\boxed{F = m \underbrace{a}_{\ddot{x}}}$

oscillators, drag, vibrations, etc

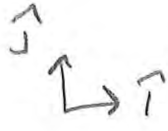
✓ (done)

2D & 3D mechanics

Force: $\vec{F} = \nearrow F$ means

a scalar F mult. by a unit vector

$\vec{F} = F \cdot \hat{a}$ $\hat{a} \equiv \nearrow$



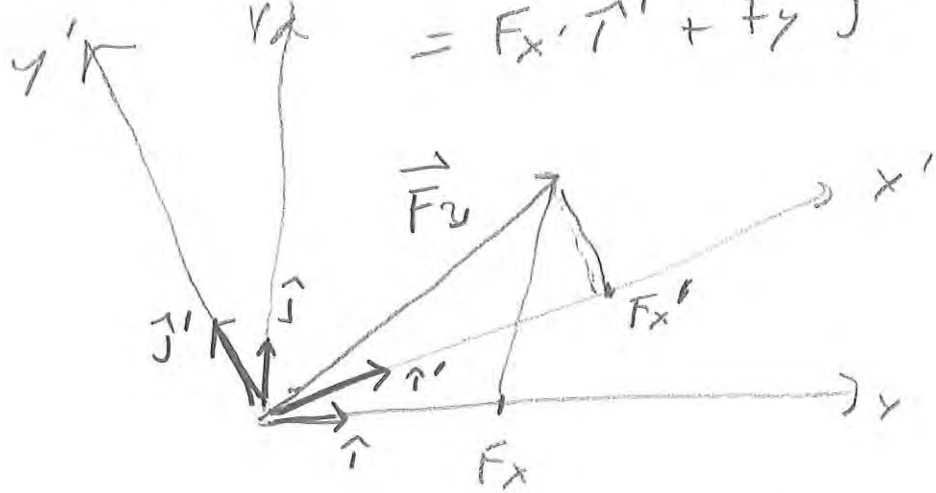
$\vec{F} = \underbrace{F_x \hat{i} + F_y \hat{j} + F_z \hat{k}}_{\text{components times unit vectors}}$

$$[\vec F] = \underbrace{[F_x,\ F_y,\ F_z]'}_{\text{``} = \vec F \text{''}}$$

List of components

$$\vec F = F_x\,\hat\imath + F_y\,\hat\jmath$$

$$= F_{x'}\,\hat\imath' + F_{y'}\,\hat\jmath'$$



$$F_x\,\hat\imath + F_y\,\hat\jmath = F_{x'}\,\hat\imath' + F_{y'}\,\hat\jmath'$$

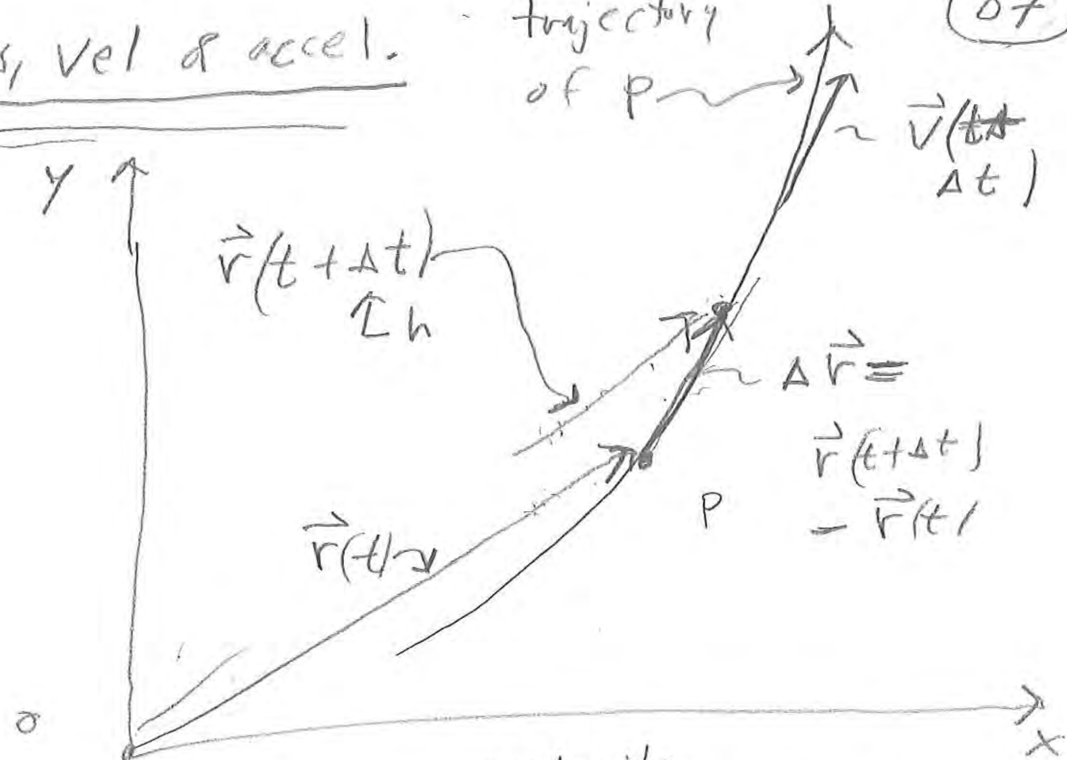
$$\boxed{\vec F = \vec F}$$

$$[F_x,\ F_y]' \neq [F_{x'},\ F_{y'}]'$$

vector = vector

but list of components ≠ list of components

Pos, vel & accel.

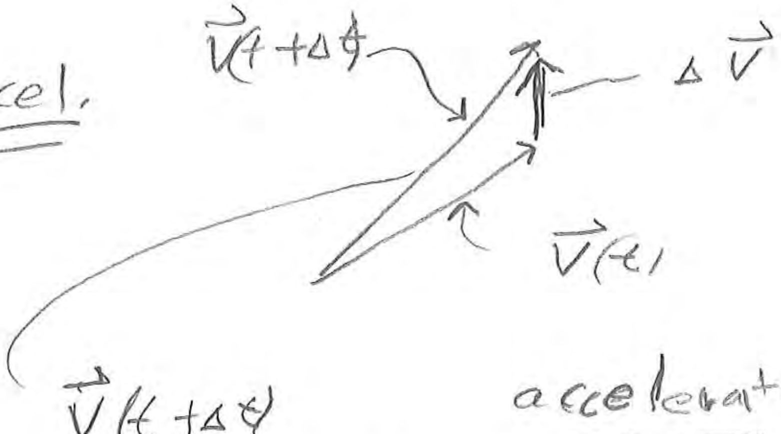


$$\vec{r}_{P/O} = \vec{r}$$

velocity

$$\boxed{\vec{V} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}}$$

accel.



acceleration

$$\boxed{\vec{a} = \frac{\Delta \vec{V}}{\Delta t}} \quad \lim_{\Delta t \to 0}$$

Components

$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

$\hat{i}, \hat{j}$ are constant

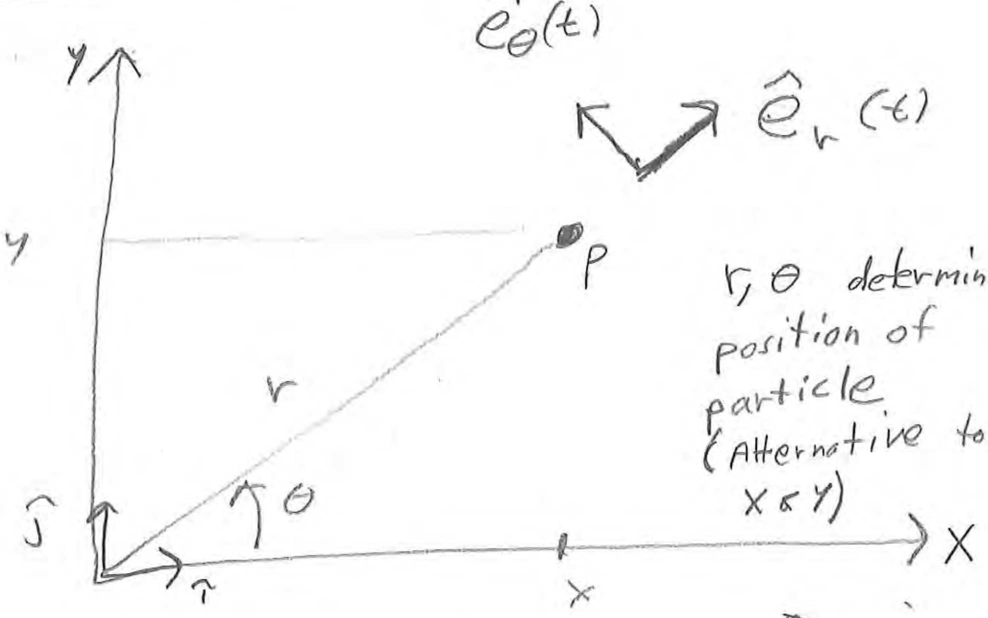
$$\vec{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$$

$$= v_x\,\hat{i} + v_y\,\hat{j}$$

$$\vec{a} = \dot{v}_x\,\hat{i} + \dot{v}_y\,\hat{j}$$

$$= \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j}$$

Polar coord.



r, θ determines position of particle (Alternative to x & y)

given $r(t)$ & $\theta(t)$
what are $\vec{v}$ & $\vec{a}$?

$\hat{e}_\theta \perp \hat{e}_r$

$\hat{e}_\theta = \hat{k} \times \hat{e}_r$

$\hat{k}$ unit vect. in z dir.

$\hat{e}_\theta$ in dir. part. moves if θ changes & r does not.

position in polar coord.

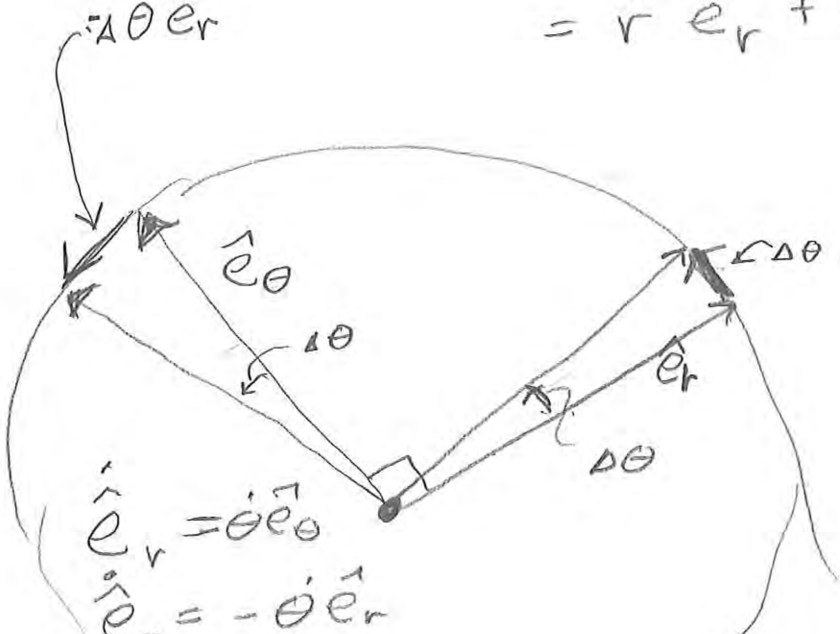
$$\vec{r} = r\hat{e}_r$$

vel.

$$\vec{v} = \frac{d}{dt}(\vec{r})$$

$$= \dot{r}\,\hat{e}_r + r\dot{\hat{e}}_r$$

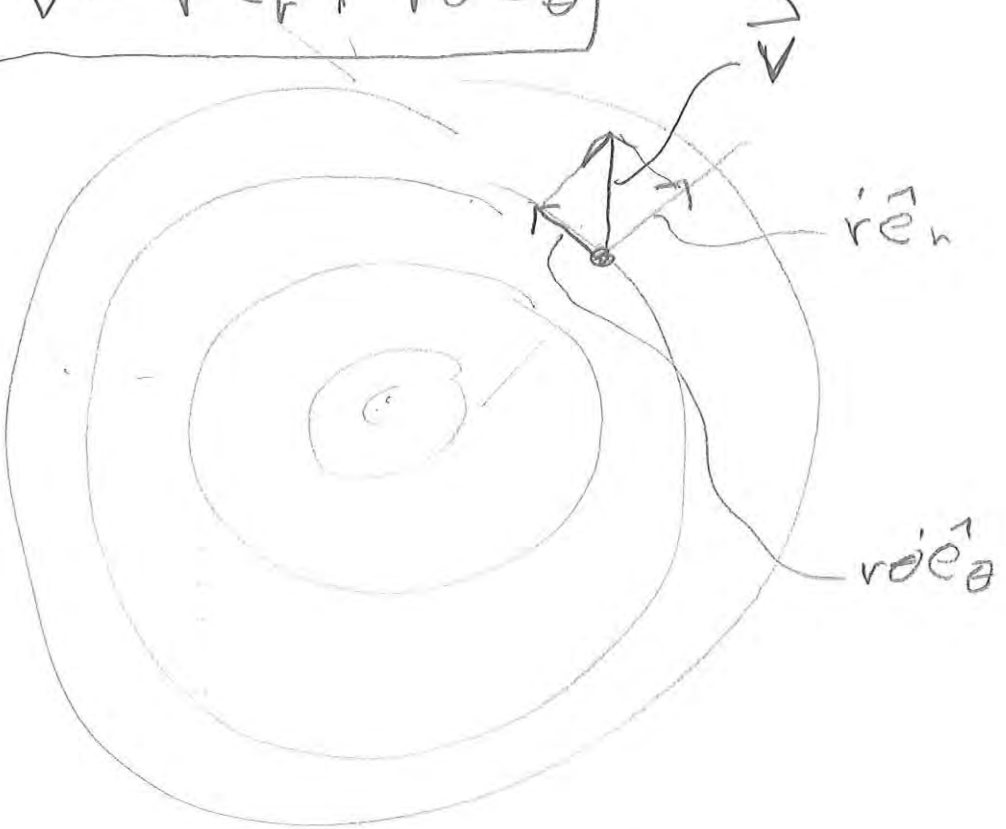
$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$



$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$

$\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$

$$\boxed{\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta}$$



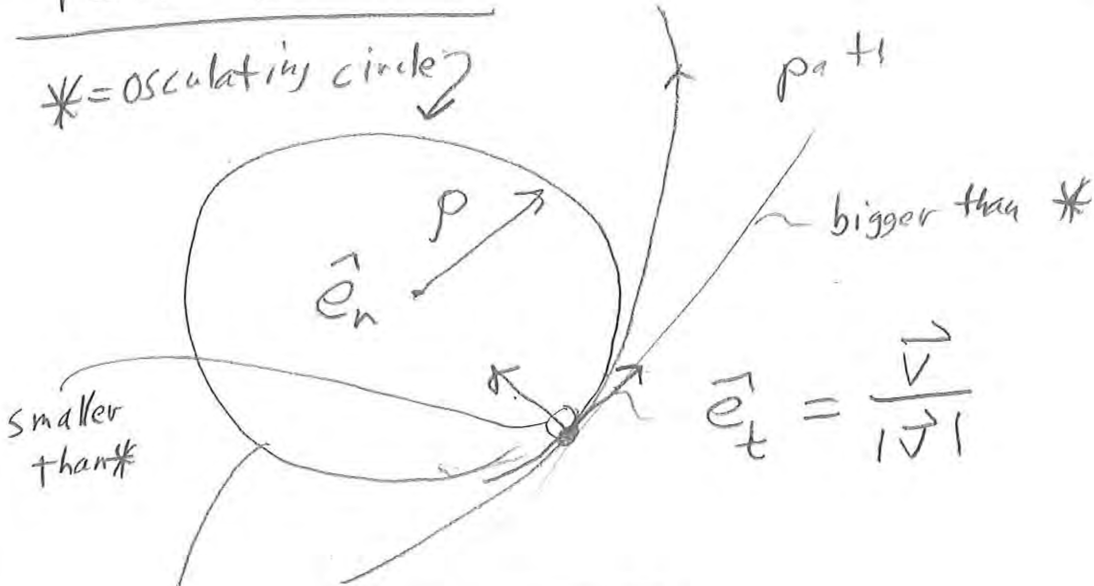
$$\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right)$$

$$\Rightarrow \boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + \underline{2\dot{r}\dot{\theta}})\hat{e}_\theta}$$

Skip steps

(see text Sect 14.1)

Path coord:



$$\hat{e}_t = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{\vec{v} = v\,\hat{e}_t}$$

$$\vec{a} = \dot{v}\,\hat{e}_t + v\,\dot{\hat{e}}_t$$

$$\boxed{\vec{a} = \dot{v}\,\hat{e}_t + v^2\,\kappa\;\hat{e}_n}$$

Centrip. Accel.

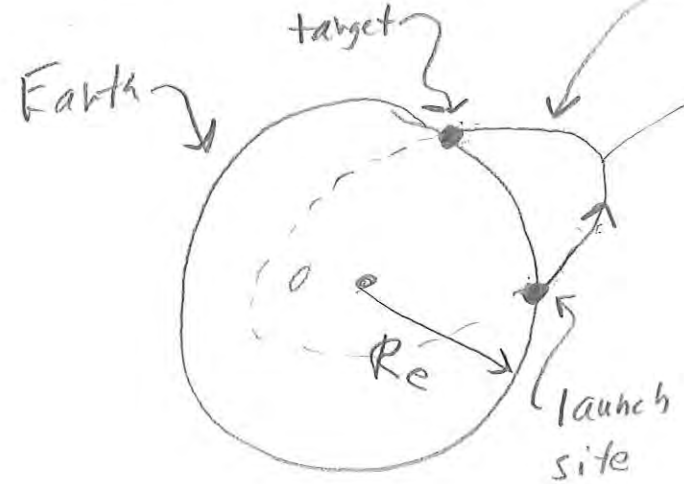
Kurvature $= 1/\rho$

radius of curvature

$\vec{a}$ = 1 part for speeding up;
+1 part for going around corners

Ex)

Nuclear War: ballistic missile



a) parab.
b) circle
c) ellipse
d) hyperbola
e) other

FBD

$$\vec{F} = F\,\hat{\lambda}$$

$$\hat{\lambda} = -\vec{r}\,/\,|\vec{r}| = -\hat{e}_r$$

$$F = \frac{mg\,R_e^2}{r^2} = \frac{GMm}{r^2}$$

G = Cavindish Const.
M = mass of Earth

$$\vec{F} = -\frac{mgR_e^2}{r^2}\frac{\vec{r}}{r}$$

$$= \frac{-mgR_e^2}{r^3}\vec{r}$$

LMB $\vec{F} = m\vec{a}$

$$\boxed{\begin{aligned} \dot{\vec{V}} &= \vec{a} = \frac{1}{m}\left(-\frac{mgR_e^2}{r^3}\vec{r}\right) \\ \dot{\vec{r}} &= \vec{V} \end{aligned}}$$

$$\dot{Z} = \begin{bmatrix} [\dot{\vec{r}}] \\ [\dot{\vec{V}}] \end{bmatrix} = \begin{bmatrix} [V] \\ [\dot{\vec{V}}] \end{bmatrix}$$

See Computer Solution of this problem, from lecture

Thms & Facts

$\vec{F} = m\vec{a}$ has solns

w/ various properties ;

Lin. Mom $\quad \int_{t_1}^{t_2} \vec{F}\,dt = m(\vec{v}_2 - \vec{v}_1)$

Power $\quad \text{Power} = \dot{E}_K$

$E_K = \frac{1}{2} m \, \vec{v}\cdot\vec{v}$

$\vec{F}\cdot\vec{v}$

$$\int P\,dt = \Delta E_K$$

$$\int \vec{F}\cdot d\vec{r} = \Delta E_K$$

If Conservative Forces

$$-\Delta E_p = \Delta E_K$$

$$\Delta(E_p + E_K) = 0$$

Angular Momentum

$$\underbrace{\vec{r}_{/0} \times \vec{F}}_{\vec{M}_{/0}} = \frac{d}{dt}\left(\vec{H}_{/0}\right)$$

$$\vec{H}_{/0} = \vec{r}_{/0} \times (m\vec{v})$$

USES of THMS.

$\Rightarrow$ * Checks

* help solve

* intuition

Thms: $\vec{L}$, $\vec{H}$, P, W, E_{TOT}

MON. March 4, 2019

TODAY

2 or more particles in space

ODE45

Thms.

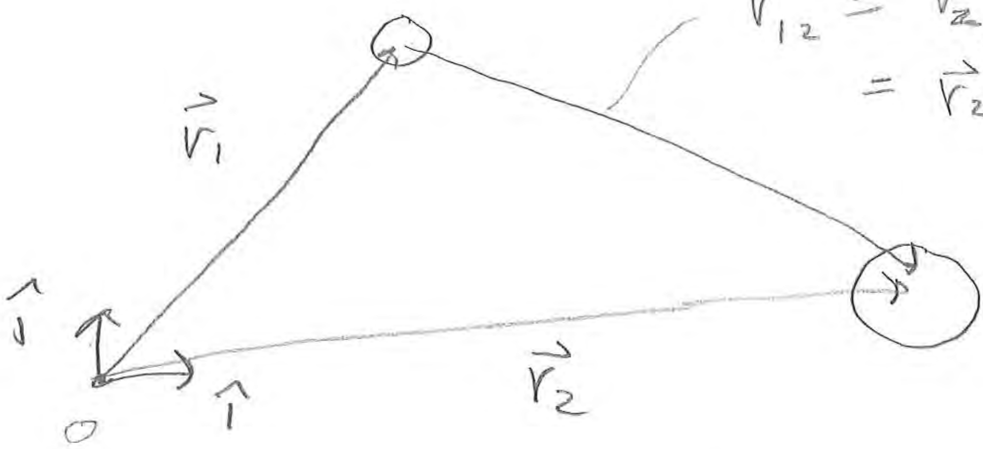
Constraints in 1D

---

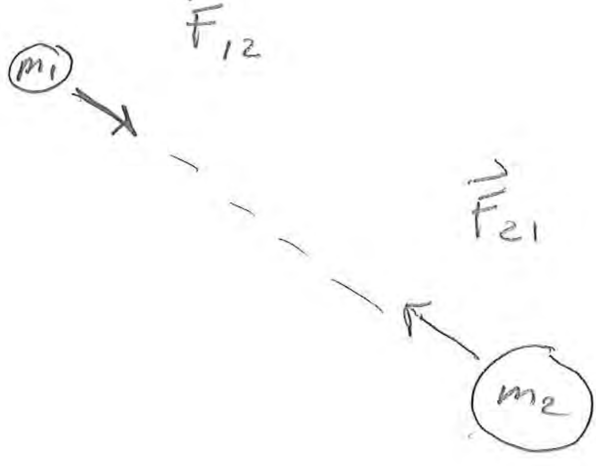
ex) 2 particles, Newtonian Gravity

$\vec{r}_{12} = \vec{r}_{2/1}$

$= \vec{r}_2 - \vec{r}_1$



FBDs



$$\vec{F}_{12} = F\,\hat{\lambda}$$

$$= \frac{G m_1 m_2}{r_{12}^2}\,\hat{\lambda}_{12}$$

$$\vec{r}_{12} / r_{12}$$

$$\vec{r}_2 - \vec{r}_1$$

Know position & vel
$\Rightarrow$ know force

$$\dot{z} = f(t, z)$$

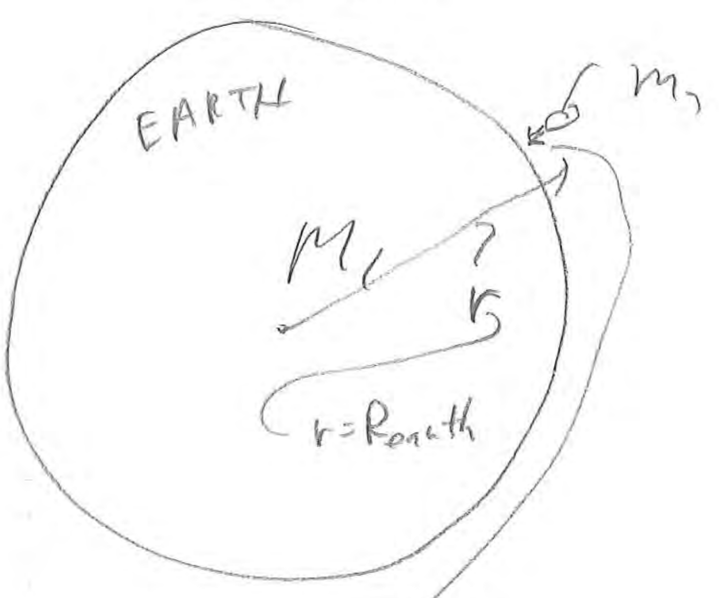
$$z = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}$$

<u>ICs</u>: $\vec{r}_1(0) = \vec{0}$, $\vec{r}_2(0) = 1\,\hat{i}$

$\vec{v}_1(0) = \vec{0}$, $\vec{v}_2(0) = 1\,\hat{j}$

$G = 1$, $m_1 = 1$, $m_2 = 2$

ASIDE! little g vs big G



$$F = \frac{m_1 m_2 G}{r^2}$$

$$= m_1 g \qquad \text{if } r = R_e$$

$$g = \frac{m_2 G}{R_{earth}^2}$$

$$G = \frac{g R_{earth}^2}{m_2}$$

For System

Lin. Mom : $\vec{L} = m_1\vec{v}_1 + m_2\vec{v}_2$

Ang. Mom /0 $\vec{H}_{/0} = \vec{r}_{1/0} \times m_1\vec{v}_1 + \vec{r}_{2/0} \times m_2\vec{v}_2$

E_K : $E_K = \frac{1}{2} m_1 \vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2} m_2 \vec{v}_2 \cdot \vec{v}_2$

E_p : $E_p \equiv -\dfrac{m_1 m_2 G}{r_{12}}$

E_{TOT} : $E_{TOT} = E_p + E_K$

QUIZ

Conserved or Not? (for system just defined)

A = yes
B = No
C = depends on ICs

$\vec{L}$? A

$\vec{H}_{/0}$? A

E_K = ? C (const. for circ. motion)

E_p = ? C (const for circ. motion)

E_{TOT} = A

⇒ MATLAB

(90/100)

* Simple motion rel. to C.O.M.

* Watch tolerances
(Don't trust anything unless it stays there when you increase accuracy.)

* ODE 45

f = @(t, z) myrhs(t, z, P)

options = odeset('AbsTol', 1e-6, 'RelTol', 1e-6)

[tarray, zarray] = ode45(f, tarray, z0, options)

(Note tarray has no effect on accuracy.)

Wed. March 6, 2019

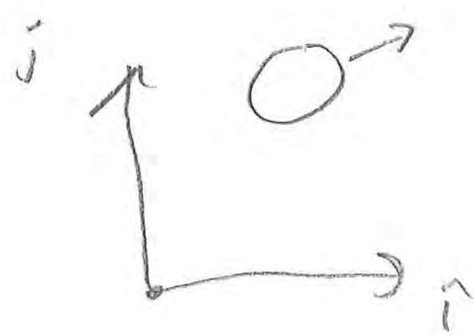
TODAY

Normal Mode Quiz (last pages)
Lots of particles
1D constrained motion

Lots of particles

See Matlab

ex)



Gen. Case
- $\vec{F}_i = m_i \vec{a}_i$ for each particle
- can calc. $\vec{F}_i$ from all $\vec{r}_i$ & $\vec{v}_i$ $\Rightarrow$ $\dot{z} = f(t, z)$

$\Rightarrow$ 100s, 1000s & billions of particles

Apply to atoms

"predicts the future" - Laplace

ex) Solid Chrystal

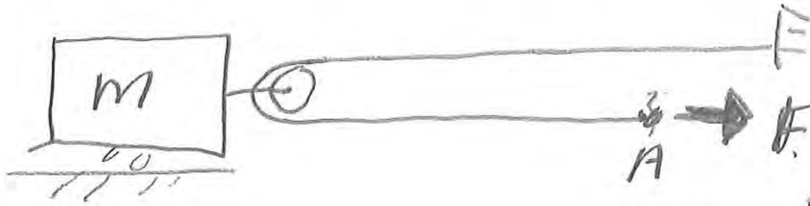
$\Rightarrow \dot{z} = f(t, z)$

1D constrained motion

a) Quiz, next pages
b) Quiz, Soln.
c) 2 masses

QUIZ

1 D, no friction
no gravity



$m = m$ $\qquad$ $F = F$

$$\frac{a_A}{a_B} = ?$$

a) 1/2

b) 1

c) 2

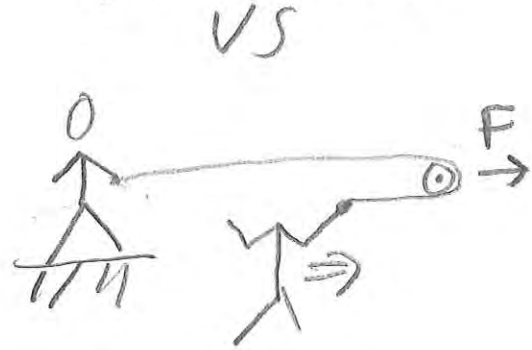
d) 4

e) 16 ✓

Lecture demo

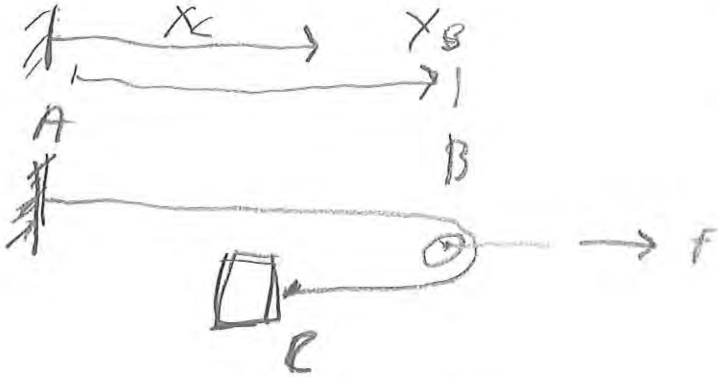
twice the force
half the motion

VS



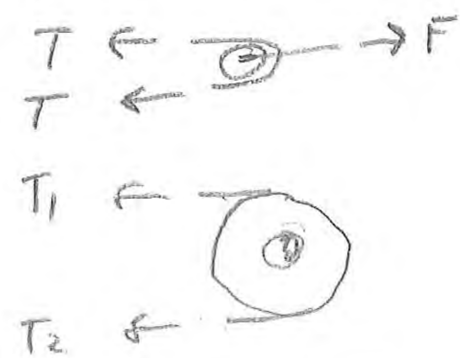
half the force
twice the motion

ex)



FBDs

$\Rightarrow T = F/2$



$\Rightarrow T_1 = T_2$

LMB $\boxed{a_c = \frac{T}{m} = \frac{F}{2m}}$

Kinematics; Length of rope = const,

$$L = x_B + (x_B - x_c) + \pi R_{pulley}$$

$$\frac{d}{dt} L = 0 = \dot{x}_B + \dot{x}_B - \dot{x}_c = 0$$

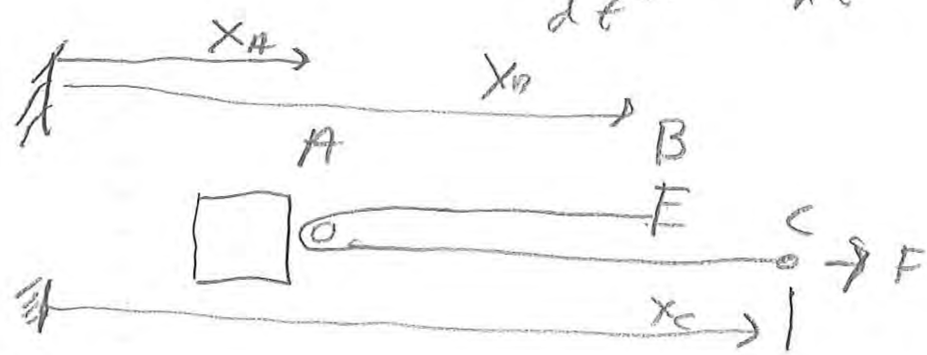
$$\Rightarrow \boxed{\ddot{x}_B = \ddot{x}_c/2 = \frac{F}{4m}}$$

$$\boxed{\ddot{x}_B = \frac{F}{4m}}$$

$f(t) = g(t)$

$$\frac{d^{37} f(t)}{dt^{37}} = \frac{d^{37} g(t)}{dt^{37}}$$

ex1



FBDs

$\Rightarrow a_A = \frac{2F}{m}$

LMB

Kinematics

$L = \text{const}$

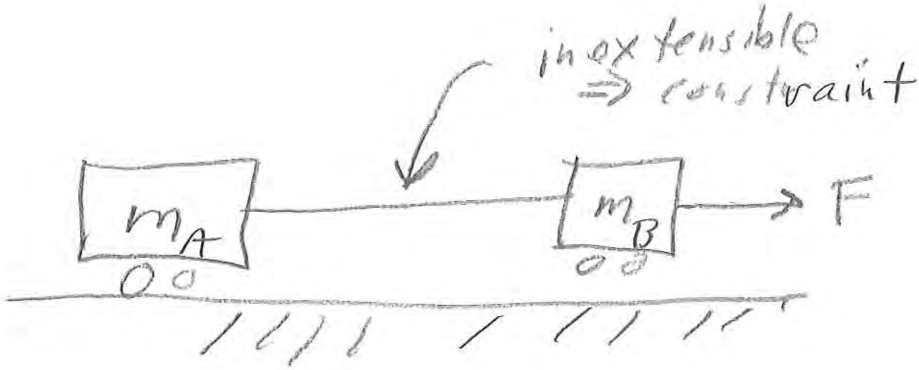
$$\text{const} = (x_B - x_A) + (x_C - x_A)$$

$$\frac{d}{dt}\{\ \} \Rightarrow 0 - \dot{x}_A + \dot{x}_C - \dot{x}_A$$

$$\Rightarrow \ddot{x}_C = 2\ddot{x}_A \Rightarrow \boxed{a_C = \frac{4F}{m}}$$

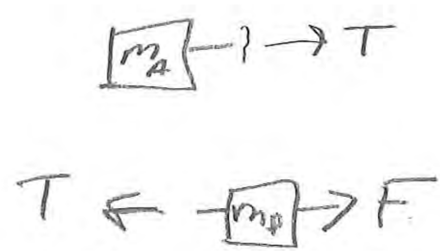
(compare to prev. example)

Ex) 1D constraint



$a_A = ?$ $a_B = ?$

FBDs



$$\underline{LMB} \Rightarrow \quad T = m_A \ddot{x}_A$$

$$LMB \Rightarrow \quad F - T = m_B \ddot{x}_B$$

Instead of Const. Law $\quad \ddot{x}_A = \ddot{x}_B$

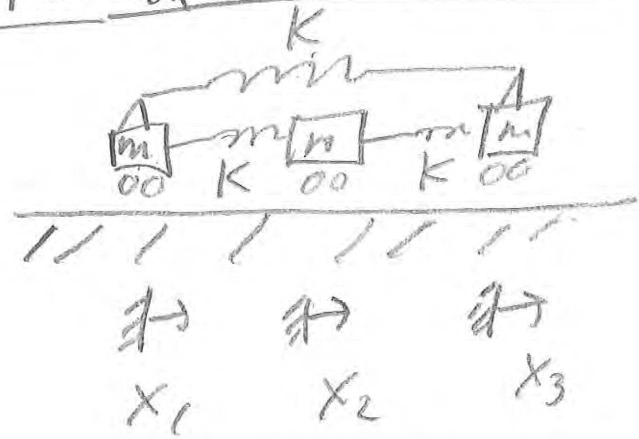
$\boxed{\text{Constraint Eqn.}}$

Solve Simult.

$$\Rightarrow \ddot{x}_A = \ddot{x}_B = \frac{F}{m_A + m_B}$$

$$\& \; T = \frac{F m_A}{m_A + m_B}$$

QUIZ on Normal Modes



$$M = m\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}; \quad K = K\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$$

$$A = M^{-1}K = \begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$$

Normal Modes?

Frequencies

See Matlab

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$\begin{bmatrix}5\\7\\-12\end{bmatrix}$$

---

* How many ind. normal, ~~[illegible]~~ modes? a) 1 b) 2, c) 3 d) 4 e) 5

* How many of above are normal modes?

a) 1
b) 2
c) 3
d) 4
e) 5

guess $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Math

check: $\underbrace{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$= 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\omega^2 = 0$

$\omega = 0$

Physical $K_{eff} = 0$ ✓

guess $A \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\omega^2 = 3$

Physical $K_{eff} = 3$ ✓

guess $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ✓

$$A \cdot v_1 = \lambda v_1$$

$$A v_2 = \lambda v_2$$

$$\Rightarrow A(6v_1 - 37v_2) = \lambda(6v_1 - 37v_2)$$ ✓

For M=diagonal, K=symmetric

$\Rightarrow A$ = symmetric

$\Rightarrow$ e-vectors mutually orthogonal

Orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are vectors whose coeff. add to 0. 2D space of e-vectors w/ $K_{eff} = 3$ is vectors w/ $\sum v_i = 0$

e.g. $[5 \;\; 7 \;\; -12]'$

TODAY: Monday March 11

Pulleys (cont'd)
2D forces in 1D motion.

PULLEYS

Use pulley to increase / decrease force on / motion of object. (4 uses). Also reverse direction.

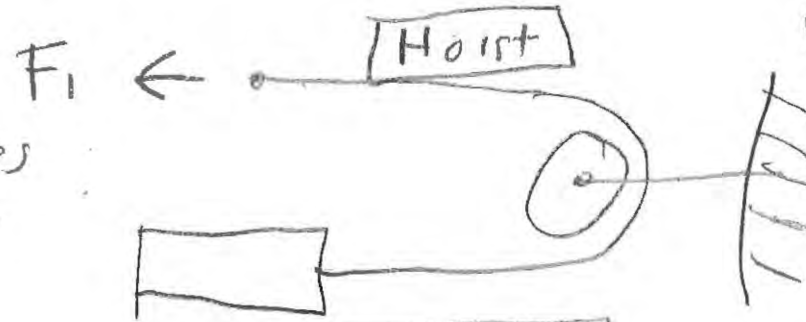
2×2

Puzzle

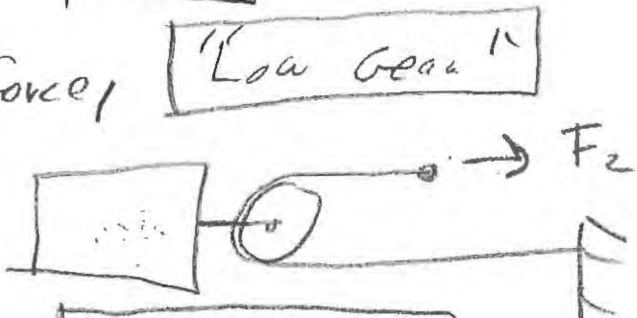
Draw 3 pulley systems that catch all uses above.

(Soln. on next page)

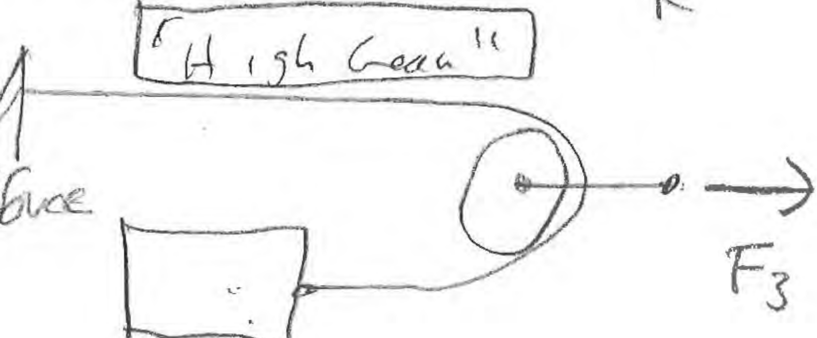
reverses motion



doubles force, halves motion



halves force doubles motion



Power balance (too)

(Power in) = (Power out)

(Applied Force) · (Applied vel.) =

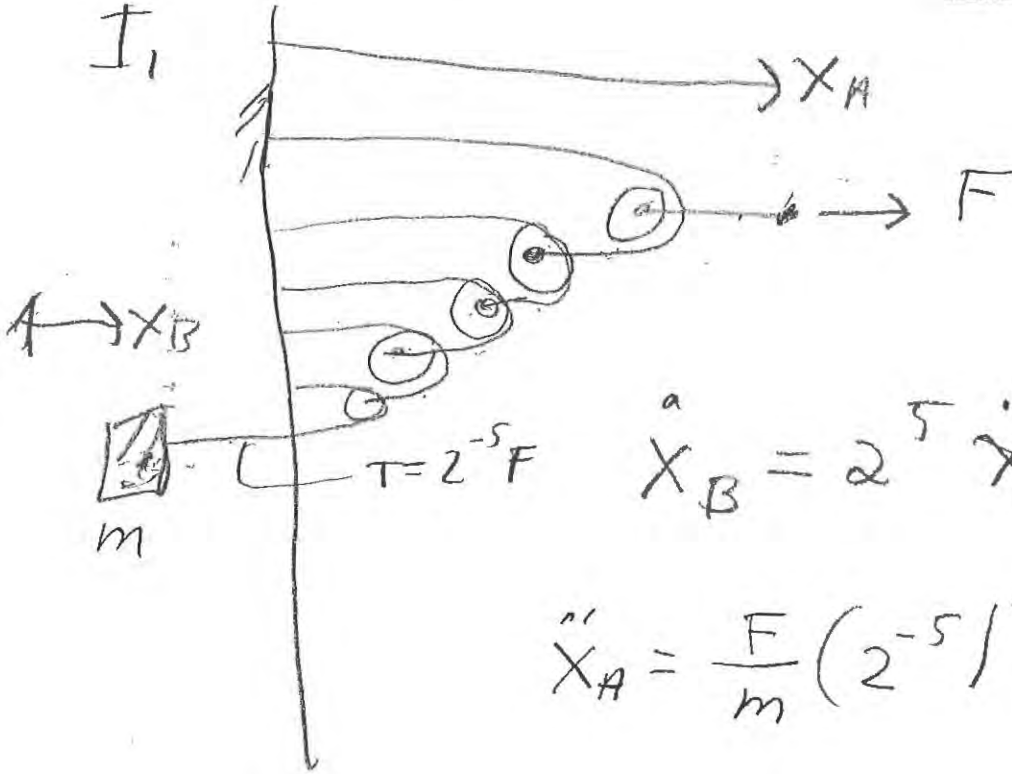
$\underbrace{(\text{Force on mass})}\cdot\underbrace{(\text{mass vel.})}$

$\Rightarrow$ Force amplification $\Leftrightarrow$ motion attenuation

Very large numbers

5 pulleys how big a number can an answer be? $\boxed{10^6}$

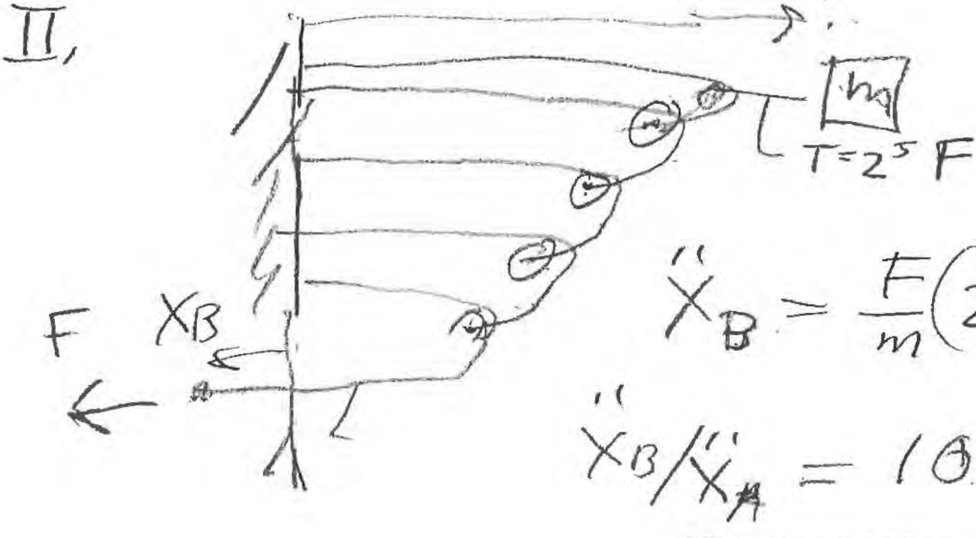
I,



$$\dot{X}_B = 2^5 \dot{X}_A$$

$$\ddot{X}_A = \frac{F}{m}\left(2^{-5}\right)^2$$

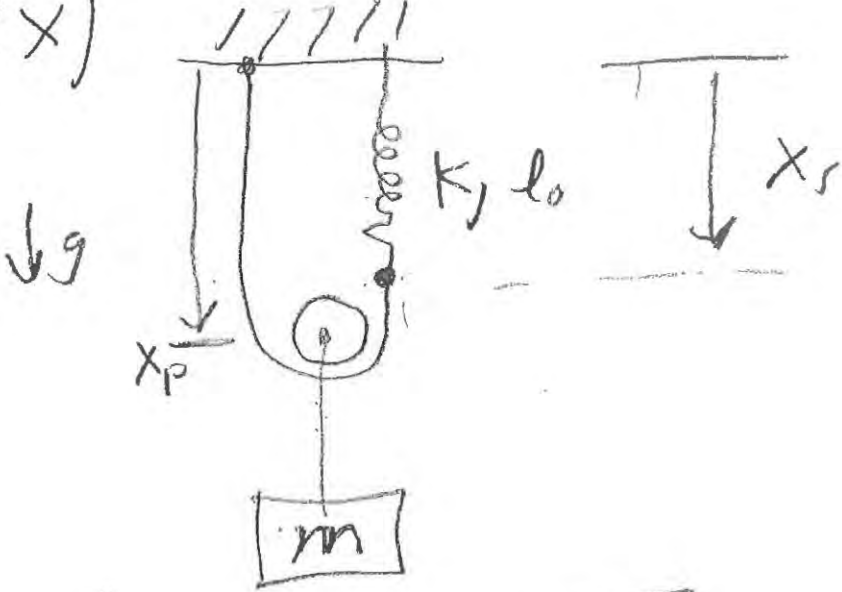
II,



$$\ddot{X}_B = \frac{F}{m}\left(2^5\right)^2$$

$$\ddot{X}_B / \ddot{X}_A = 10^6$$

ex)



$$\omega = c\sqrt{k/m}$$

a) $c = 1$

b) $c = 2$

c) $c = 1/2$

d) $c = 1/4$

e) $c = 4$

Soln on next pages,

ex) (cont'd)

Kinematics

$$a_m = \ddot{x}_p$$

$$l_r = \text{const.} = x_p + (x_p - x_s)$$

$$2x_p = x_s + l_r$$

length of rope

$$x_s = 2x_p - l_r$$

$$\Delta l_s = x_s - l_0$$

rest length of spring

$$\boxed{\Delta l_s = 2x_p - l_r - l_0}$$

one factor of 2

FBD:

LMB

$$\sum \vec{F} = m\vec{a}$$

$\{\}\cdot\hat{T}$

LMB

!

$$mg - 2T_s = m\ddot{x}$$

$k(\Delta l)$

!

$2x_p - l_p - l_s$

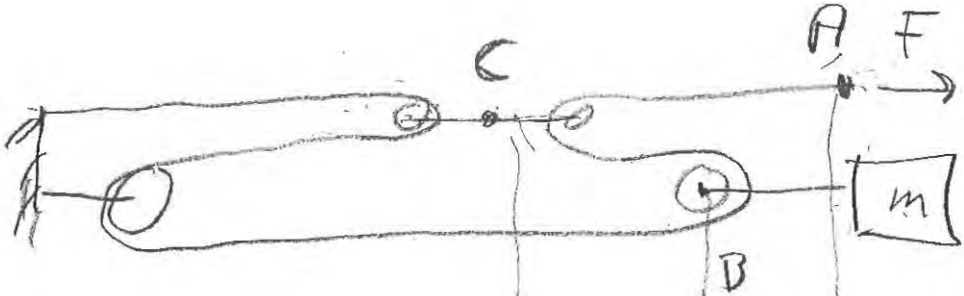
$$m\ddot{x} + 4kx = \begin{pmatrix}\text{const}\\ \text{stuff}\end{pmatrix}$$

$2\times 2 = 4$

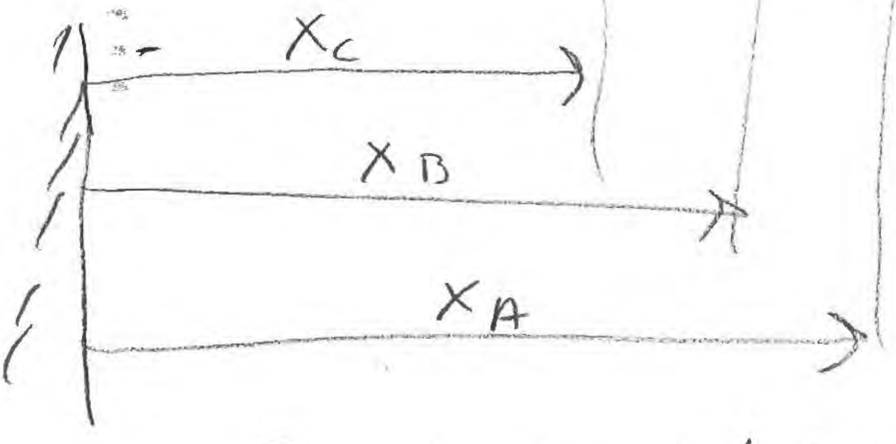
$$\Rightarrow \omega = \sqrt{4k/m} = \boxed{2\sqrt{k/m}}$$

soln. to puzzle on page 113.

ex) A partially indeter-minate problem



$l = \text{const}$



B moves w/ mass

$$l = \text{const} = 2x_C + (x_B - x_C)$$
$$+ x_B$$
$$+ x_A - x_C$$

1 eqn for 3 unknowns

Odd problem (cont'd) (117)

Can't solve for motion of

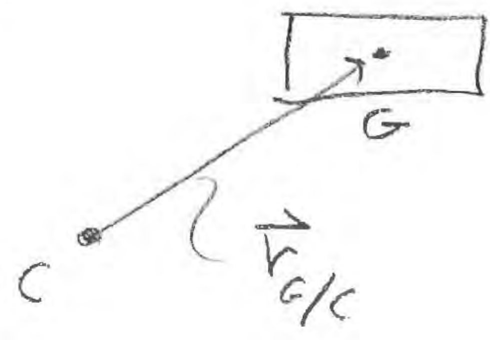
x_C. Not determinable using simple methods.

Rigid objects in 2D w/ 1D motion

((Car braking problems))

Key eqn:

For rigid object & any pt. C



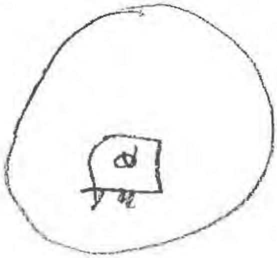
$$\boxed{\sum \underline{M}^{ext}_{/C} = \vec{r}_{G/C} \times m\,\vec{a}_{G/\mathcal{F}} + I^{G} \alpha \hat{K}} \quad (*)$$

$\alpha = 0$ for no-rotation problems

$\mathcal{F}$ixed frame

<u>recall</u>

A)



"$M = I\alpha$"

"$T = J\dot{\omega}$"

B)

$\vec{r} \times \vec{F} = \vec{r} \times (m\vec{a})$

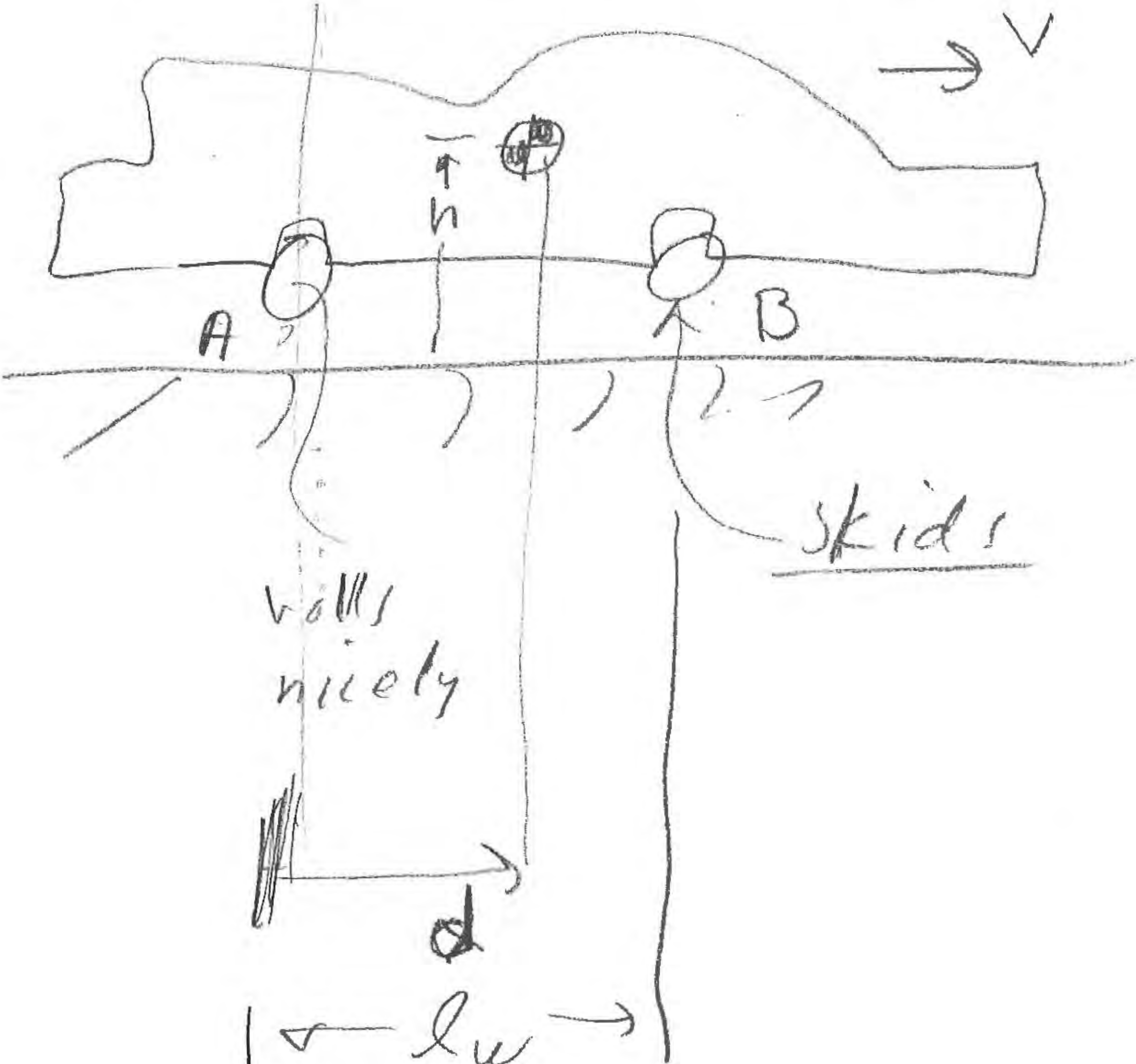
particle motion

$(A) + (B) \Rightarrow *$

intuitive aid for eqn $*$

| Eqn $*$ is key for rest of semester |

ex) Front Wheel Braking



assume no rotation

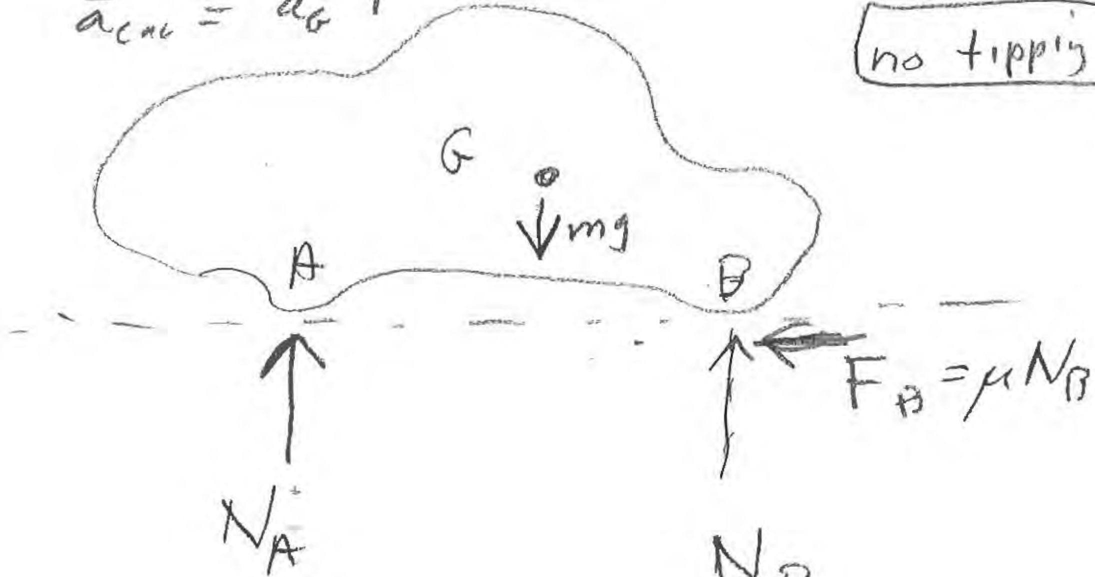
$\alpha = 0$

$a_{car} = ?$ will be $a_G < 0$

$\vec{a}_{car} = a_G \hat{\imath}$

$\hat{\jmath} \uparrow \rightarrow \hat{\imath}$

no tipping



Unknowns $\quad a_G = $ scalar

$F_f, \ N_A, N_B$

4 Eqs

Eqs: $F_B = \mu N_B$ (1)

LMB: $(N_A + N_B)\hat{\jmath}$ (2)

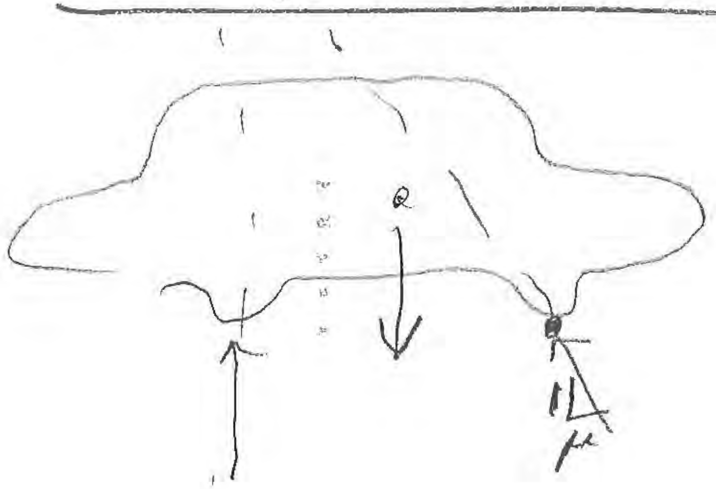
$-mg\hat{\jmath} - F_B\hat{\imath} = m a_G \hat{\imath}$

AMB/G: $\vec{r}_{B/G} \times N_B \hat{\jmath} + \vec{r}_{A/G} \times N_A \hat{\jmath}$ (1)
$+ \vec{r}_{B/G} \times (F_f \hat{\imath}) = I^G \alpha \hat{K}$

$\Rightarrow$ 4 eqs & 4 unknowns

Short Cut

H = pt. at intersection of lines of action of forces at A & B.



$$\sum \vec{M}_{/H} = \vec{r}_{G/H} \times m(a_G \hat{i}) + \cancel{I\alpha}^{\,0} \hat{k}$$

N_A, N_B, F_B drop out.

$\Rightarrow$ 1 eqn in 1 unknown a_G

(note $-mg\hat{j}$ is in eqn. but it is known)

Wed. March 13, 2019

TODAY

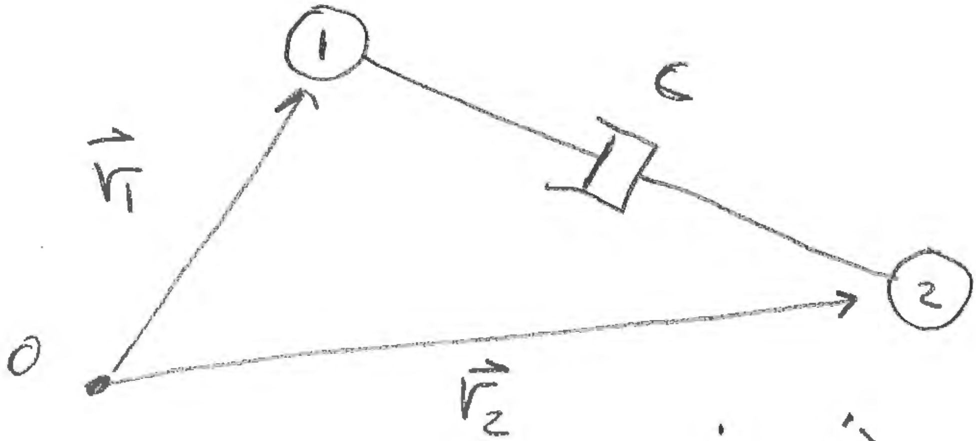
Quiz particle in space

Pulley reviews

Car braking example

(Notes here complete the algebra that was not done in lecture for car braking problem)

Quiz



Given $\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2, c$

Find force on 1. ?

$\boxed{\text{3 minutes}}$

Answers on next page, you will have 30 s to pick one,

[Don't write components explicitly.]

force on ① = $\vec{F}$ = ? 124

Define

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

① $\vec{F} = ?$

$$\dot{\vec{r}} = \dot{\vec{r}}_2 - \dot{\vec{r}}_1$$

②

$$r = |\vec{r}|$$

$$\hat{\lambda} = \vec{r} / |\vec{r}|$$

OPTIONS

a) $\vec{F} = c(\vec{r} \cdot \dot{\vec{r}})\vec{r} / r^2$

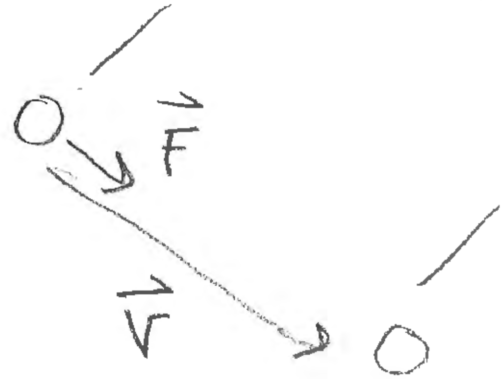
b) $\vec{F} = c(\hat{\lambda} \cdot \dot{\vec{r}})\hat{\lambda}$

c) $\vec{F} = \dfrac{c(\vec{r}_2 - \vec{r}_1) \cdot (\dot{\vec{r}}_2 - \dot{\vec{r}}_1)(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^2}$

d) 2 of above

e) all 3 of above ✓

Soln 1



$\vec{r} = \vec{r}_2 - \vec{r}_1$

$\vec{r}_{12} = \vec{r}_{2/1}$

$\dot{\vec{r}} = \dot{\vec{r}}_2 - \dot{\vec{r}}_1$

$$\ell = |\vec{r}| = \sqrt{\vec{r}\cdot\vec{r}} = r$$

$$\boxed{\vec{F} = \underbrace{c\,\dot{\ell}}_{\text{scalar part}}\ \hat{\lambda}} \quad ① \qquad \hat{\lambda} = \vec{r}/|\vec{r}| = \frac{\vec{r}}{r}$$

$$\dot{\ell} = \frac{d}{dt}(\ell) = \frac{d}{dt}\sqrt{\vec{r}\cdot\vec{r}}$$

$$= \frac{1}{2\sqrt{\vec{r}\cdot\vec{r}}}\frac{d}{dt}(\vec{r}\cdot\vec{r})$$

$$= \frac{1}{2r}(\vec{r}\cdot\dot{\vec{r}} + \dot{\vec{r}}\cdot\vec{r}) =$$

$$= \frac{\dot{\vec{r}}\cdot\vec{r}}{r} \quad ②$$

① & ② $\Rightarrow$

$$\vec{F} = c\,\dot{\ell}\,\hat{\lambda}$$

$$= c\,\frac{\dot{\vec{r}}\cdot\vec{r}}{|\vec{r}|}\,\frac{\vec{r}}{|\vec{r}|} = c(\dot{\vec{r}}\cdot\hat{\lambda})\hat{\lambda}$$

Soln 2 (Quiz cont'd)

hard part is $\dot{\ell} = ?$

Note: $\{ \ell^2 = \vec{r} \cdot \vec{r} \}$

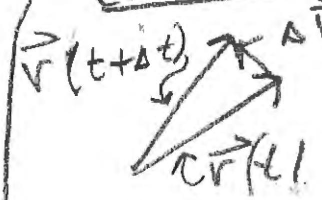
$$\frac{d}{dt}\{\ \} \Rightarrow 2\ell\dot{\ell} = \vec{r}\cdot\dot{\vec{r}} + \dot{\vec{r}}\cdot\vec{r}$$

$$\dot{\ell} = \frac{\vec{r}\cdot\dot{\vec{r}}}{\ell}$$

$\ell = r = |\vec{r}|$

$$\boxed{\dot{\ell} = \dot{\vec{r}}\cdot\hat{\lambda}}$$

example



$\Longleftrightarrow$ $\boxed{\dot{\ell} = |\dot{\vec{r}}|}$ ✗

wrong

$\dot{\vec{r}} \neq \vec{0}$, yet $\dot{\ell} = 0$!

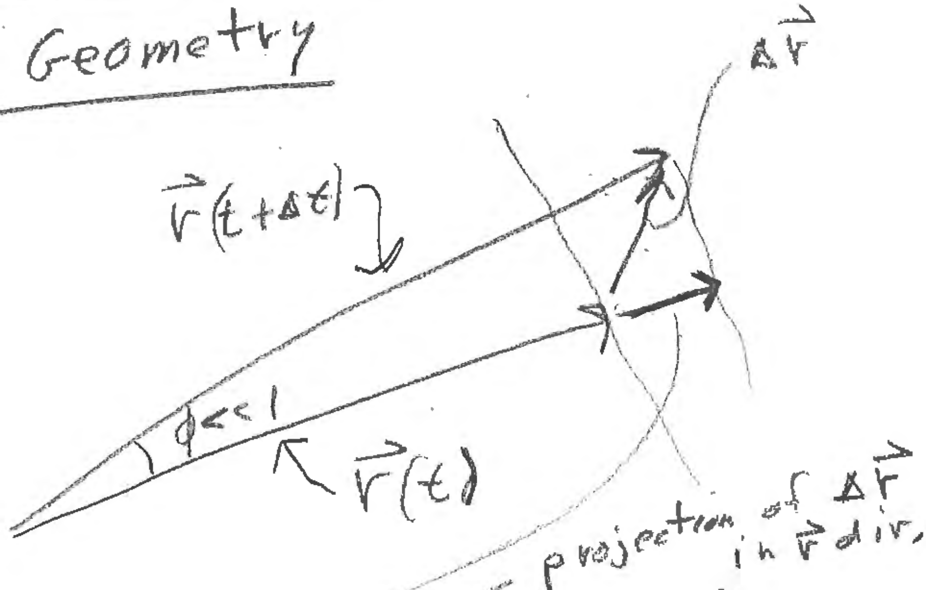
$$\frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}|\dot{\vec{r}}|$$

Ignore this: talking to student

WATCH OUT FOR THIS MISTAKE.

Soln 3 to quiz

USE Geometry



$$\Delta l \cong (\Delta\vec{r}) \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$\dot{l} = \frac{\Delta\vec{r}}{\Delta t} \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$\boxed{\dot{l} = \dot{\vec{r}} \cdot \hat{\lambda}}$$

Student asked: "What is $\Delta\vec{r}$?"

Answer $\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$

= change in $\vec{r}$ in time Δt

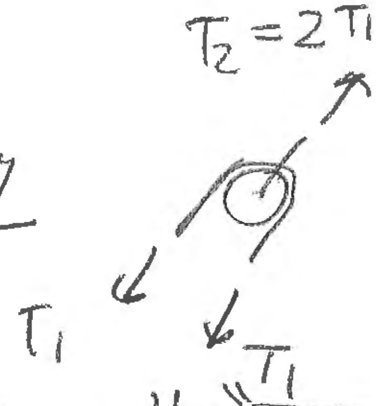
```
%Particles in space Example (lecture Mar 13, 2019
```

```
for x = 1:10
    disp(x)
end
```

```
%Given r1, r1dot, r2, r2dot
c= rand(1);
r1    = rand(3,1); r2    = rand(3,1);
r1dot = rand(3,1); r2dot = rand(3,1);

%Find a vector in the direction r12
%with magnitude   c * d/dt (|r12|)

r    = r2    - r1;       % rel position
rdot = r2dot - r1dot;    % rel velocity

lambda = r/norm(r);      % uinit vector

Fscalar = c * dot(rdot,r)/norm(r); %see lecture notes

Fvector = Fscalar * lambda  % the answer, a vector
```

Pulleys (review)

FBD of each pulley



Assumptions: "ideal pulley"
round,
good bearings,
& negligible inertia

Inextensible rope

each rope: $\ell = \text{const}$

$\Rightarrow$ write length(s) in terms of key positions of masses, connection points

If concerned with v & a & not position

$\Rightarrow$ can neglect πR

& other constant terms,

Another "trick"

power balance to solve problems,

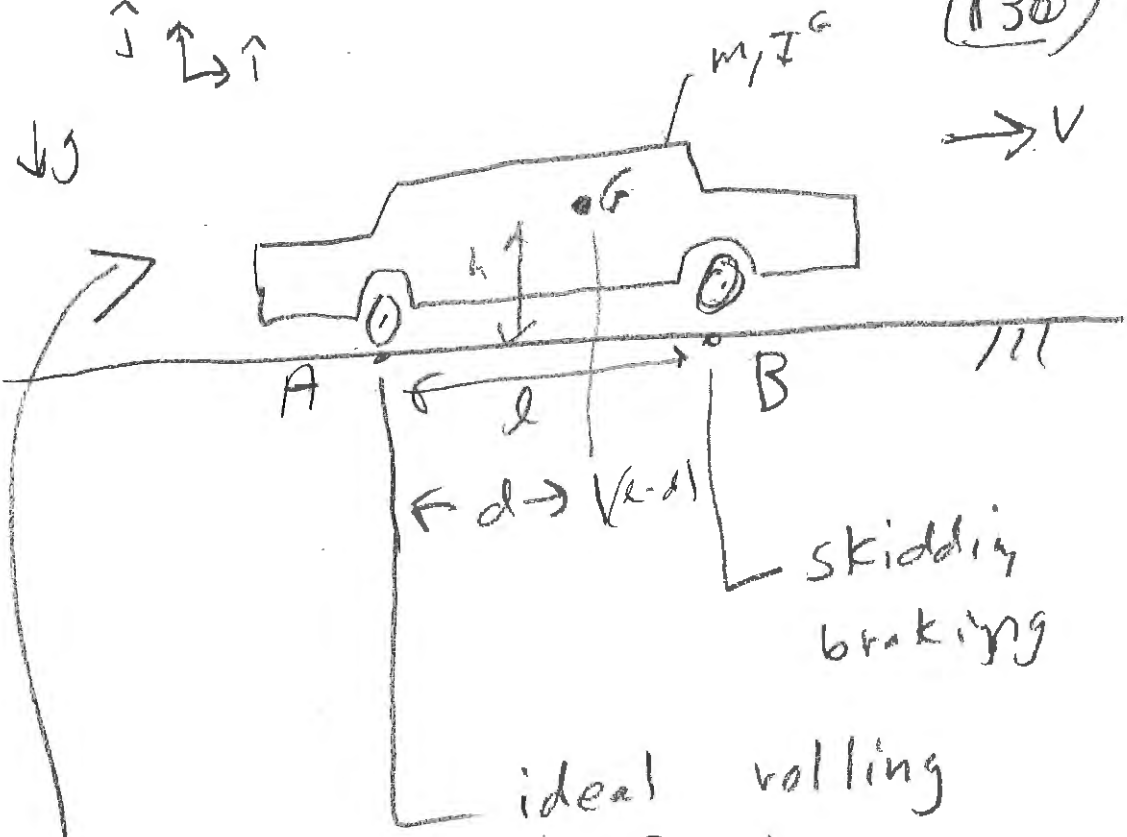
$$\boxed{P_{in} = \dot{E}_K}$$

for system, includg masses

And to check problems.

$$\boxed{P_{in\ total} = 0}$$

for massless pulley system w/out masses

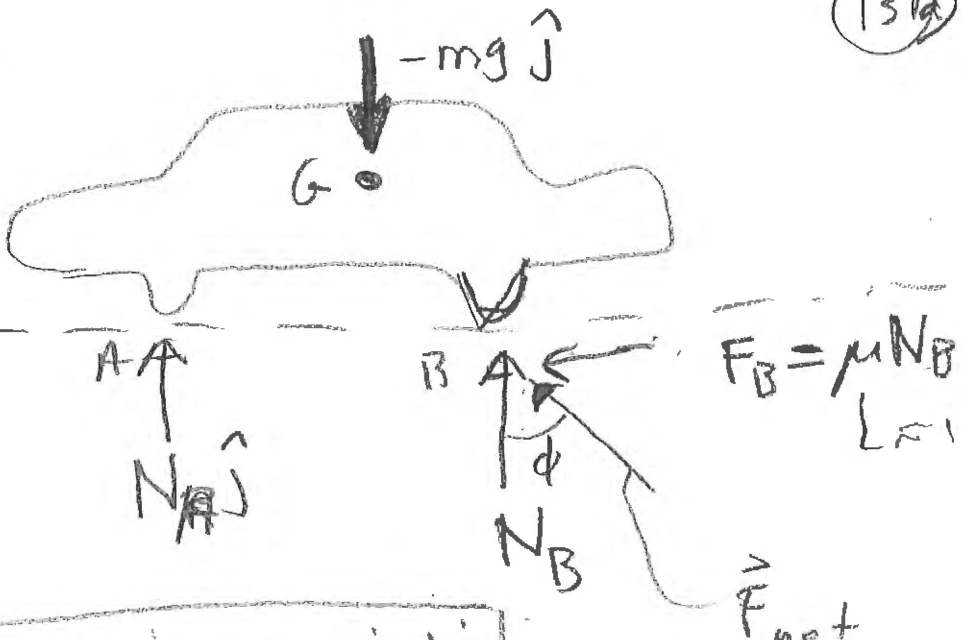


round
bearings good
negligible inertia

CAR BRAKING PROBLEM

$$a_G = ?$$

FBD



ASIDE about Friction

$\tan\phi = \mu$

↑ friction angle
equiliv. way to express
μ

$\mu = 1 \iff \phi = \pi/4$

Kinematics: $\omega = 0, \; \dot{\omega} = 0$

$\vec{a}_G = a_G\,\hat{\imath}$

ie. No rotation &
G moves horizontally

13 lb

LMB

$$\sum \vec{F}_i^{ext} = m_{tot} \vec{a}_G$$

$$\left\{ \boxed{N_A \hat{j} + N_B \hat{j} - mg\hat{j} - \underbrace{F_B}_{(\mu N_B)} \hat{i} = m a_G \hat{i}} \right\} \circledast$$

$$\{*\}\cdot\hat{i} \Rightarrow \quad -F_B = m a_G \qquad (1)$$

$$\{\ \}\hat{j} \Rightarrow \quad N_A + N_B = mg \qquad (2)$$

4 unknowns N_A, N_B, F_B, a_G

$$F_B = \mu N_B \qquad (3)$$

Need another eqn.

AMB/G:

$$\sum \vec{M}_{/G} = I^G \cancel{\alpha}^{\,0} \hat{k}$$

$$\underbrace{\vec{r}_{A/G}}_{-d\hat{i}-h\hat{j}} \times N_A\hat{j} + \overbrace{\vec{r}_{B/G}}^{(l-d)\hat{i}-h\hat{j}} \times (N_B\hat{j} - F_B\hat{i})$$

$$= \vec{0} \qquad (4)$$

$AMB_{/G}$ (cont'd)

$$\{(4)\}\cdot\hat{k} \Rightarrow \quad -dN_A + (\ell-d)N_B - F_B h = 0 \qquad (4)$$

Need to Solve (1)-(4) for a_G.

$$(2) \Rightarrow \qquad N_A = mg - N_B \qquad (5)$$

Subst. (5) into (4)

$$-d\underbrace{(mg - N_B)}_{N_A} + (\ell - d)N_B - \overbrace{\mu N_B}^{F_B} h = 0 \qquad (6)$$

Solve (6) for N_B:

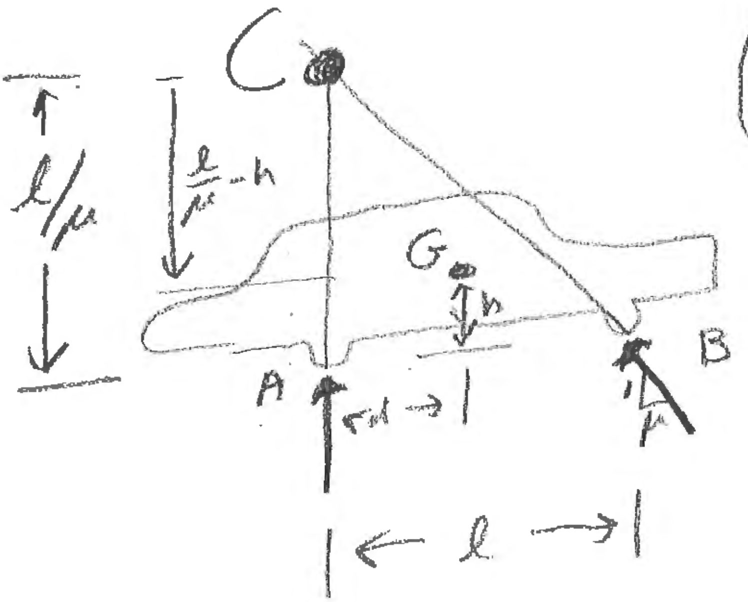
$$N_B = \frac{dmg}{d + (\ell - d) - \mu h} \qquad (7)$$

$$(7)\ \&\ (3) \Rightarrow \boxed{a_G = \frac{-dg}{\ell/\mu - h}}$$

Alternative approach to con

Wheel Forces have no moment about C.

So, AMB/C is one eqn. in one unknown, a_G.



AMB/C $\quad \sum \vec{M}_{/C}^{ext} = \vec{r}_{G/C} \times m\vec{a}_G + I^G \cancel{\alpha}^{0} \hat{k}$

$a_G \hat{\imath}$

$\vec{r}_{G/C} \times -mg\hat{\jmath}$

$-\left(\frac{\ell}{\mu}-h\right)\hat{\jmath}+d\hat{\imath}$

$$-d\,mg = \left(\frac{\ell}{\mu}-h\right) m a_G$$

$$\boxed{a_G = \frac{-d \cdot g}{\ell/\mu - h}}$$

Comment on Soln.

We had soln:

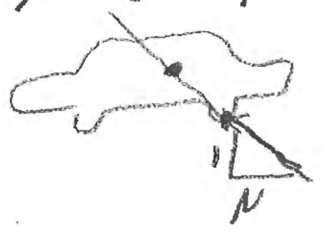
$$a_G = \frac{-dg}{\ell/\mu - h}$$

$$N_B = \frac{dmg}{\ell - \mu h}$$

$$N_A = mg - N_B$$
$$= \left(\frac{(\ell - \mu h) - d}{\ell - \mu h}\right) mg$$

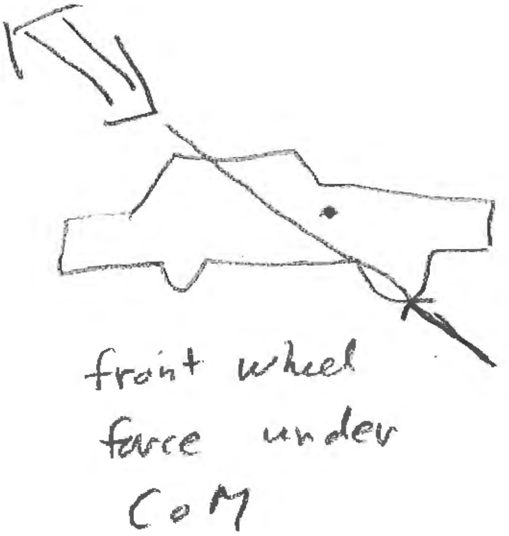
Note: $N_A = 0$ if $\ell - \mu h - d = 0$
$\Rightarrow \mu = (\ell - d)/h$

friction force goes through G = CoM $\Rightarrow$



Then, also
$$N_B = mg$$
$$a_G = -\mu g$$

for $\mu > \frac{l-d}{h}$



then $N_A < 0 \Rightarrow$ NONSENSE

Car doesn't tip is invalid assumption in this case

TODAY March 18, 2019

Quizes (Review)

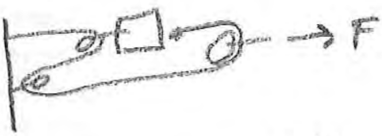
Circular Motion

---

3 Quizzes were done first. Here they are at the end of this lectures notes

3 quizes

1. $\vec{F} = -c\,\vec{v}\,|\vec{v}|^{1/2}$

2. 

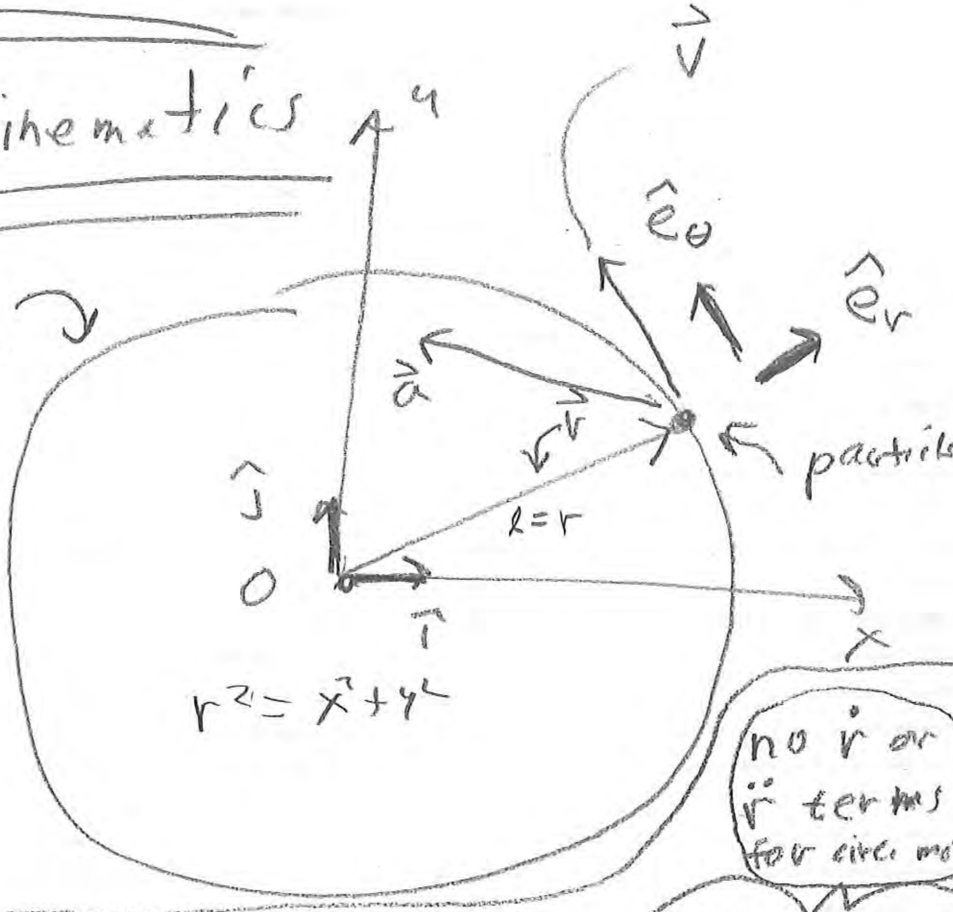
3) 

CIRCULAR MOTION

Particles

Kinematics

Circle



no $\dot r$ or $\ddot r$ terms for circ. motion

position $\vec r = x\hat\imath + y\hat\jmath = r\hat e_r$

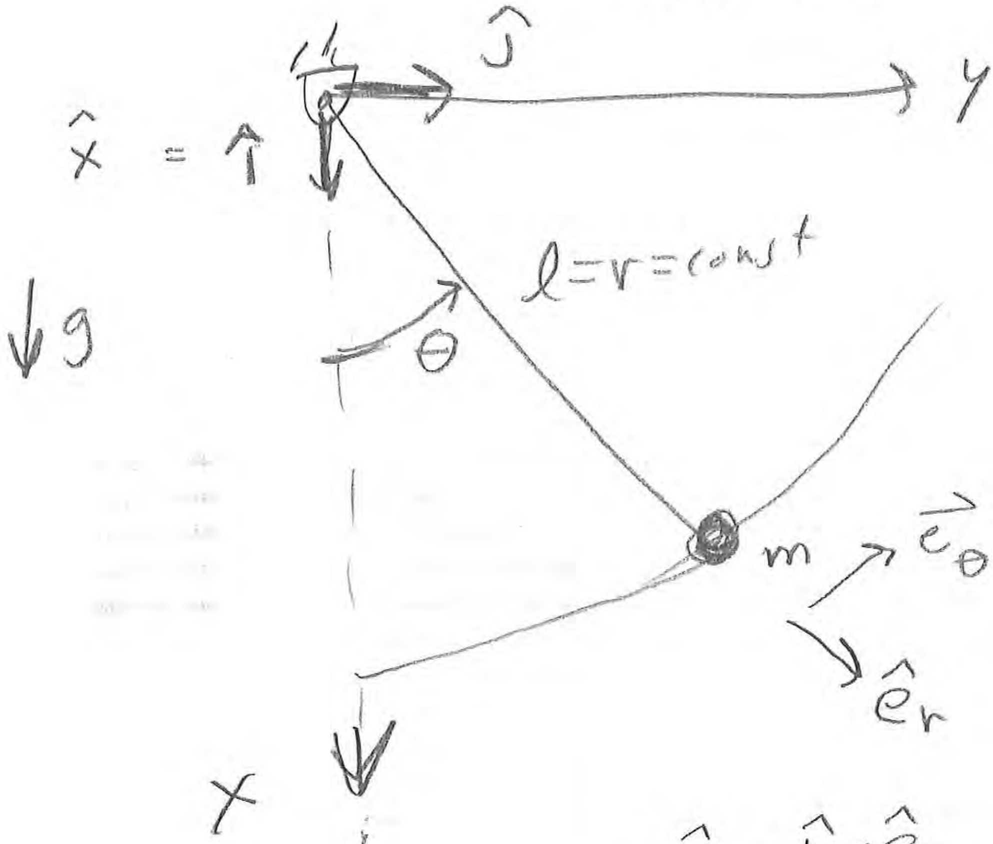
vel. $\vec v = \dot x\hat\imath + \dot y\hat\jmath = r\dot\theta\hat e_\theta = v_\theta\hat e_\theta = v\hat e_\theta$

accel $\vec a = \ddot x\hat\imath + \ddot y\hat\jmath = a_\theta\hat e_\theta + \frac{-v_\theta^2}{r}\hat e_r$

$= r\ddot\theta\hat e_\theta + -r\dot\theta^2\hat e_r$

Pendulum: 1037 ways

↑ slight exaguration



$\hat{e}_\theta = \hat{k} \times \hat{e}_r$

$\hat{e}_r = \vec{r}/|\vec{r}|$

$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

FBDs

Method 1

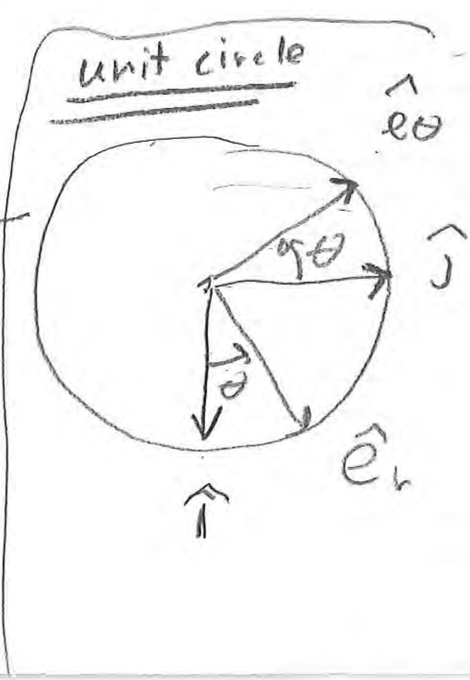
LMB:

$$\sum \vec{F} = m\vec{a}$$

$$\left\{ +mg\hat{\imath} - T\hat{e}_r = m\left(r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r\right) \right\}$$

$\{\ \}\cdot\hat{e}_\theta \Rightarrow$

$$\cancel{m}g\underbrace{\hat{\imath}\cdot\hat{e}_\theta}_{= -\sin\theta} = \cancel{m}r\ddot{\theta}$$



$$\Rightarrow -g\sin\theta = r\ddot{\theta}$$

$$\boxed{\ddot{\theta} = \frac{-g}{r}\sin\theta}$$

EoM

Method 2

LMB

$$\left\{ mg\hat{\imath} - T\hat{e}_r = m(r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r) \right\}$$

$$\boxed{\ddot{\theta} = \frac{-g}{r}\sin\theta}$$

$$\hat{e}_r \times \{ \ \} \Rightarrow \text{(same thing)}$$

Method 3

AMB/O

Mom. of Inertia of pt. mass is zero

$$\sum \vec{M}_{/O} = \underbrace{\vec{r}_{G/O}}_{r\hat{e}_r} \times m\underbrace{\vec{a}_G}_{\underbrace{(r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r)}_{\text{circular motion}}} + \cancel{I}^{\,0}\ddot{\theta}\hat{k}$$

$$\sum \vec{M}_{/O} = r\hat{e}_r \times (mg\hat{\imath} - T\hat{e}_r)$$

$$\Rightarrow \boxed{\ddot{\theta} = \frac{-g}{r}\sin\theta \quad \text{again}} \quad \text{(again)}$$

Method 4

Cons. of Energy

$$E_{TOT} = \text{const.}$$

$$\text{const.} = \overbrace{E_p}^{mgh} + \overbrace{E_K}^{\frac{1}{2}mv^2}$$

$$\left\{ \text{const} = \overbrace{-mgr\cos\theta}^{mgh} + \frac{1}{2}(r\dot{\theta})^2 m \right\}$$

$$\frac{d}{dt}\{\ \} \Rightarrow \boxed{\ddot{\theta} = \frac{-g}{r}\sin\theta}$$

(again)

Method 5

Power Balance

$$P = \dot{E}_K \qquad \frac{d}{dt}\left(\frac{(r\dot{\theta})^2 m}{2}\right)$$

$$P = \vec{V} \cdot (mg\hat{\imath}) \qquad \vec{V} = r\dot{\theta}\hat{e}_\theta$$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{r}\sin\theta}$$

(again)

Method 6

use cart coord

LMB: $\vec{F} = m\vec{a}$

$$mg\hat{\imath} - T\hat{e}_r = m(\ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath})$$

$$\hat{e}_r = \left(\frac{x}{r}\,\hat{\imath} + \frac{y}{r}\right)\hat{\jmath}$$

T, $\ddot{x}$, $\ddot{y}$ as unknowns

Have 2 eqs for 3 unknowns

another eqn: $l = \text{const}$

$$\Rightarrow \quad l^2 = \text{const}$$

$$\Rightarrow \left\{ x^2 + y^2 = \text{const} \right\}$$

$\frac{d^2}{dt^2}\{\ \} \Rightarrow$ rel. bet. $x, \dot{x}, \ddot{x}$ / $y, \dot{y}, \ddot{y}$

3rd eqn. $\rightarrow$

[TO BE CONT'D IN ANOTHER LECTURE]

Quiz 1

Drag force on particle is prop. to $v^{3/2}$. Write $\vec{F}_D$ as a vector: $\vec{F} = ?$

Write Matlab commands for

$F = ?$
$\uparrow$ 2x1

Given $\vec{v}$, c, $v = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$

Q1: $\vec{F}$ | F, Matlab

a) $\vec{F} = -cV^{3/2}$ ✗	$\overbrace{\texttt{F = -c*V^(3/2)}}^{\text{MATLAB error}}$ ✗
b) $\vec{F} = -c\lvert\vec{v}\rvert^{1/2}\vec{v}$ ✓	`F = -c*V*norm(V)` $\underbrace{\texttt{V*norm(V)}}_{\propto V^2}$ ✗ ✗
c) $\vec{F} = -c\vec{V}^{3/2}$ ✗	`F = -c*norm(V)^...` `(3/2)*V` ✗
d) $\vec{F} = -c\lvert\vec{V}\rvert^{3/2} * \vec{V}$ ✗	`F = -c*V/norm(V)...` `^(3/2)` ✗
e) NONE	OF ABOVE ✓

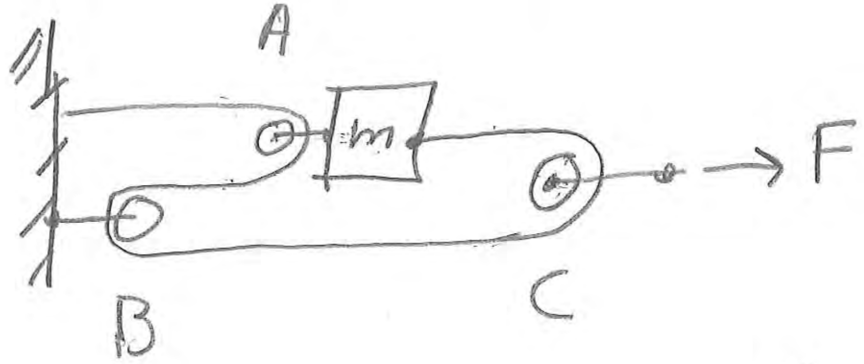
Correct Matlab

```
F=-c*V*...
(norm(V))^(1/2)
```

`V.^(3/2)`

$$= \begin{bmatrix} V_x^{3/2} \\ V_y^{3/2} \end{bmatrix} \quad ✗$$

QUIZ 2 $\ddot{x}_C = ?$ 145



a) $F/(4m)$ ✓

b) $F/(2m)$

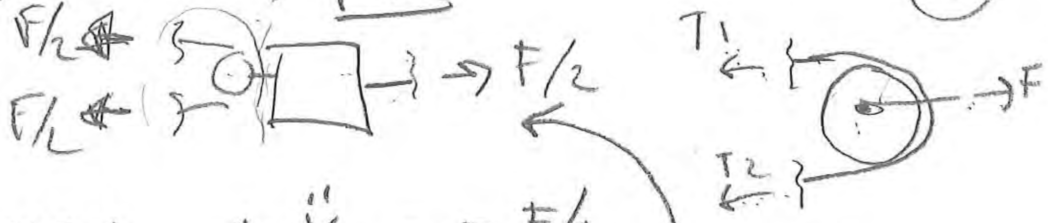
c) F/m

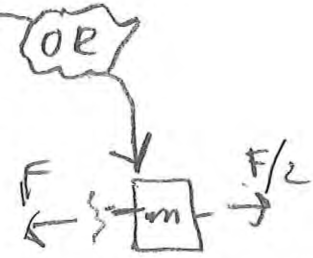
d) $-F/m$

e) $-F/(2m)$

Q2:

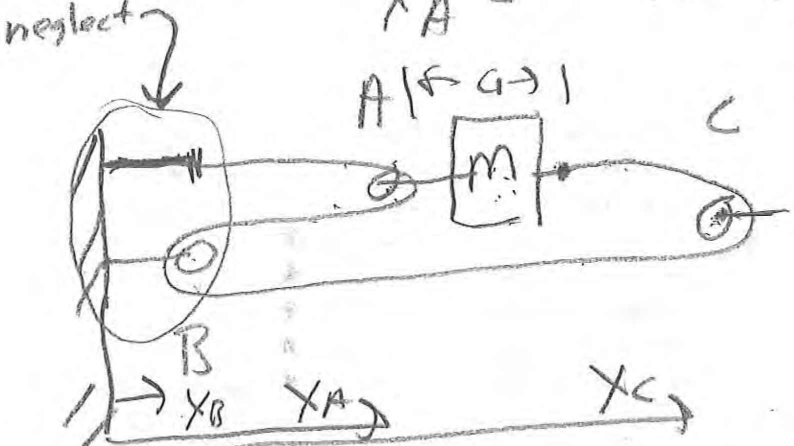
FBDS





LMB: $m\ddot{x}_A = -F/2$

$\ddot{x}_A = -F/2m$



$$
\begin{aligned}
\text{Const} &= \text{length} \\
&= x_A + (x_A - x_B) + x_B \\
&\quad + (x_C - x_B) + x_C - (x_A + c_1)
\end{aligned}
$$

$\frac{d^2}{dt^2}\{\}$

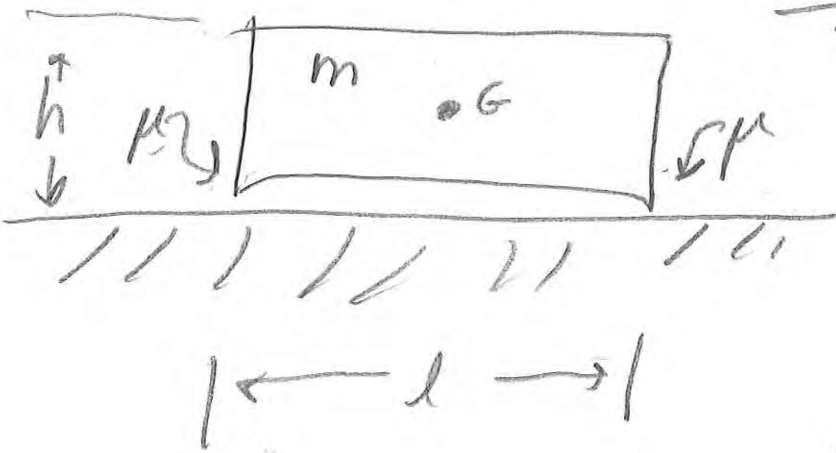
$\Rightarrow \quad 0 = \ddot{x}_A + 2\ddot{x}_C$

$\ddot{x}_C = -\ddot{x}_A/2 = \boxed{F/4m}$

QUIZ 3

$\downarrow g$ $\hat{j}\uparrow \rightarrow \hat{i}$

$\rightarrow v > 0$



$a_G = ?$ $a_G = \ddot{x}_G$

a) $-\mu g$ ✓

b) $-\mu/(2g)$

c) $-\mu(h/l)g$

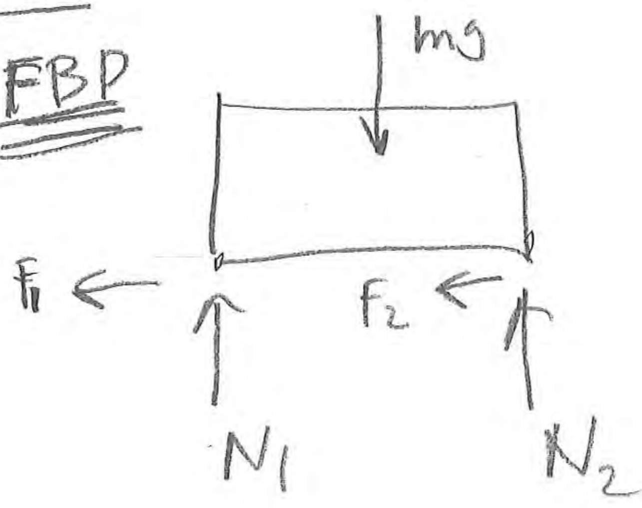
d) $-\mu g(h/2 - l/2)/l$

e) $-\mu g/3$

Q3

(148)

FBP



Friction: $F_1 = \mu N_1$

$F_2 = \mu N_2$

$\{LMB\}\cdot\hat{j}$

$$\Rightarrow (N_1 + N_2) - mg = m\cancel{\ddot{y}_G}^{\,0}$$

$$N_1 + N_2 = mg$$

$$\Rightarrow F_1 + F_2 = \mu m g$$

$$\{LMB\}\cdot\hat{i} \Rightarrow -F_1 - F_2 = m\ddot{x} \quad ✓$$

$$\boxed{\ddot{x} = -\mu \cancel{m} g / \cancel{m}}$$

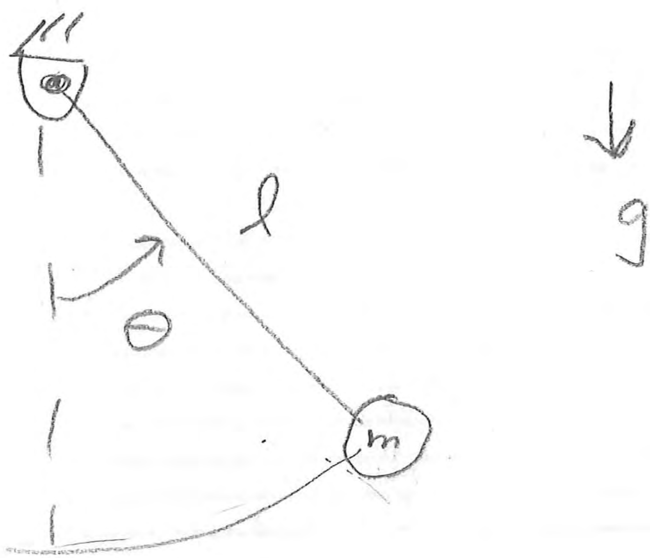
Lecture 16, March 20, 2019

TODAY

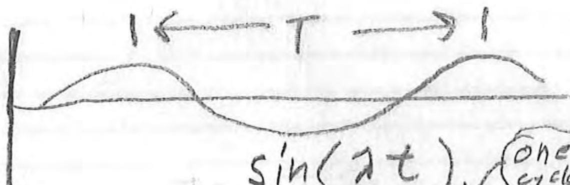
QUIZ, Rotations

See also computer demo on Rotations.

QUIZ



Assuming small $\theta \ll 1$



$\sin(\underbrace{\lambda t}_{2\pi})$ one cycle

a) $\lambda = g/\ell$

b $\lambda = \sqrt{g/\ell}$ ✓

c) $\lambda = 1$

d) $\lambda = \sqrt{\ell/g}$

e) $\lambda = \ell/g$

$T = 2\pi/\lambda$

$\lambda T = 2\pi$

$\lambda = 2\pi/T$

why "λ"?

omega (ω)

overloading

ω = ang. vel

α = freq.

ODE:

$$\boxed{\ddot{\theta} = -\frac{g}{\ell}\sin\theta}$$

small angle approximation

$\theta << 1$

$$\Rightarrow \quad \sin\theta \approx \theta - \frac{\theta^3}{3!} + \cancel{\frac{\theta^5}{5!}} \cdots$$

$$\Rightarrow \quad \boxed{\ddot{\theta} = -\frac{g}{\ell}\,\theta}$$

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\theta = 0}$$

$A\sin\left(\sqrt{\frac{g}{\ell}}\,t\right)$

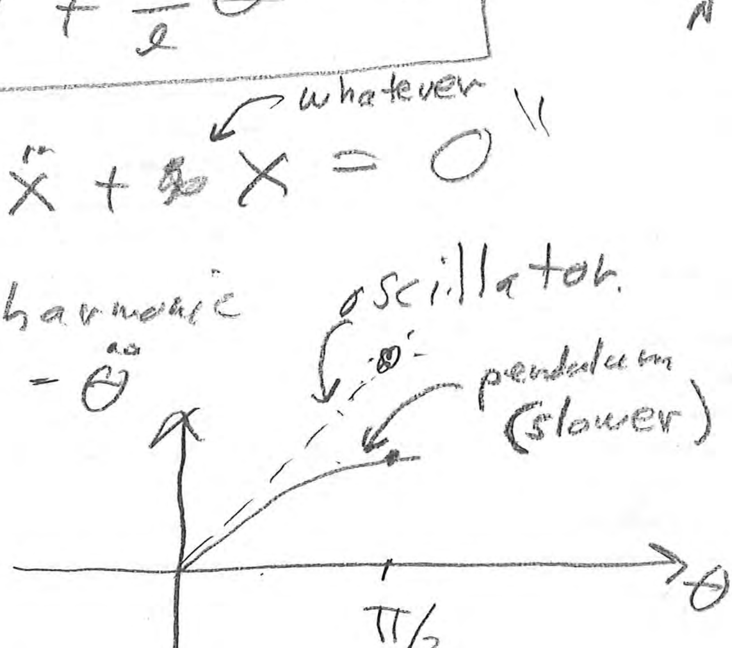
$\uparrow$

λ

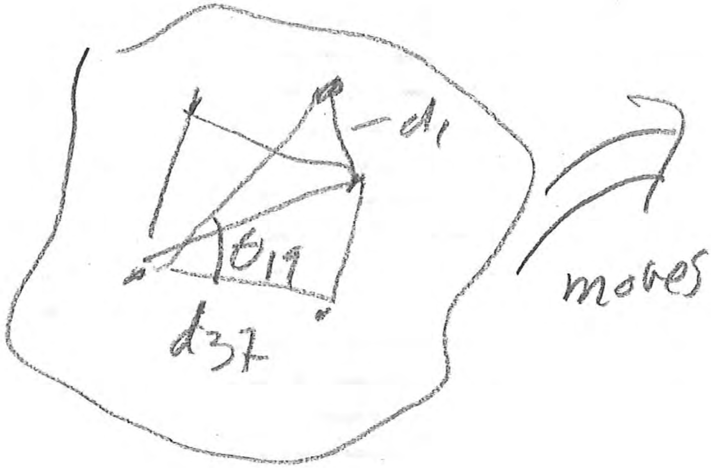
whatever

" $\ddot{x} + \cancel{\ } x = 0$ "

$\Rightarrow$ harmonic oscillator



Rigid objects

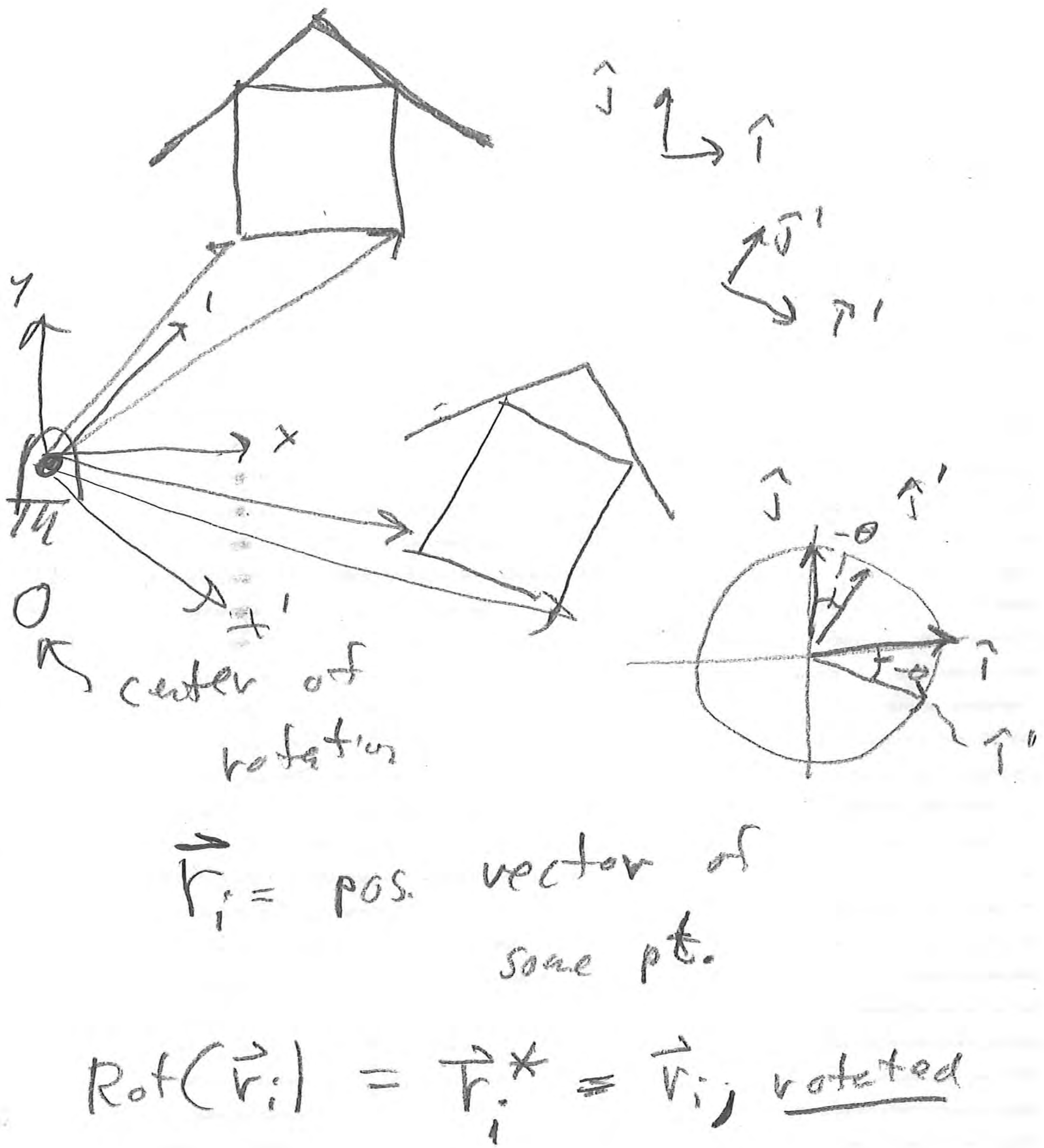


All ~~length~~ lengths &
all angles marked in
object are const.

rigid object

Review

no rotation
(breaking car problem)



$\vec{r}_i$ = pos. vector of some pt.

$Rot(\vec{r}_i) = \vec{r}_i^* = \vec{r}_i$, rotated typical pt.

$$\vec{r}_i \xrightarrow{\text{rotation}} \vec{r}_i^{\,*}$$

orig. coords — new coords

$$\vec{r}_i = x_i \hat{\imath} + y_i \hat{\jmath}$$

TWO WAYS TO LOOK AT SAME THING

$$\xrightarrow[\text{rotate}]{} \Rightarrow \vec{r}_i^{\,*} = x_i^* \hat{\imath} + y_i^* \hat{\jmath}$$

new

$$= x_i \hat{\imath}' + y_i \hat{\jmath}'$$

$$x_i^* = ? \qquad y_i^* = ?$$

$$\boxed{\vec{r}_i^{\,*} = \vec{r}_i^{\,*}}$$

Laurie Anderson "Let X=X"

$$\left\{ x_i^* \hat{\imath} + y_i^* \hat{\jmath} = x_i \hat{\imath}' + y_i \hat{\jmath}' \right\}$$

$$\{\,\} \cdot \hat{\imath} \Rightarrow x_i^* = x_i \underbrace{\hat{\imath}' \cdot \hat{\imath}}_{\cos\theta} + y_i \underbrace{\hat{\jmath}' \cdot \hat{\imath}}_{-\sin\theta}$$

$\{\hat{i}, \hat{j}\} \Rightarrow$

$$y_i^* = \sin\theta \, x_i + \cos\theta \, y_i$$

$$\begin{bmatrix} x_i^* \\ y_i^* \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} x_i \\ y_i \end{bmatrix}}_{\text{one pt}}$$

$$\begin{bmatrix} \text{New house coords} \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \underbrace{\left[\begin{array}{c|c|c|c} x_1 & x_2 & \cdots & \cdots \\ y_1 & y_2 & & \end{array}\right]}_{\text{diff. pts. on picture: 1, 2, 3, ...}}$$

$$\left[\begin{array}{c|c|c} x_1^* & x_2^* & \cdots \\ y_1^* & y_2^* & \end{array}\right]$$

Eqs. of Motion

for circular motion

physics classes

$$\sum \vec{M}_{/O} = \begin{cases} I^O \dot{\omega} \hat{k} \\ \boxed{\vec{r}_{G/O} \times m\vec{a}_G + I^G \dot{\omega} \hat{k}} \end{cases}$$

our main eqn.

origin of rotation

$$I^G = \begin{cases} \sum r_{i/G}^2 \, m_i \\ \int r_{/G}^2 \, dm \end{cases}$$

Polar moment of inertia about G

$$\sum \vec{M}_{/C} = \underbrace{\left.\begin{array}{l} \sum_i \vec{r}_{i/C} \times m_i \vec{a}_c \\ \int \vec{r}_{/C} \times \vec{a}\, dm \end{array}\right\}}_{\dot{\vec{H}}_{/C}}$$

always

all dyn.

$$\dot{\vec{H}}_{/C} = \overbrace{\vec{r}_{G/C} \times m \vec{a}_G}^{\text{this is all if no rotation}} + I^G \dot{\omega} \hat{k}$$

$\boxed{\text{2D rigid objects}}$

α = angular accel.

~~Rotation~~

Lecture 17, March 25, 2019

TODAY

QUIZ (not)

1) m, G, I

2) Rotational Dynamics

$$\boxed{\text{"Let } X = X\text{"}}$$

Laurie Anderson music

Distribution of mass

How much?: m_{TOT}

Where?: G, $\vec{r}_G$

How spread out?: I^G, (I^O)

$M_{TOT} = \Sigma$ add up, one way or another

m particle

$m_1 + m_2$ 2 particles

$\sum m_i$ system of particles

$\int dm$ continuum

$\int \rho \, dV$ volume

mass per unit volume

$\int \rho_s \, dA$ surface flat or curved

m per unit area

$\int \rho_\ell \, ds$ curve

arc length

mass per unit length

associative rule of addition

$\sum m_I$ system of systems

$$\underbrace{(m_1 + m_2 \dots)}_{m_I} + \underbrace{(m_{37} + m_{38} \dots \dots)}_{m_{II}}$$

Center of mass

"Center of Gravity", G, CoM,

Average pos. of mass in system, "weighted" by mass.

scores

Analogy: test scores: $x_1, x_2 \ldots$

$x_{Ave} = ?$

$= x$ one test

$x_{Ave} = \dfrac{x_1 + x_2}{2}$ 2 tests

$x_{Ave} = \dfrac{\sum x_i}{n}$ lots of tests

\# of stud. w/ score

score

$$x_{Ave} = \frac{n_1 x_1 + n_2 x_2 + \ldots n_m x_n}{n_{stud\,tot}}$$

if n_i students got score x_i

back to CoM : pos. x

analog to last scene

$$m_{tot} X_G = \begin{cases} X & \text{one part} \\ x_1 m_1 + x_2 m_2 & \text{2 part.} \\ \sum m_i x_i & \text{collection} \\ \int x \, dm & \text{continuum} \\ \int_V x \rho \, dV & \text{Volume} \\ \int x \rho_A \, dA & \text{Surface} \\ \int x \rho_s \, ds & \text{Curve} \\ m_I X_I + m_{II} X_{II} + m_{III} X_{III} \text{ etc} & \text{System of systems} \end{cases}$$

Vector position of G: $\vec{r}_G$

$$m_{TOT}\,\vec{r}_G = \left.\begin{cases} \sum m_i \vec{r}_i \\[2em] \int \vec{r}\, dm \\[2em] \underbrace{\sum \vec{r}_I^{\,G}\, m_I}_{\text{system of systems}} \end{cases}\right\}\ \text{main cases}$$

Center of mass rel. to some position A: $\underbrace{\vec{r}_{G/A} = \vec{r}_{AG}}$

$$m_{TOT}\,\vec{r}_{G/A} = \sum \vec{r}_{i/G}\, m_i$$

$$\vec{r}_{o/G} = ?$$

$$\underbrace{m\,\vec{r}_{G/G}}_{\text{expect this to be zero}} = \sum \vec{r}_{i/G}\, m_i$$

$$= \sum (\vec{r}_i - \vec{r}_G)\, m_i$$

doesn't vary w/ i

$$= \underbrace{\sum \vec{r}_i\, m_i}_{\shortparallel} - \sum \vec{r}_G\, m_i$$

$$= m_{TOT}\,\vec{r}_G - \vec{r}_G \underbrace{\sum m_i}_{m_{TOT}}$$

$$= m_{TOT}\,\vec{r}_G - m_{TOT}\,\vec{r}_G$$

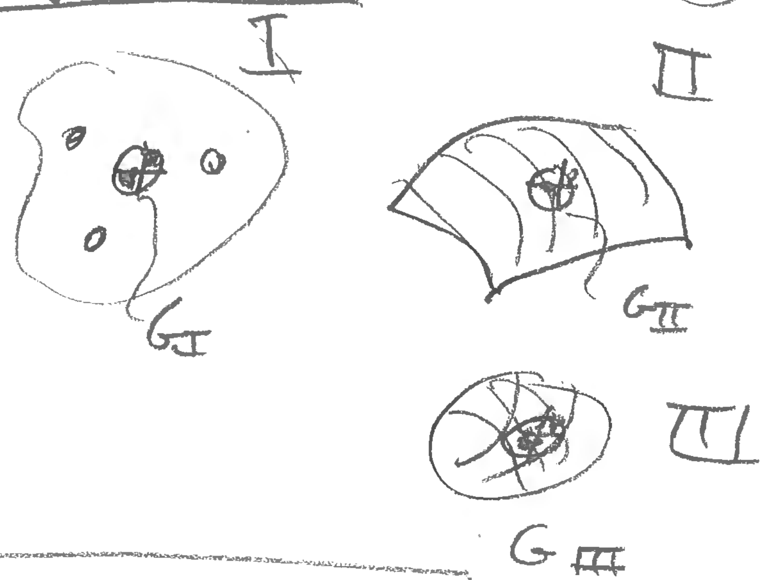
$$= \vec{0}$$ ✓ as expected

useful math for

$\vec{H}$ & E_K

decomposing ang. mom. ↑ Koenig's Thms

Apply to subsystems



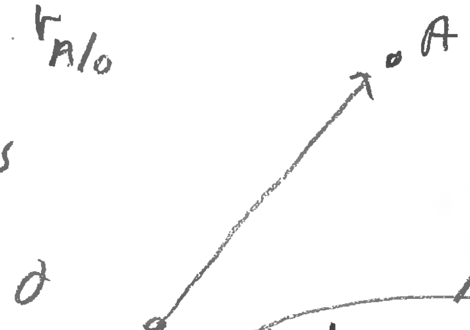
$$\boxed{m_{TOT}\,\vec{r}_{G\,TOT} = \sum m_I\,\vec{r}_{GI}}$$

Very useful

ASIDE

$\vec{r}_A = \vec{r}_{A/O}$

means



$\vec{r}$ vs $[\vec{r}_{A/O}]_{xyz}$

simple & vague

precise & confusing

Moment of inertia

(How spread out is mass)

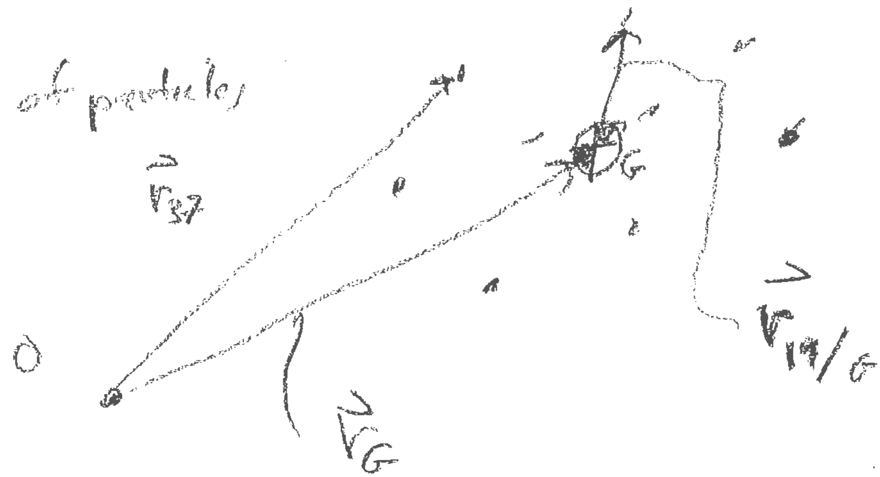
$\underline{\underline{I}}^G$ = moment of inertia rel. to center of mass (of object)

Applies to a $\boxed{\text{single rigid object}}$

(in 2D it's a scalar)

particle $\quad I^G = 0$

coll. of particles

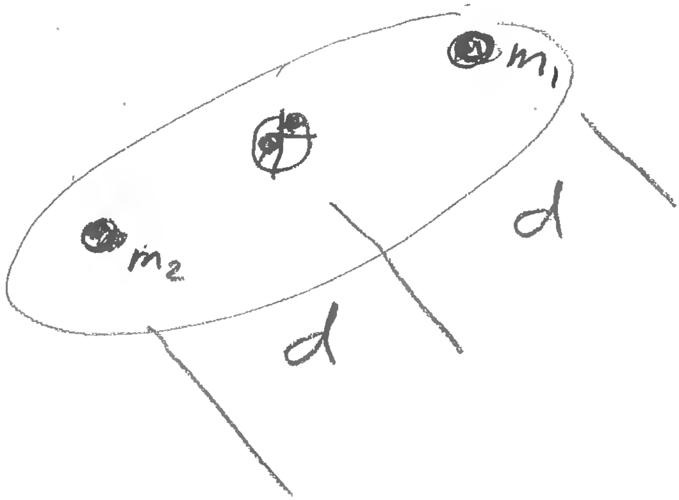


$$\boxed{I^G = \sum m_i \underbrace{r_{i/G}^2}_{|\vec{r}_{i/G}|^2}}$$

Continuum

$$I^G = \int r_{/G}^2 \, dm$$

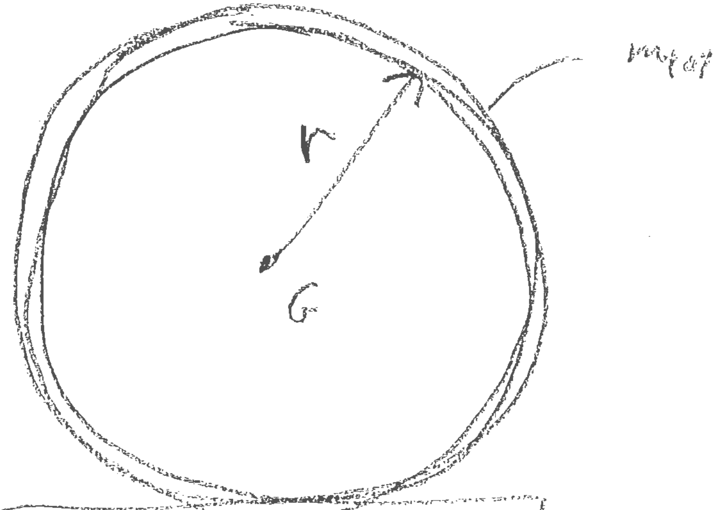
ex) 2 identical masses



$$I^G = m_1 d^2 + m_2 d^2$$

$$\boxed{I^G = m_{TOT} \, d^2}$$

ex) ring



$$\boxed{I^G = r^2 m_{TOT}}$$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$I^G = \int r^2 dm$$

$$I^G = \int_0^{2\pi} r^2 \underbrace{\gamma \underbrace{r\,d\theta}_{ds}}_{dm}$$

$$I^G = \int_0^{2\pi} r^3 \gamma\, d\theta = 2\pi r^3 \gamma$$

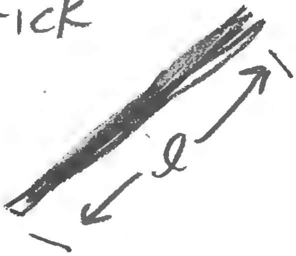
$$I^G = m_{TOT}\, r^2$$

$$\llcorner 2\pi\gamma$$

as expected

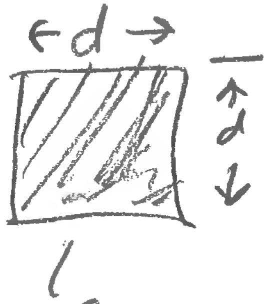
Know these integrals

stick



$I^G = ml^2/12$

$I^G = mr^2/2$

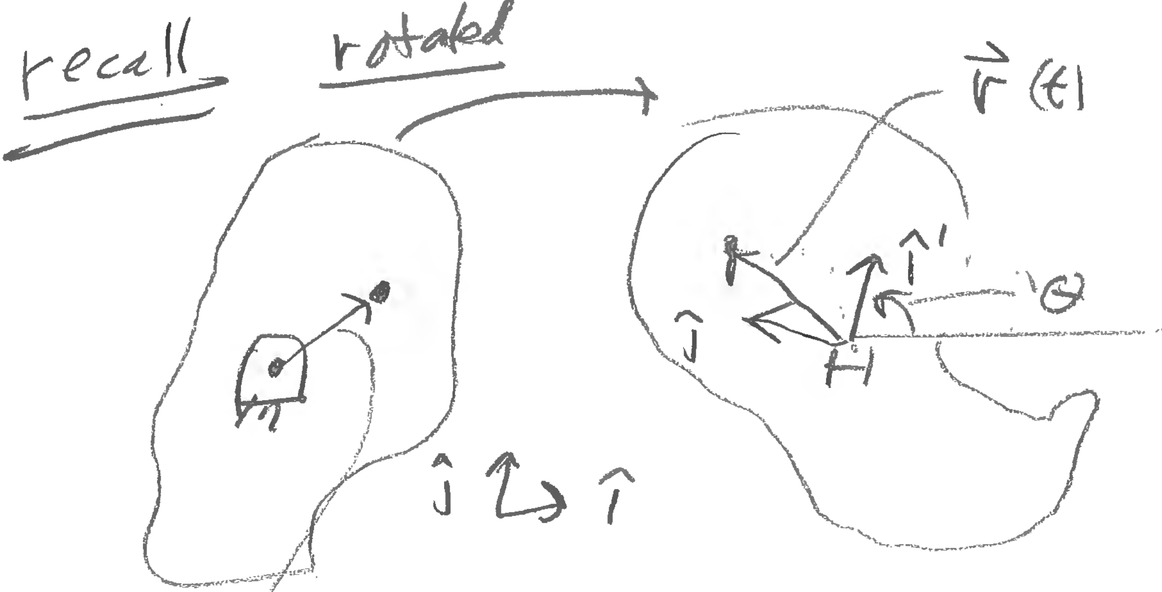


$I^G = md^2/6$

$I^G = \frac{m}{12}(d^2+h^2)$

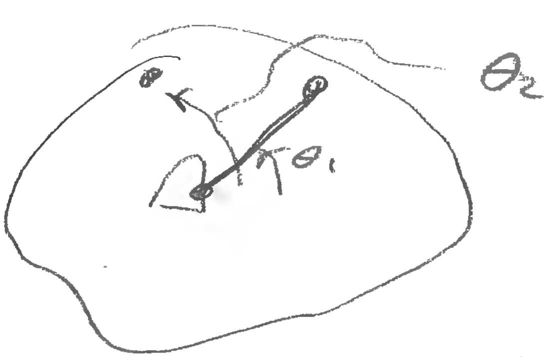
Back to Kinematics

Rotation of rigid objects



$$\vec r^{\,0} = x_0\,\hat\imath + y_0\,\hat\jmath$$

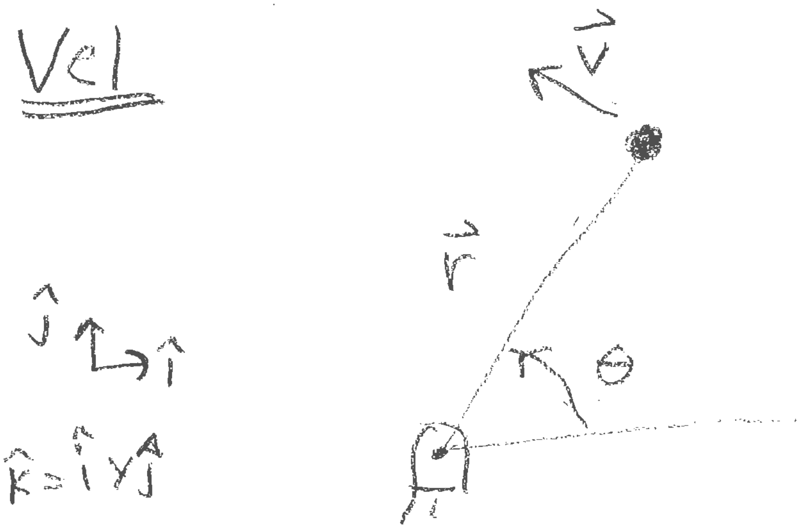
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



$\theta_1 \neq \theta_2$

$\dot\theta_1 = \dot\theta_2$

Vel



$$\vec{V} = r\dot{\theta}\hat{e}_\theta$$

θ for that particle

$$= \dot{\theta}\left(\hat{k} \times \vec{r}\right)$$

same dir,
same mag,
$\Rightarrow$ same vector
"McCarthy"

$$= \left(\dot{\theta}\hat{k}\right) \times \vec{r}$$

$$= \vec{\omega} \times \vec{r}$$

$$\llcorner \dot{\theta}\hat{k}$$

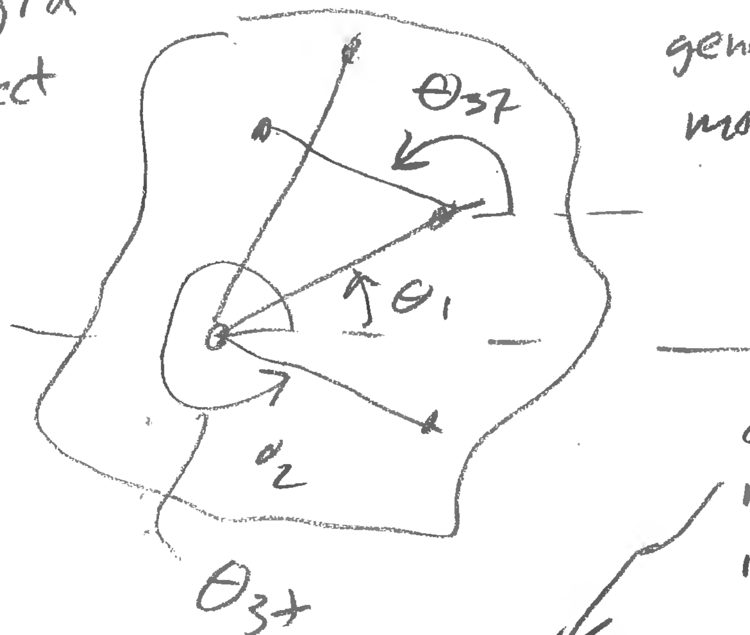
$$\boxed{\vec{\omega} = \text{angular velocity of object}}$$

$$\omega = |\vec{\omega}|$$

More specific

rigid object

even for general motion



any line marked in object

$$\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_{37} = \omega$$

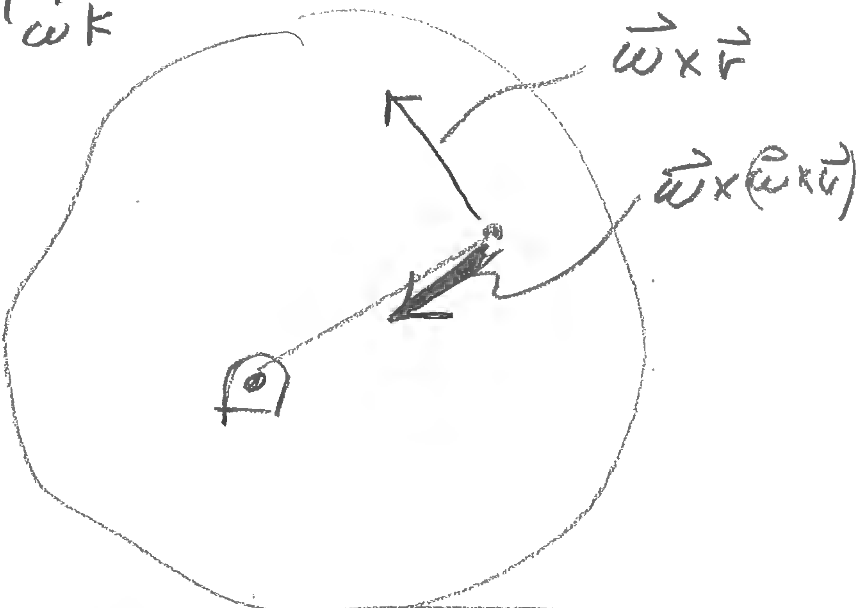
ω = rot. rate of every line on object.

Circ. motion acceleration

$$\vec V = \vec\omega \times \vec r_{/o}$$

$$\dot{\vec V} = \vec a = \dot{\vec\omega}\times\vec r + \vec\omega \times \underbrace{\dot{\vec r}}_{(\vec\omega\times\vec r)}$$

$$= \underbrace{\dot{\vec\omega}}_{\dot\omega\hat k}\times\vec r + \underbrace{\vec\omega\times(\vec\omega\times\vec r)}_{-\omega^2\vec r}$$



$$\boxed{\vec a_{/o} = \dot{\vec\omega}\times\vec r - \omega^2\vec r}$$

circ. motion

AMB/c

For any FBD

$$\sum \vec{M}_{/c}^{ext} = \begin{cases} \vec{r}_{i/c} \times (m_i \vec{a}_i) = \dot{\vec{H}}_{/c} \\ \int \vec{r}_{/c} \times \vec{a}\, dm = \dot{\vec{H}}_{/c} \\ \sum_{\text{subsystem}} \dot{\vec{H}}_{I/c} \quad \text{of subsystems} \\ \frac{d(\vec{H}_{/c})}{dt} \quad \text{for a fixed pt. at } C \end{cases}$$

$\vec{H}$ = angular momentum

("$\vec{L}$" in Physics)

Key fact:

derived result.
(deriv. in book)

gen. formula for 2D is gen. motion

$$\sum \vec{M}_{/C} = \underbrace{\left(\vec{r}_{G/C} \times m_{tot} \vec{a}_G + I^G \dot{\omega} \hat{k} \right)}_{\sum \vec{r}_{i/C} \times m_i \vec{a}_i}$$

any 2D motion of rigid object

$$\sum \vec{M}_{/O} = I^O \dot{\omega} \hat{k}$$

for rot. about fixed pt. O

BEWARE

Advice: Ignore I^O unless you understand.

Rotational Dynamics
Gen. motion
Today

Wed, March 27 2019

2D rigid objects

General facts:

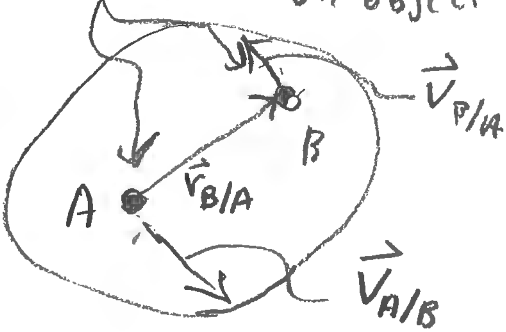
$$\boxed{\vec{V}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}}$$

$\vec{V}_{B/A} = \vec{V}_B - \vec{V}_A$

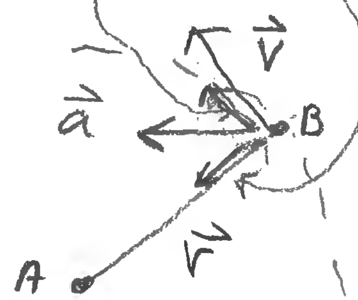
$\vec{\omega} = \dot{\theta}\hat{k}$

$\dot{\theta}$ = rate of change of any angle on object

any 2 pts, fixed on object



$$\vec{a}_{B/A} = \underbrace{\dot{\vec{\omega}} \times \vec{r}_{B/A}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})}_{= -\omega^2 \vec{r}_{B/A}}$$



accel. towards center of circle due to change of direction

acc. tan to circle due to change of speed

Moment of Inertia

$$I^G = \int |\vec{r}_{/G}|^2 dm = \int r_{/G}^2 dm$$

polar mass moment of Inertia about G

$\underbrace{CoM = G}_{\text{Center of Mass at } G}$: $m_{tot}\vec{r}_G = \begin{cases} \int \vec{r}\, dm \\ \sum \vec{r}_i m_i \end{cases}$

Mechanics

$\vec{L}$ = Lin mom $\equiv \sum m_i \vec{v}_i$

LMB : $\boxed{\sum \vec{F}_j^{ext} = \underbrace{\sum m_i \vec{a}_i}_{\dot{\vec{L}}}}$

2D rigid object

$$\boxed{\sum \vec{F}^{ext} = m_{tot}\vec{a}_G}$$

and all other systems.

However, on rigid object, G is fixed w.r.t. object. A pt. "on" the object.

AMB: $\sum \vec{M}^{ext}_{/C} = \frac{d}{dt} \vec{H}_{/C}$

pt. C is fixed in space

$= \sum \vec{r}_{i/C} \times m_i \vec{a}_i$ ✓

C doesn't have to be fixed in space

$$\boxed{\vec{H}_{/C} = \sum \vec{r}_{i/C} \times m_i \vec{v}_{i/C}}$$ = Ang Mom. About C

Rigid Object 2D

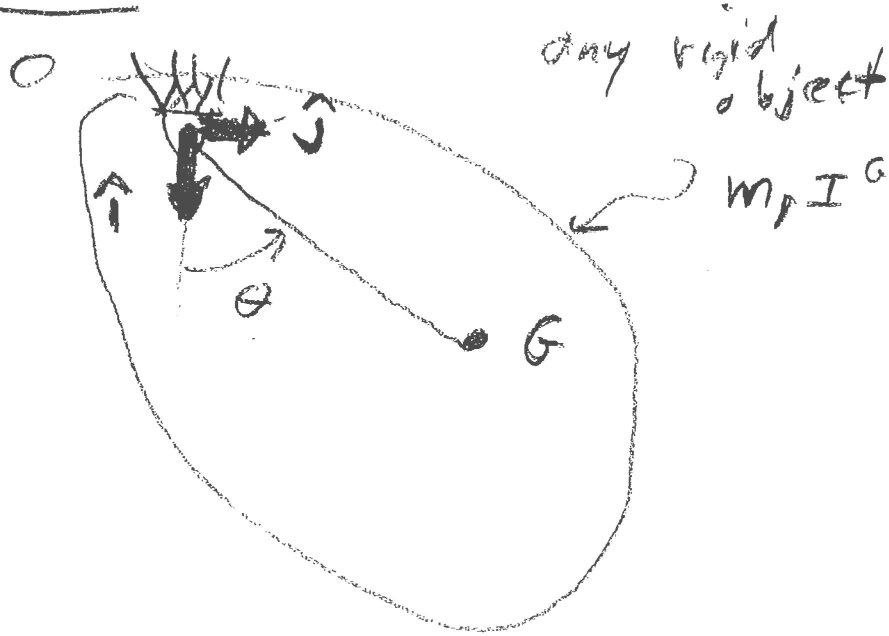
$$\boxed{\sum \vec{M}^{ext}_{C}} = \dot{\vec{H}}_{/C} = \boxed{\vec{r}_{G/C} \times m \vec{a}_G + I \alpha \hat{k}}$$

$\alpha = \ddot{\theta}$

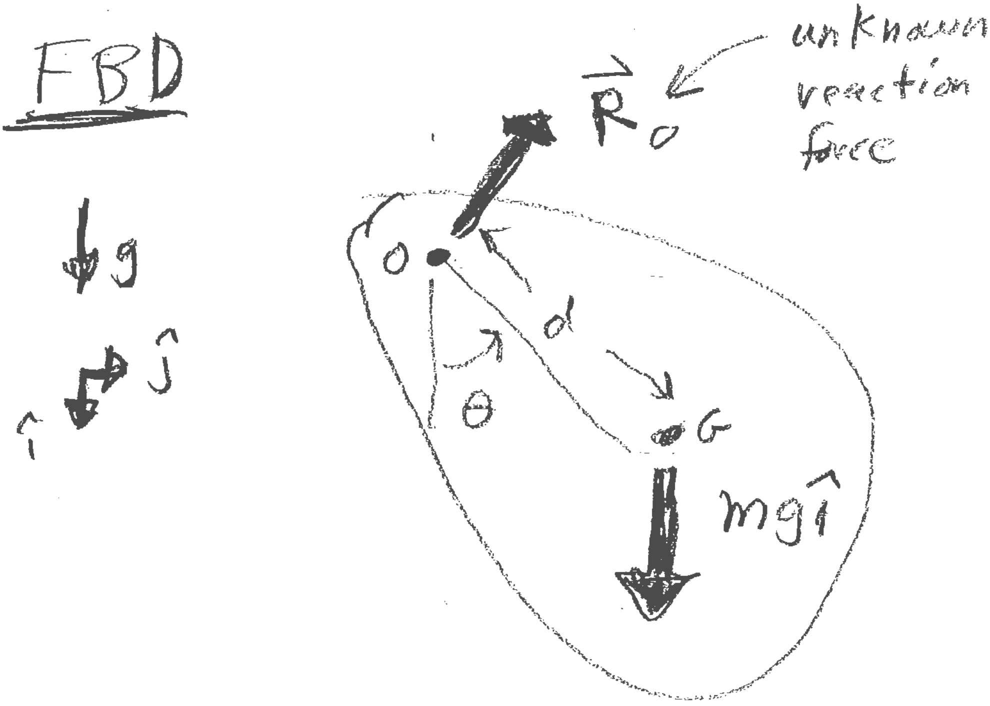
angle of line marked in object

does not have to be fixed in space

Pendulum



FBD



EoM ?

Ans: $\sum \vec{M}_{/0} = \overbrace{\vec{r}_{G/0}}^{d\hat{e}_r} \times m\vec{a}_G + I\alpha\hat{k}$

$\vec{a}_G$: $(d\ddot{\theta}\hat{e}_\theta + d\dot{\theta}^2\hat{e}_r)$

$\vec{r}_{G/0} \times mg\hat{i}$

$\vec{r}_{G/0}$: $d\underbrace{(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})}_{\hat{e}_r}$

$$\left\{ -mgd\sin\theta\hat{k} = d^2\ddot{\theta}m\hat{k} + I^G\ddot{\theta}\hat{k} \right\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow -mgd\sin\theta = \underbrace{\left(I^G + md^2\right)}_{[I^0]}\ddot{\theta}$$

aside

$$\boxed{\ddot{\theta} = \frac{-mgd}{\underbrace{I^G + md^2}}\sin\theta}$$

$$\boxed{\ddot{\theta} = -c\sin\theta}$$

<u>Recall</u>: simple pend

$c = \frac{g}{l} = g/d$

New idea:

mean square distance of mass from G

$$r_{gyr}^2 \equiv (\text{"radius of gyration"})^2$$

$$= I^G/m$$

$$\Rightarrow \quad I^G = m\, r_{gyr}^2$$

$$\ddot{\theta} = \frac{-\cancel{m} g d}{\cancel{m} r_{gyr}^2 + \cancel{m} d^2} \sin\theta$$

$$= \frac{-g}{d} \underbrace{\left(\frac{1}{1 + \left(r_{gyr}/d \right)^2} \right)}_{\text{shape factor}} \sin\theta$$

ex) pt. mass: $r_{gyr} = 0$

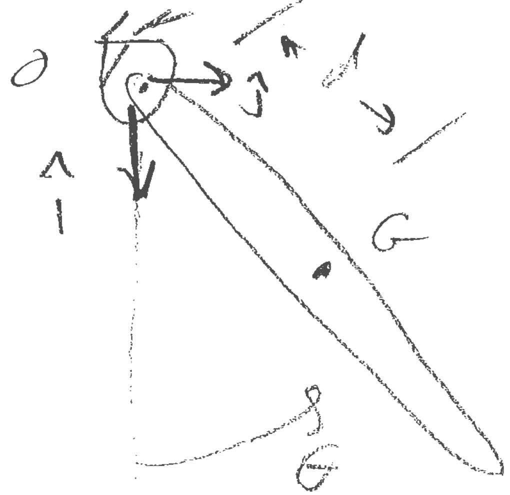
shape factor $= 1$

ex)

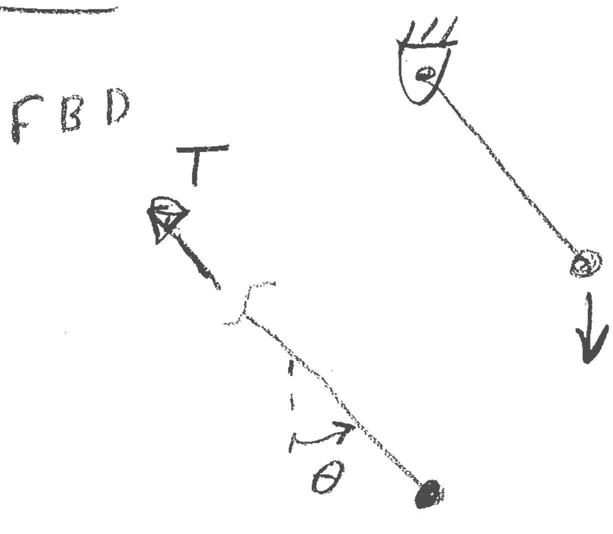
Stick as pendulum

line mass

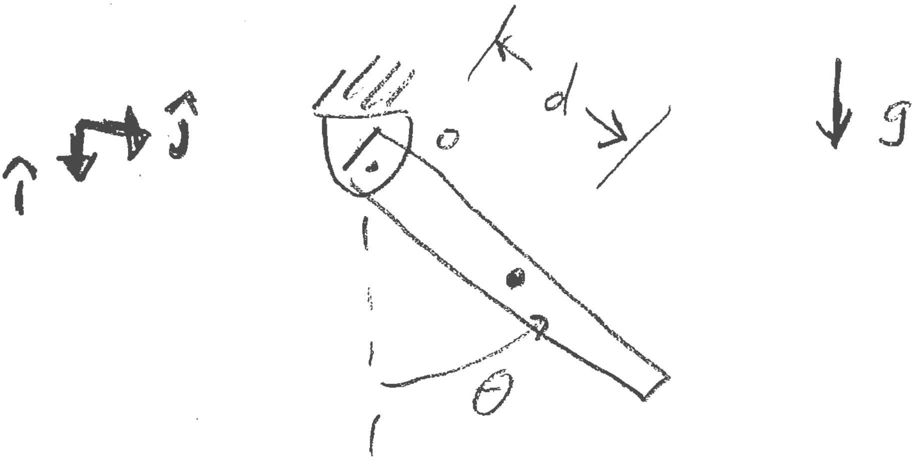
uniform



Recall: pt. mass pendum

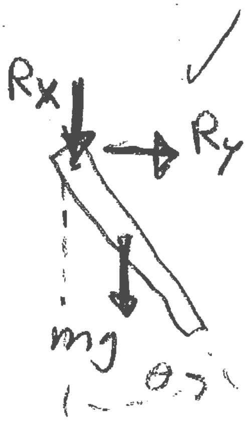


Stick-pendulum

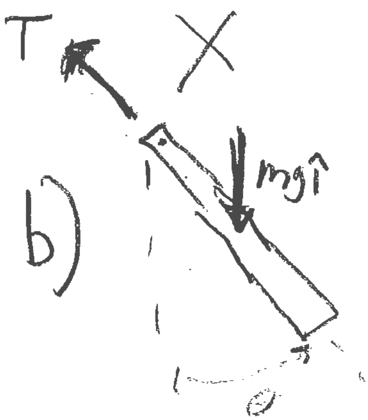


Which FBD is wrong?

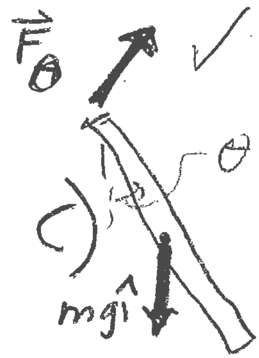
a)



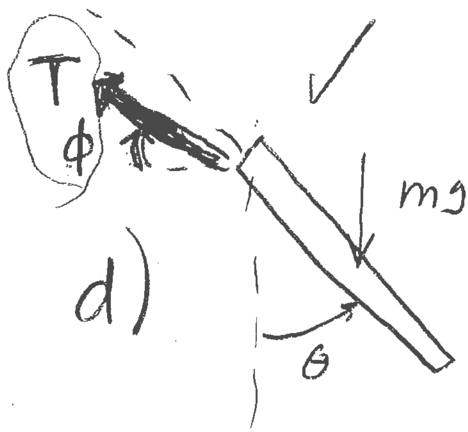
b)



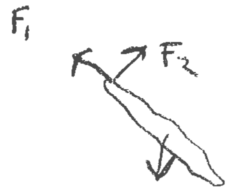
c)

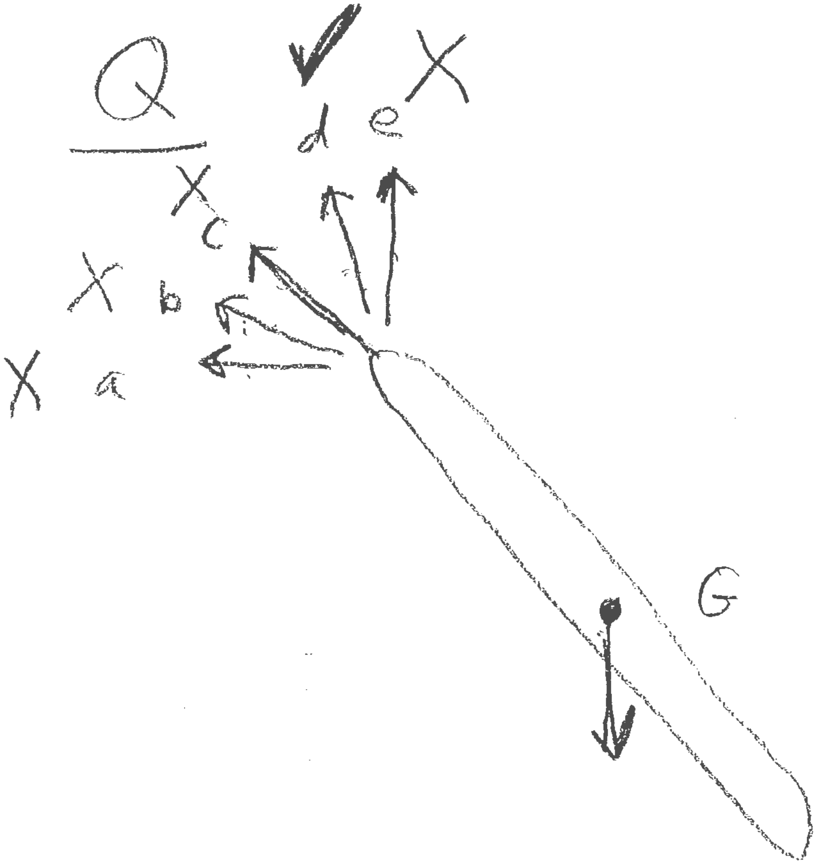


d)



e) None, all are O.K.





a) stick has vert. accel $\neq g$
$\Rightarrow$ needs vert. comp. (a) has none
$\Rightarrow$ X

$\ddot{\theta} > 0$

b) $AMB_{/G} \Rightarrow$ CCW ang. accel, but
$\ddot{\theta} < 0$ in this config w. $\theta > 0$
$\Rightarrow$ X

c) $AMB_{/G} = 0$ if no torques. But
$\ddot{\theta} \neq 0 \Rightarrow$ torque-free C is wrong!

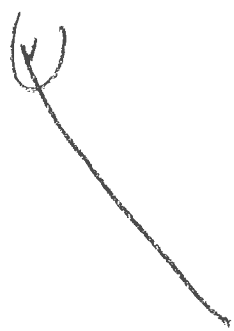
d) all o.k.

e) Doesn't give known-to-exist horiz. accel.

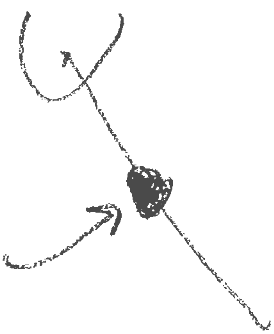
Q

$\downarrow g$

Uniform Stick



Uniform Stick w/ pt. mass added 1/2 way down



frequency of oscillation changed?

a) faster ?

b) Same ?

c) Slower ?

Note

$$\ddot{\theta} = \frac{-g}{\ell}\sin\theta$$

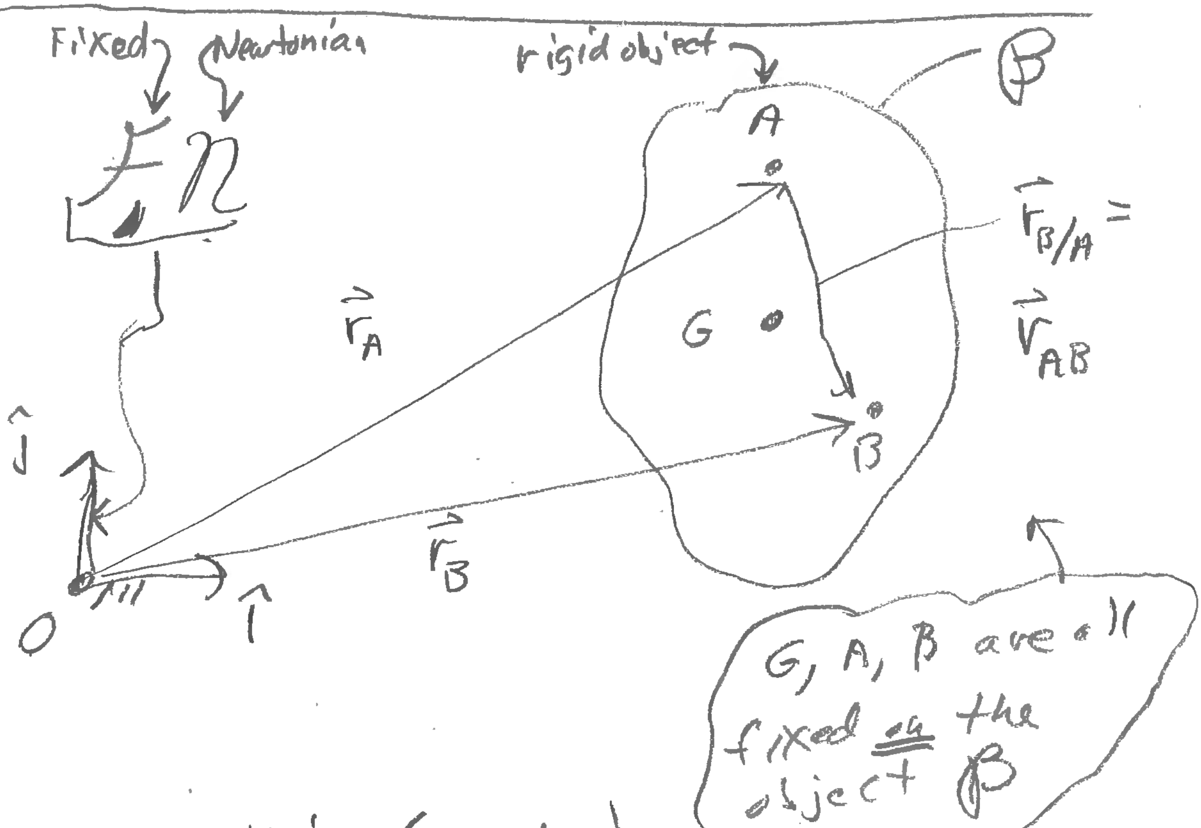
$$\ddot{\theta} = -\frac{g}{\ell}\underbrace{\left(\frac{1}{1+(r_{gyr}/d)^2}\right)}_{<1}\sin\theta$$

swings slower than pt. mass

Adding pt. mass slows the pendulum.

TODAY

General motion of 2D rigid object



Kinematics (geometry)

Know these

Pos. $\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$

Vel. $\vec{v}_B = \vec{v}_A + \boxed{\vec{v}_{B/A}}$, $\quad \vec{v}_{B/A} = \omega\hat{k} \times \vec{r}_{B/A}$

acc $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$, $\quad \vec{a}_{B/A} = -\omega^2 \vec{r}_{B/A} + \dot{\omega}\hat{k} \times \vec{r}_{B/A}$, $\quad \dot{\omega} = \alpha$

Rotation

Be able to derive this

$$\boxed{\underbrace{\begin{bmatrix} X_{B/A} \\ Y_{B/A} \end{bmatrix}}_{\text{present state}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \underbrace{\begin{bmatrix} X_{B/A}^{(0)} \\ Y_{B/A}^{(0)} \end{bmatrix}}_{\text{ref. state}}}$$

Motion Quantities

Know these

Lin. Mom.:

$$\vec{L} = m\vec{V}_G$$

$$\dot{\vec{L}} = m\vec{a}_G$$

Ang. Mom.:

$$\vec{H}_C = \vec{r}_{G/C} \times m\vec{V}_G + I^G \omega \hat{K}$$

$\vec{H}_{/C}$

C is a fixed pt.

$$\dot{\vec{H}}_{/C} = \underbrace{\vec{r}_{G/C} \times m\vec{a}_G}_{\text{motion of CoM.}} + \underbrace{I^G \dot{\omega} \hat{K}}_{\text{Motion rel. to CoM}}$$

$\dot{\omega} = \alpha$

works for pts not fixed

~~Laws of Mech.~~

$$\text{Kin En} = E_K = \frac{1}{2} m |\vec{V}_G|^2 + \frac{1}{2} I^G \omega^2$$

motion of CoM
+ motion rel. to CoM

$$\boxed{\text{SEE HUGE TABLE 1 IN BACK OF BOOK}}$$

e.g. row 7

other useful tables there also

LAWS of MECH

LMB: $\sum \vec{F}^{ext} = \dot{\vec{L}}$

$\vec{L} = m\vec{a}_G$

AMB: $\sum \vec{M}_{/C}^{ext} = \dot{\vec{H}}_{/C}$

any pt. C

$$\boxed{\vec{r}_{G/C} \times m\vec{a}_G + I^G \alpha \hat{k}}$$

Conservative forces $\Rightarrow E_p$

Cons of En. $E_p + E_K = const$

Non-Cons. Forces (any forces)

Power Balance:

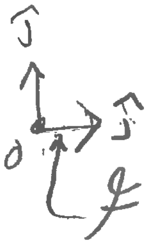
$$\text{Power in} = \dot{E}_K$$

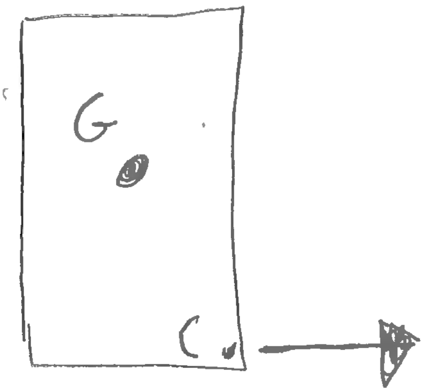
$$\sum \vec{F} \cdot \vec{v} = \dot{E}_K$$

particle velocities

$\vec{V}_G(0) = \vec{0}$, $\vec{\omega}(0) = 0\hat{k}$

Q: QUIZ





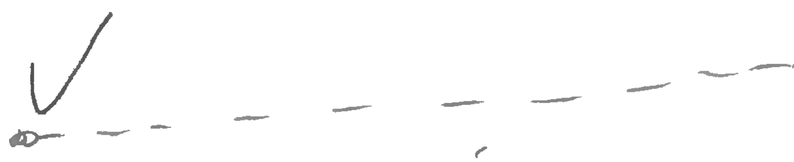
No other forces than this

$\vec{F} = \underbrace{F_0}_{\text{const}} \hat{I}$

Path of G ?

↑ trajectory

a) 

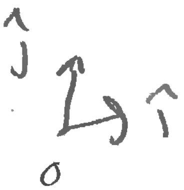
b) 

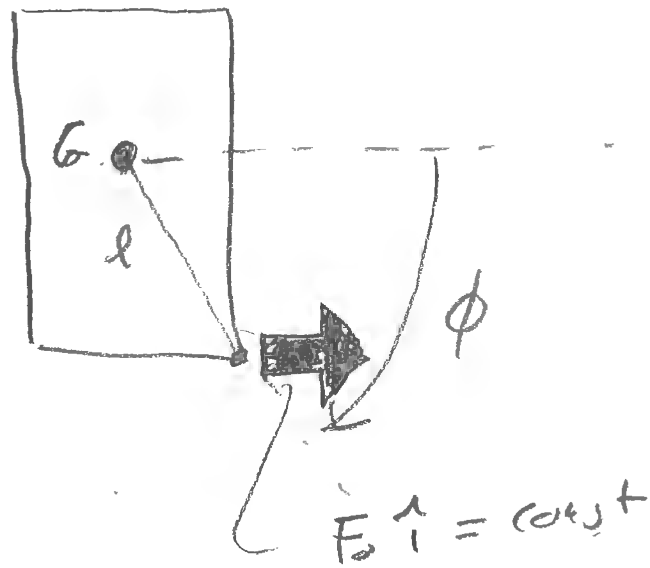
c) 

d) 

e) None of above

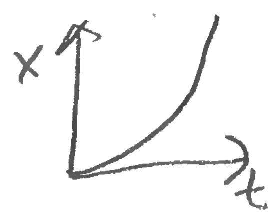
FBD





LMB: $\sum \vec{F}^{ext} = \dot{\vec{L}}$

$$\{ F_0 \hat{i} = m \vec{a}_G \}$$



$\{\,\}\cdot\hat{i}$ $\quad F_0 = m \ddot{x}_G$

$$\ddot{x}_G = F_0/m$$

$$x_G = \tfrac{1}{2}(F_0/m)\, t^2$$

$\{\,\}\cdot\hat{j}$ $\quad 0 = m \ddot{y}_G$

See next page

$\Rightarrow y = 0$ (all t)

$\Rightarrow$ Quiz Ans. $= (b)$

Why $y(t) = 0$?

$\ddot{y} = 0$ (1) , $\underbrace{\dot{y}(0) = 0, \quad y(0) = 0}_{I.C.'s}$

$\int (1)\, dt$ $\quad \dot{y}(t) = \underbrace{\int_0^t \ddot{y}\, dt'}_{0} + \underbrace{\dot{y}(0)}_{0} = 0$

$\dot{y}(t) = 0$ $\qquad \overbrace{y(0) = 0}^{I.C.}$

integrate again, use

$\Rightarrow \boxed{y(t) = 0}$

What about AMB?

$\underline{AMB}/G$ $\quad \sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$ $\qquad$ C.C.W

$\{\ \}\cdot\hat{k}$ $\quad l \sin\phi\, F_0 = I^G \dot{\omega}$ $\quad +\phi = C.W$

$L - \ddot{\phi}$

$$\boxed{\ddot{\phi} = -\frac{l F_0}{I^G} \sin\phi}$$ JUST LIKE A pendulum

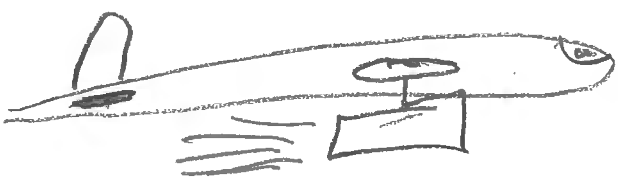
We can do a million problems now

Hard parts: * Geometry;
* Constit. Laws

ex)

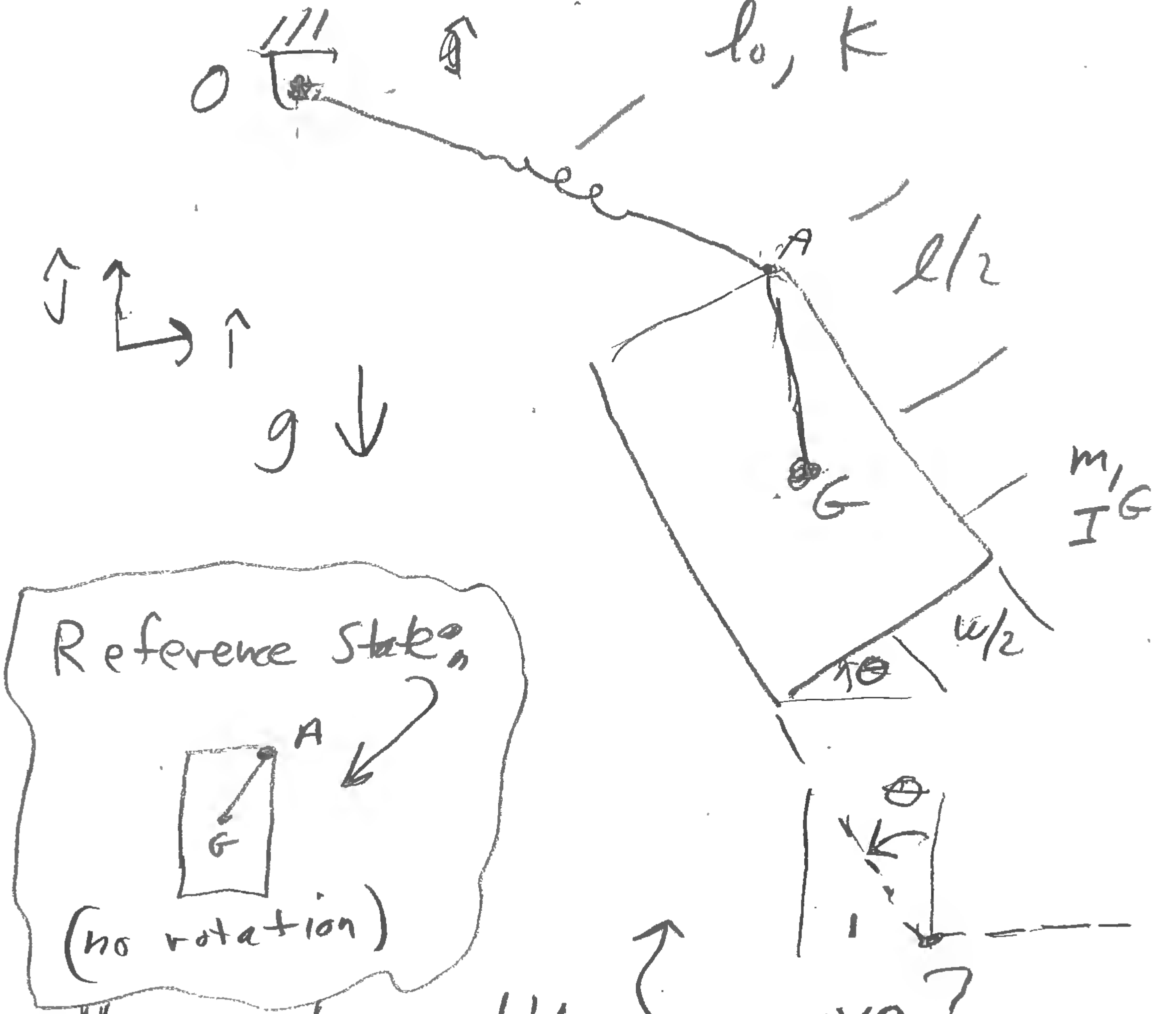
car w/ suspension

ex)



2D Flight Dynamics

Bungy Jumping



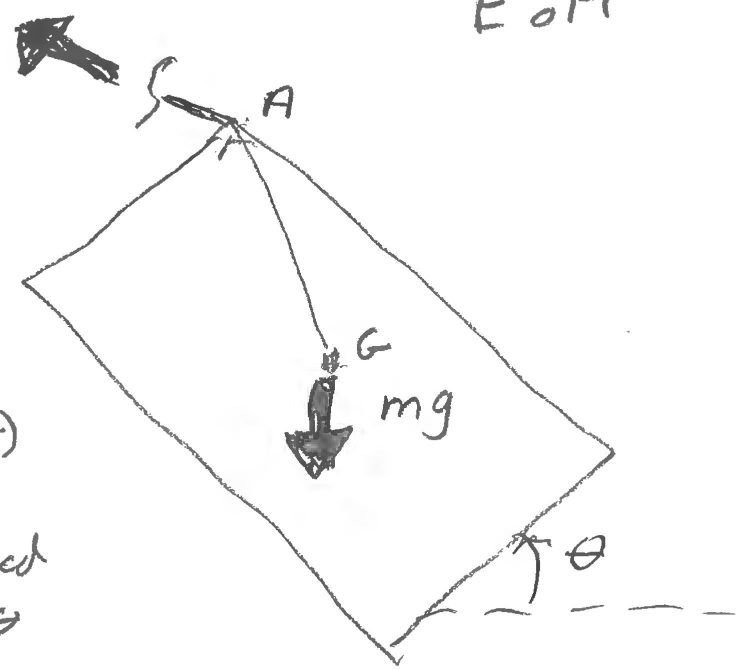
How does this move?

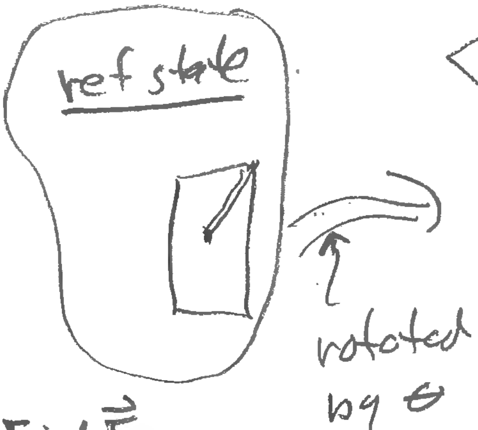
$\Rightarrow$ Find E.O.M

$\Rightarrow$ Given $\vec{r}_G, \vec{V}_G, \theta, \omega$ & parameters

$$\boxed{\underline{\text{Find}} \quad \vec{a}_G, \quad \dot{\omega}}$$ ← those are EoM

FBD





Find $\vec{F}_s$

$$\vec{F}_s = T_s \,\hat{\lambda}_{AO} = -T_s \,\hat{\lambda}_{OA}$$

$$\vec{r}_{A/O} = \vec{r}_{G/O} + \vec{r}_{A/G}$$

$$\vec{r}_{A/G} = x_{A/G}\,\hat{i} + y_{A/G}\,\hat{j}$$

present state

$$\begin{bmatrix} x_{A/G} \\ y_{A/G} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R} \begin{bmatrix} w/2 \\ l/2 \end{bmatrix}$$

ref state $\begin{bmatrix} x_{A/G}(0) \\ y_{A/G}(0) \end{bmatrix}$

We now know, given θ, $\vec{r}_G$:

$$\Rightarrow \quad \vec{r}_{A/O}$$

and $\hat{\lambda}_{A/O} = \vec{r}_{A/O} / |\vec{r}_{A/O}|$

Tension

$$T_s = K(l - l_0)$$

$$l = |\vec{r}_{A/O}|$$

$$\Rightarrow \quad \boxed{\vec{F}_s = -T_s \hat{\lambda}_{OA}}$$

See next page

LMB

$$\sum \vec{F} = m\vec{a}_G$$

$$-mg\hat{j} + \vec{F}_s = m\vec{a}_G$$

2 eqs $\quad \boxed{\dot{\vec{V}}_G = \frac{1}{m}\left[\vec{F}_s - mg\hat{j}\right]}$

2 eqs $\quad \boxed{\dot{\vec{r}}_G = \vec{V}_G}$ — Kinematics

* Aside on $\vec{F}_s$

Same calc as before, different lay out

$$\vec{F}_s = -T_s \hat{\lambda}_{OA}$$

$$\hat{\lambda}_{OA} = \vec{r}_{OA} / |\vec{r}_{OA}|$$

$$T_s = k(l - l_0)$$

$$\vec{r}_{OA} = \vec{r}_G + \vec{r}_{A/G}$$

$$l = |\vec{r}_{OA}|$$

$$\vec{r}_{A/G} = X_{GA}\hat{\imath} + Y_{GA}\hat{\jmath}$$

$$\begin{bmatrix} X_{GA} \\ Y_{GA} \end{bmatrix} = R \begin{bmatrix} X_{GA}^{ref} \\ Y_{GA}^{ref} \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} X_{GA}^{ref} \\ Y_{GA}^{ref} \end{bmatrix} = \begin{bmatrix} w/2 \\ L/2 \end{bmatrix}$$

On Computer: work from bottom up

END ASIDE

AMB

$$\sum \vec{M}_{/G} = \dot{\vec{H}}_G$$

$$\vec{r}_{A/G} \times \vec{F}_s = I^G \dot{\omega} \hat{k}$$

know know

$\theta, \hat{F}$

1 eq $\Rightarrow$ $\boxed{\dot{\omega} = \frac{1}{I^G}\left[\quad\right]}$

1 eq $\Rightarrow$ $\boxed{\dot{\theta} = \omega}$ $\leftarrow$ kinematics

$\Rightarrow$ 6 ODEs

w/ init. conds. $\Rightarrow$ Full motion

Solve ODES

$\Rightarrow x_G(t), \; y_G(t), \; \theta(t)$

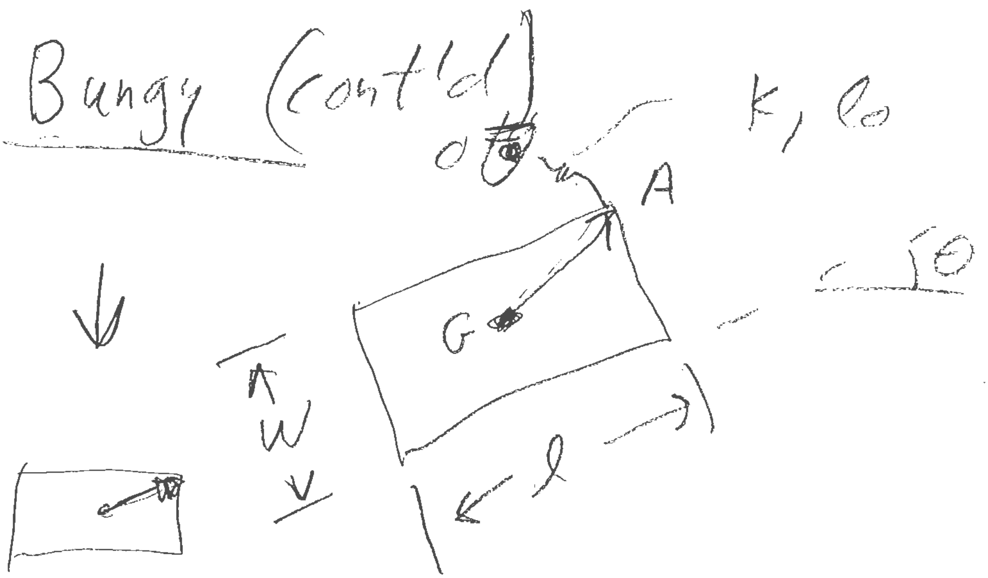
$\Rightarrow$ animation ☺

TODAY

~~Bungy Jumping on Matlab~~
Impulse - momentum
Rolling
(Collisions)

Heat Treatment of Aluminum

Bungy (cont'd)

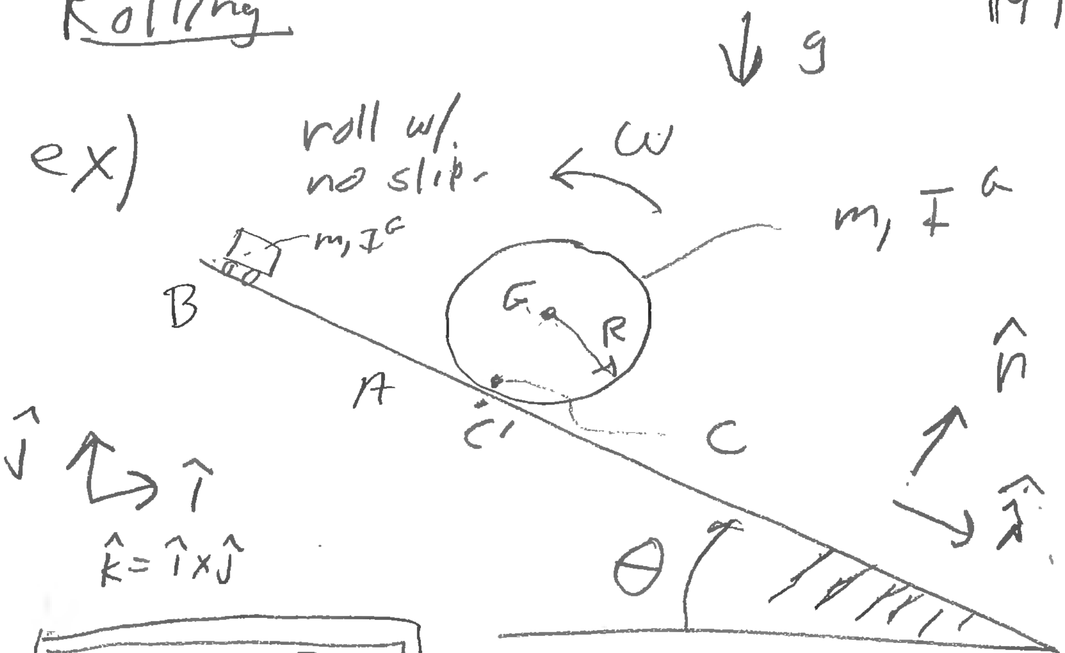


See notes from last lecture
(parted, expanded)

Key: Finding force of spring on block

Rolling

ex)



$$\boxed{\vec{a}_G = ?}$$

$$\boxed{\vec{V}_G = V_{Gx}\hat{\lambda}}$$

Kinematics

$$\vec{V}_C = \underbrace{\vec{V}_C}_{\vec{V}_{C'} = \vec{0}}$$

$$\vec{0} = \vec{V}_C$$

$$= \vec{V}_G + \vec{V}_{C/G}$$

$$= V_{Gx}\hat{\lambda} + \underbrace{\vec{\omega}}_{\omega\hat{k}} \times \underbrace{\vec{r}_{C/G}}_{-R\hat{n}}$$

$\hat{k} \times \hat{n} = -\hat{\lambda}$

$$\left\{ \vec{0} = V_{Gx}\hat{\lambda} + \omega R\hat{\lambda} \right\}$$

Rolling constraint

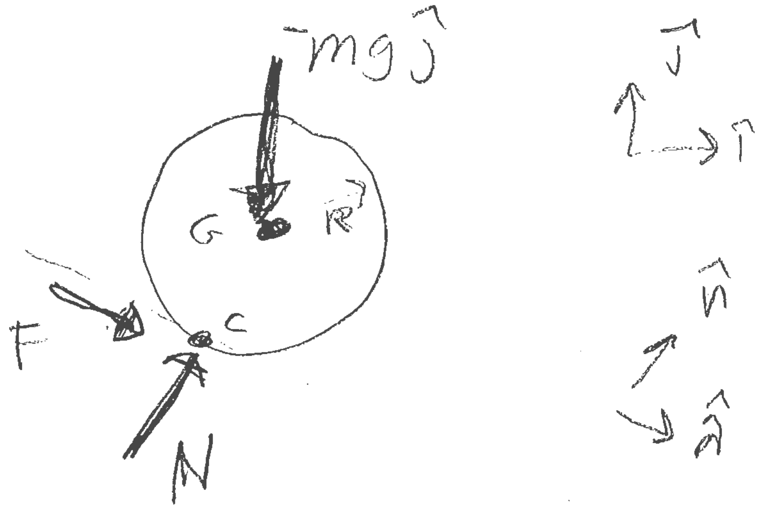
$\{\ \}\cdot\hat{\lambda} \Rightarrow \boxed{V_{Gx} = -R\omega}$ ①

$\frac{d}{dt}$ ① $\Rightarrow \boxed{a_{Gx} = -R\dot{\omega}}$

Note: $\vec{a}_C \neq \vec{a}_{C'}$!

FBD

m, I^G



AMB/C:

$$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

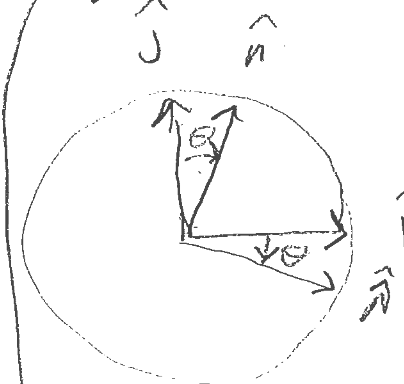
$$\vec{r}_{G/C} \times -mg\hat{j} = \vec{r}_{G/C} \times m\vec{a}_G + I^G\alpha\hat{k}$$

Subst. in: $\vec{a}_G = a_{Gx}\hat{\lambda}$ $\quad a_{Gx} = -R\alpha$

$\vec{r}_{G/C} = R\hat{n}$

$$\Rightarrow \quad R\hat{n} \times -mg\hat{j} = R\hat{n} \times ma_{Gx}\hat{\lambda} + I_G \frac{-a_{Gx}}{R}\hat{k}$$

vector aside



$\hat{n} \times \hat{j} = \sin\theta\,\hat{k}$

$\hat{n} \times \hat{\lambda} = -\hat{k}$

$$\Rightarrow \left\{ -R mg \sin\theta\,\hat{k} = -R m a_{Gx}\hat{k} - I_G \frac{a_{Gx}}{R}\hat{k} \right\}$$

$\{\ \}\cdot\hat{k} \Rightarrow$

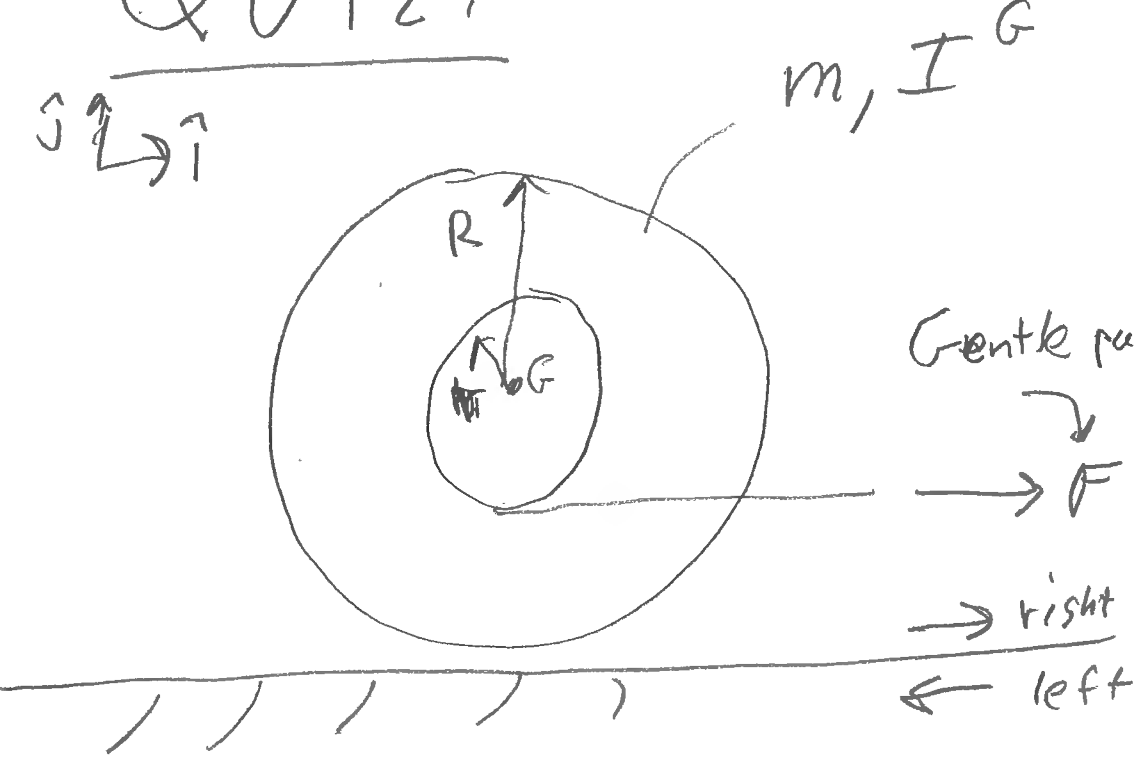
$$R mg \sin\theta = \left(Rm + \frac{I_G}{R}\right) a_{Gx}$$

$$\boxed{a_{Gx} = \frac{R mg \sin\theta}{R m + I_G/R}} \quad \boxed{A}$$

B) <u>Special case</u>! $I_G = 0 \Longleftrightarrow$ block on ramp

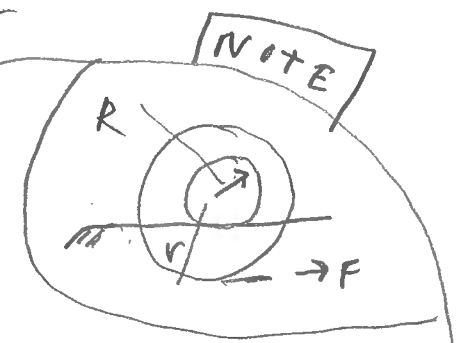
$$\boxed{a_{Gx} = g\sin\theta} > \boxed{A}$$

QUIZ!



$a_{Gx} \Rightarrow$

1) to right ✓
2) to left
3) doesn't move
5) depends on I^G & m
6) depends on r, R

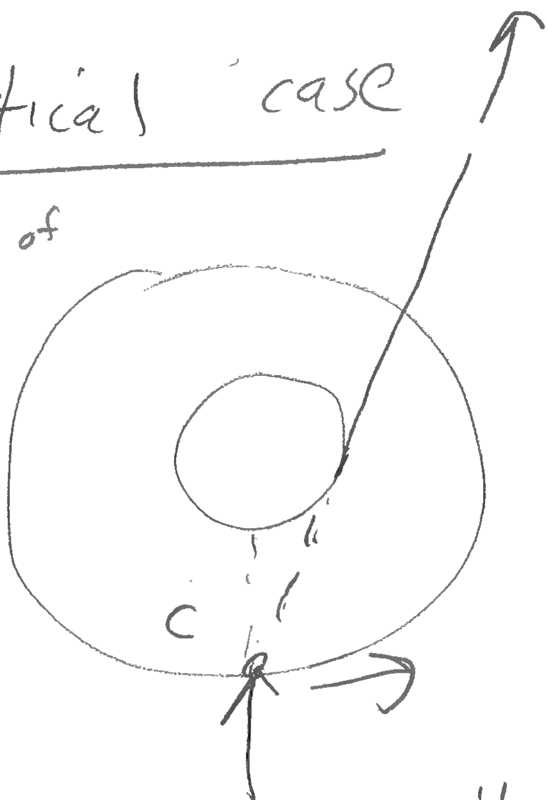


(actually, if $r > R$ dir. changes)

FBD

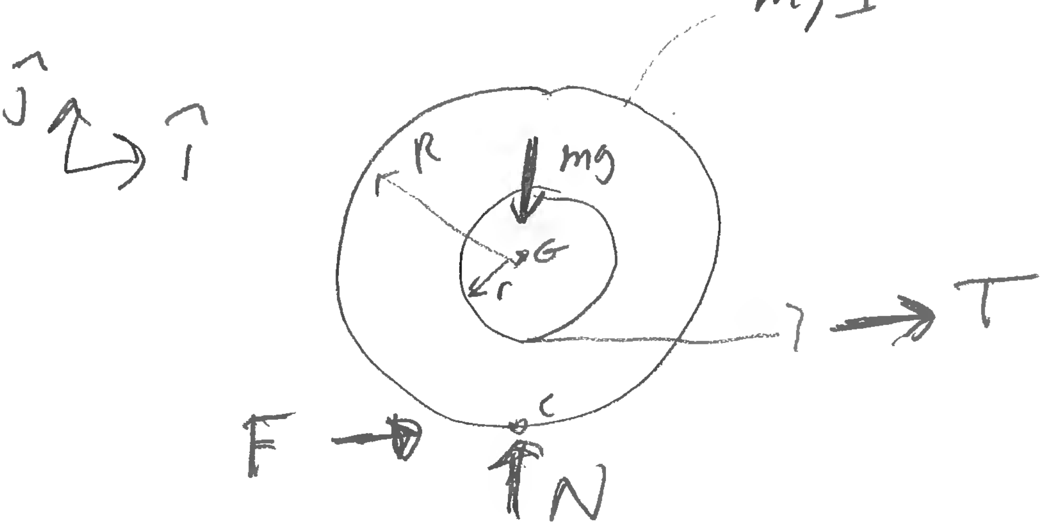
Critical case

line of action of film goes through pt C



$\sum \vec{M}_{/C} = \vec{0}$ doesn't go right or left

Level pulling



AMB/c

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

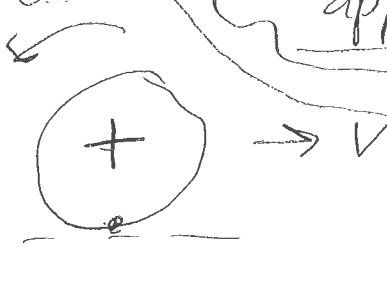
$$-(R-r)T\hat{k} = \underbrace{\vec{r}_{G/c}}_{R\hat{j}} \times m\overbrace{\vec{a}_G}^{a_{Gx}\hat{i}} + I\underbrace{\alpha}_{-\frac{a_G}{R}}\hat{k}$$

$$-(R-r)T\hat{k} = -Rma_{Gx} - I^G\frac{a_G}{R}\hat{k}$$

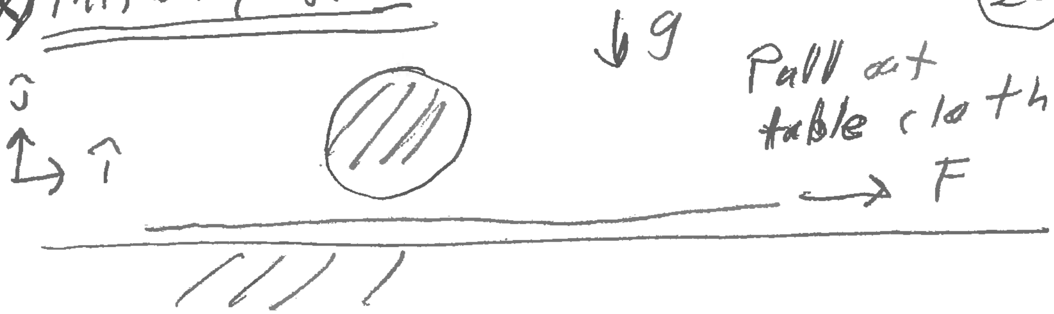
$$\boxed{a_{Gx} = \frac{(R-r)}{Rm + I^G/R}\, T}$$

moves in dir. of applied force

ASIDE



ex) Initially still



after cloth pulled out eventually rolls, which way?

$\vec{V}_G = V_G \hat{\imath}$

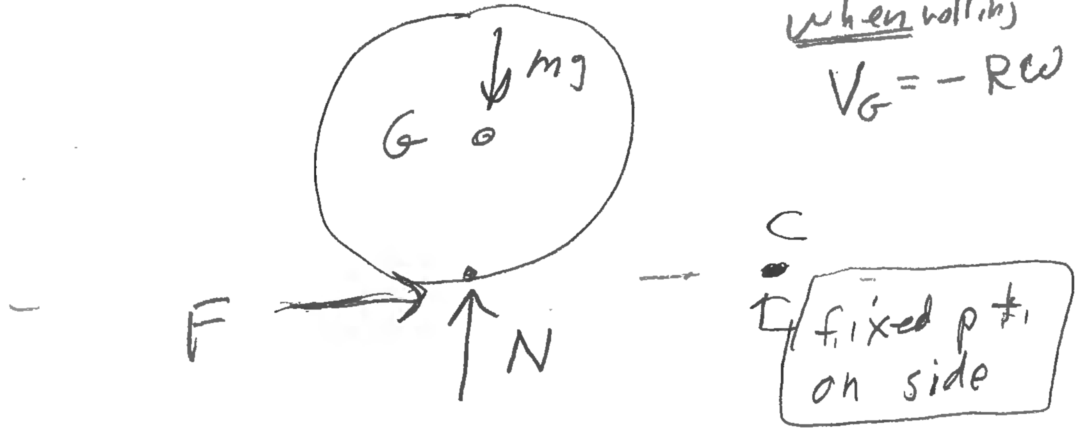
$\vec{a}_G = a_G \hat{\imath}$

kinematics

when rolling

$V_G = -R\omega$

FBD



LMB $\sum \vec{F} = m\vec{a}$

$$\{F\hat{\imath} + (N-mg)\hat{\jmath} = m a_x \hat{\imath}\}$$

$$\{\ \}\cdot\hat{\jmath} = \boxed{N = mg}$$

AMB $/_C$

↑

fixed pt. in space

$$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

$$= \vec{r}_{G/C} \times m\vec{a}_G + I^G \alpha \hat{k}$$

$$\vec{0} = R m a_{Gx} \hat{k} + I^G \alpha \hat{k}$$

Note: "$-$" missing in lecture, fixed here

$$\Rightarrow \{-R m a_{Gx} + I^G \alpha = 0\}$$

$\int dt \Longrightarrow$

$V_G(0)=0$

$\omega(0)=0$

$$\boxed{-R m V_{Gx} + I^G \omega = 0} \quad (1)$$

Rolling Const. recall

$$\boxed{R\omega + V_{Gx} = 0} \quad (2)$$

2 eqs for V_{Gx} & ω

homogeneous, nonsingular

$\Longrightarrow$ as observed

$$\boxed{V_{Gx} = \omega = 0}$$

no rolling

TODAY

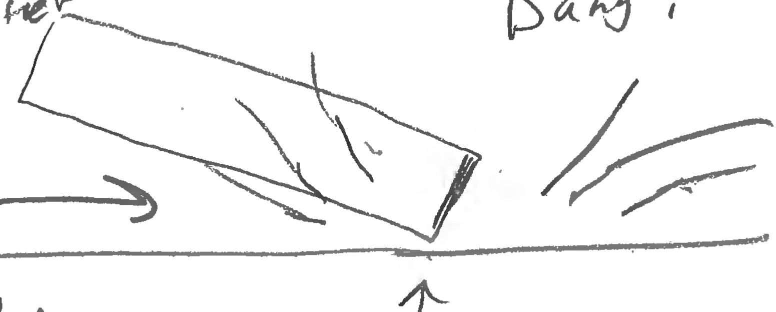
Collisions of rigid objects

t^- just before coll.

t^+ just after

Bang!

during →



Basic ideas

* Collision forces are big

  ⇒ neglect non-collision forces

  ⇒ don't show non-coll. forces on collision FBD

* Short time ⇒ config. doesn't change BUT vels. do change.

  from t^- to t^+
  $\vec{v}^+ = \vec{v}^-$
  $\theta^+ = \theta^-$

* Integrate eqs in time

  "F = ma" ⇒ "Impulse = m(Δv)"

ex) uniform stick

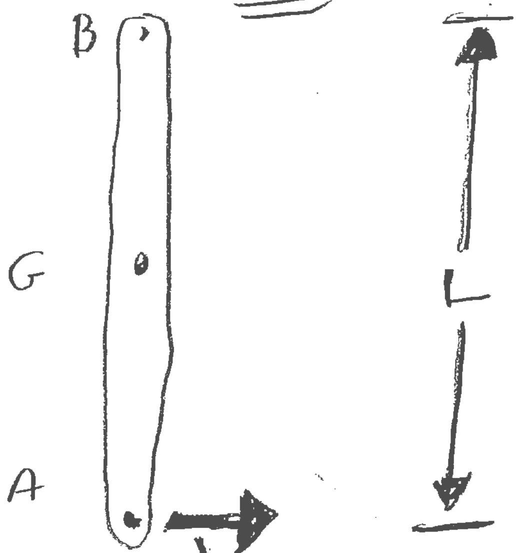
$\vec{V}_G(0) = \vec{0}$

$\dot{\theta}(0) = \vec{0}$

FBD

$m,\ I^G$

$\hat{j}$ $\hat{i}$



given

$$\underbrace{\int F\,dt}_{\text{"}I\text{", "}p\text{"}} = \text{Impulse}$$

$\vec{V}_B^{\,+} = ?$ $\omega^+ = ?$

LMB

$$\int \sum \vec{F}\,dt = m\Delta \vec{V}_G$$

$$\int F\,dt\,\hat{i} = m\Delta \vec{V}_G$$

$$= m\Big(\underbrace{\vec{V}_G(t^+)}_{\vec{V}_G^{\,+}} - \underbrace{\cancel{\vec{V}_G(t^-)}}_{\vec{V}_G^{\,-}}\Big) \quad \vec{0}$$

$$\int F\,dt\,\hat{i} = m\vec{V}_G^{\,+}$$

2 pts

$$\boxed{\vec{V}_G^{\,+} = \int F\,dt\;\hat{\imath}\;/m}$$

AMB /G

note

$$\int \sum \vec{M}_{/G}\,dt = \Delta \vec{H}_{/G}$$

$$\vec{r}_{A/G} \times \left(\int F\,dt\;\hat{\imath}\right) = \vec{H}_G^{\,+} - \cancel{\vec{H}_G^{\,-}}$$

$\vec{0}$

$$\vec{r}_{A/G} = -\frac{L}{2}\hat{\jmath}$$

$$\frac{L}{2}\int F\,dt\;\hat{K} = \dot{\theta}^{+} I^{G}\,\hat{K}$$

$$\boxed{\dot{\theta}^{+} = \frac{\left(\int F\,dt\right)\frac{L}{2}}{I^{G}}}$$

$$\vec{V}_B^{\,+} = V_B^{+}\;\hat{\imath}$$

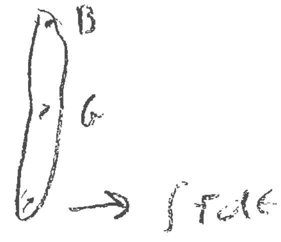
$V_B^{+} > 0$ (a)

< 0 ✓ (b)

$= 0$ (c)

R09

$$\vec V_B^{+} = \vec V_G^{+} + \vec V_{B/G}^{+}$$



$$= \frac{\int F dt}{m}\,\hat\imath$$

$$\frac{L}{2}\,\hat\jmath$$

$$+ \ \vec\omega^{+} \times \vec r_{B/G}$$

$$L \quad \dot\theta^{+}\hat k$$

$$\frac{\int F dt}{I^G}$$

$$\dot{\hat e}_r = \vec\omega \times \hat e_r$$

$$\dot{\hat\imath}' = \vec\omega \times \hat\imath'$$

$$\dot{\vec Q} = \vec\omega \times \vec Q$$

$$=$$

$$= \int F dt \left[\frac{1}{m} - \frac{L^2/4}{I^G} \right] \hat\imath$$

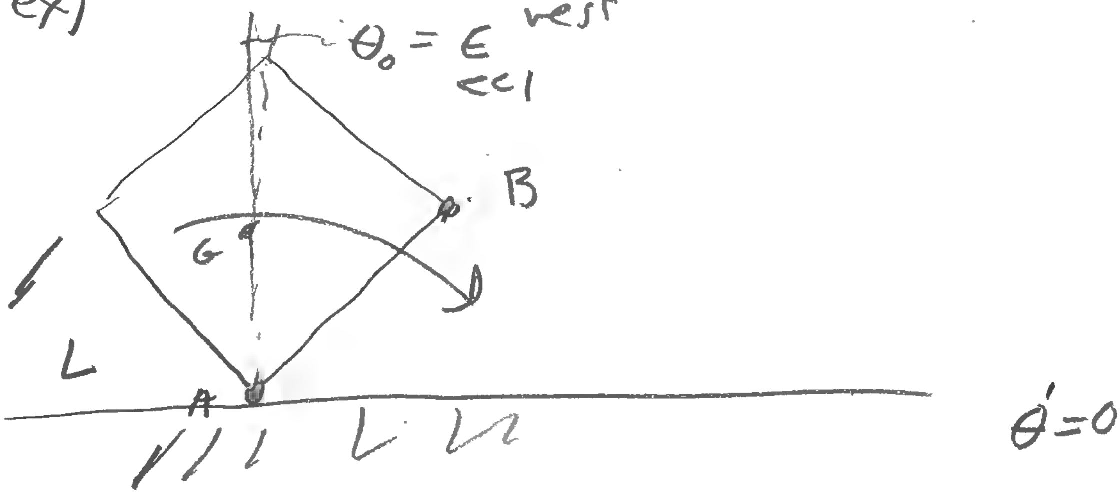
always: $I^G \leq m L^2/4$

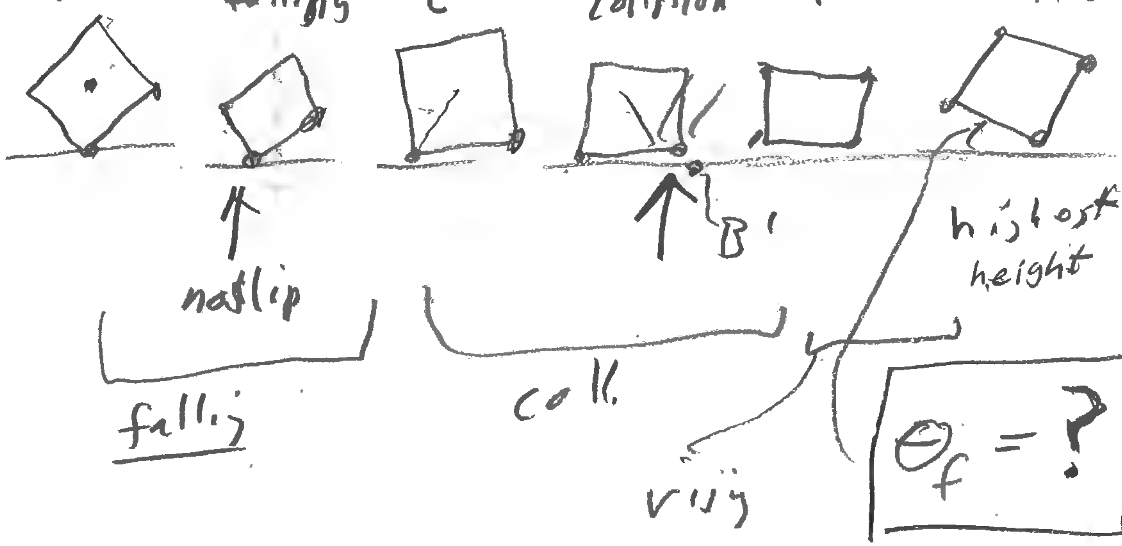
ex) $I^G = mL^2/4 \Rightarrow V_B^{+} = 0$

most extreme

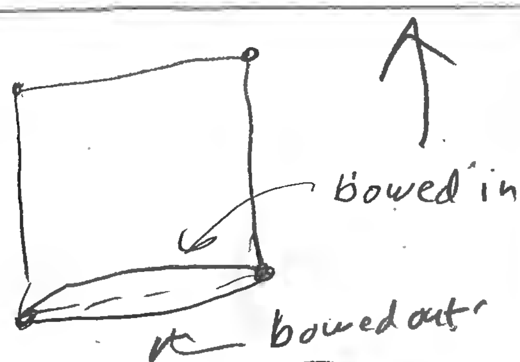
$$\boxed{V_B^{+} \leq 0}$$

ex)





$$\boxed{\theta_f = ?}$$



very different!

Solve 3 sub-problems

1) Falling ; energy cons.

$\Rightarrow \dot{\theta}^-$

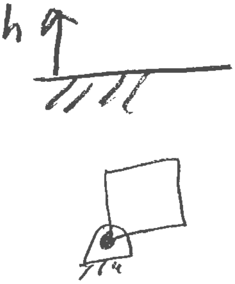
2) Collision ;

$\underbrace{AMB_{/B}}_{\text{no net moment impulses}} \Rightarrow \dot{\theta}^+$

3) rising ; energy cons. again

Steps, to solve

1) Falling ; En. Cons.



$$E^0_{tot} = E^{t^-}_{tot}$$

$$\cancel{E_K^0} + E_p^0 = E_K^- + E_p^-$$

E_p^0: $\frac{L}{2}\sqrt{2}\, mg$; E_p^-: $\frac{L}{2} mg$

$$\underbrace{\frac{L}{2}\sqrt{2}\, mg = \frac{1}{2} m |\vec{V}_G^-|^2 + I^G \dot{\theta}^{-2}/2}_{}$$

1 eqn for $\dot{\theta}^-$

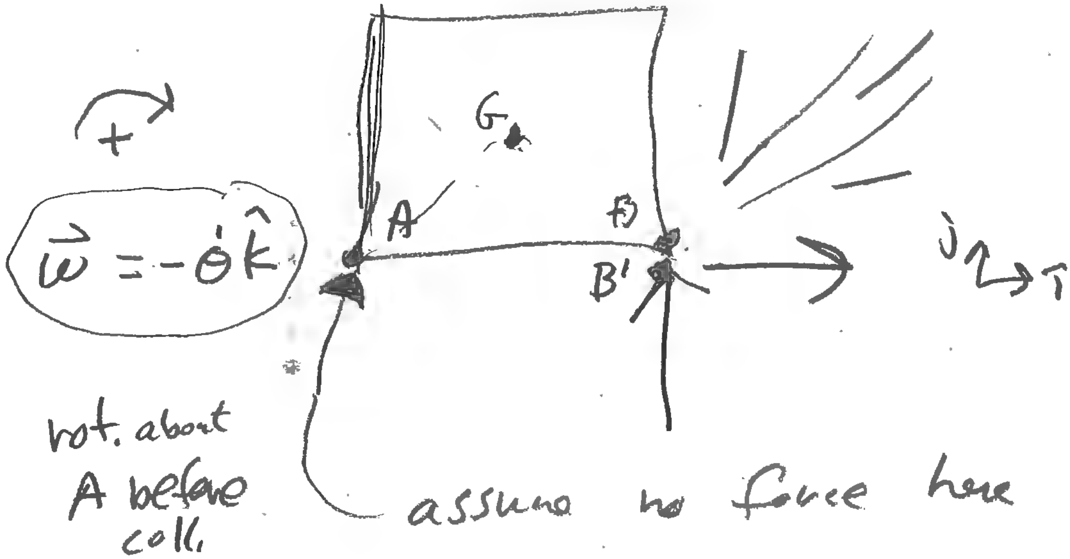
$|V_G^-| = \dot{\theta}\sqrt{2}\frac{L}{2}$

$\dot{\theta}^-$ known ✓ (208)

$\dot{\theta}^- = \sqrt{\text{mess}}$

2) Collision

Collision FBD



AMB$/_{B'}$

$$\int \underbrace{\cancel{\sum \vec{M}_{/B'} dt}}_{\vec{0}} = \Delta \vec{H}_{B'}$$

$$= \vec{H}^+_{/B'} - \vec{H}^-_{/B'}$$

$$\vec{H}^-_{/B} = \overbrace{\vec{r}^-_{G/B'} \times \vec{V}^-_G m}^{=\vec{0}} + I^G \overbrace{\omega^-}^{-\dot{\theta}^-} \hat{k}$$

$\vec{r}_{G/B'} = \frac{L}{2}(-\hat{i}+\hat{j})$

$\vec{V}_G^- = \vec{\omega}^- \times \vec{r}_{G/A}$, with $\vec{\omega}^- = -\dot{\theta}^-\hat{k}$

$\vec{r}_{G/A} = \frac{L}{2}(\hat{i}+\hat{j})$

$$\vec{H}_B^{+} = \underbrace{\vec{r}_{G/B}}_{\frac{L}{2}(-\hat{i}+\hat{j})} \times \underbrace{\vec{v}_G^{+}}_{\underbrace{\vec{\omega}^{+}}_{-\dot{\theta}^{+}} \times \underbrace{\vec{r}_{G/B}}_{\frac{L}{2}(-\hat{i}+\hat{j})}} m + I^G \underbrace{\omega^{+}}_{-\dot{\theta}^{+}} \hat{k}$$

$$\vec{H}_O^{-} = \vec{H}_B^{+} \Rightarrow \boxed{\dot{\theta}^{+}}$$

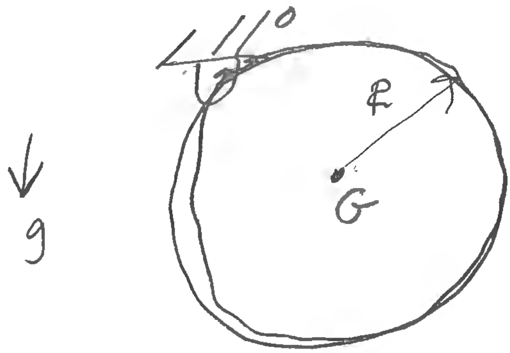
3) En. Cons. (again)

$$\Rightarrow \quad \theta^{final} \simeq 1.2^{\circ}$$

Today

pos, vel, accel 3 ways
cartesian, polar, path, rotating frames.

First: Prelim hoop

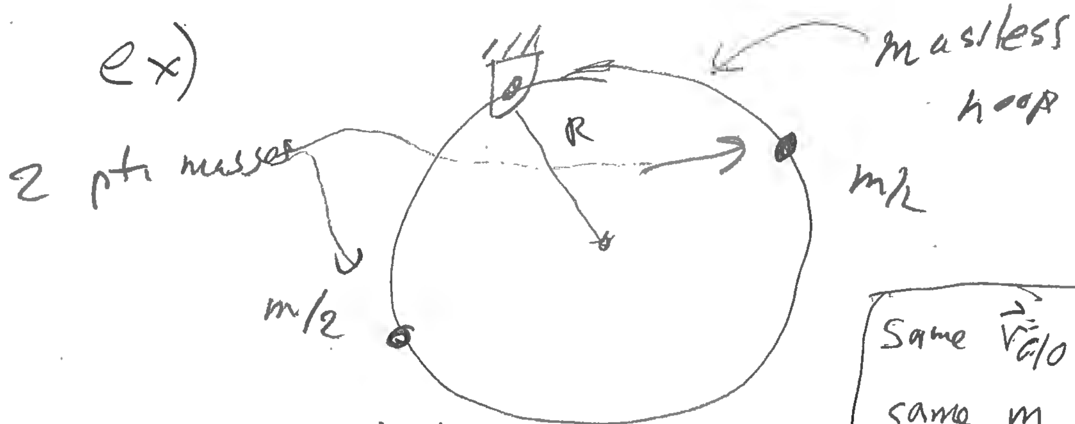


FBD

Dynamics fully determined by these things

$$|\vec{r}_{G/O}| = R$$
$$\text{mass} = m$$
$$I^G = mR^2$$

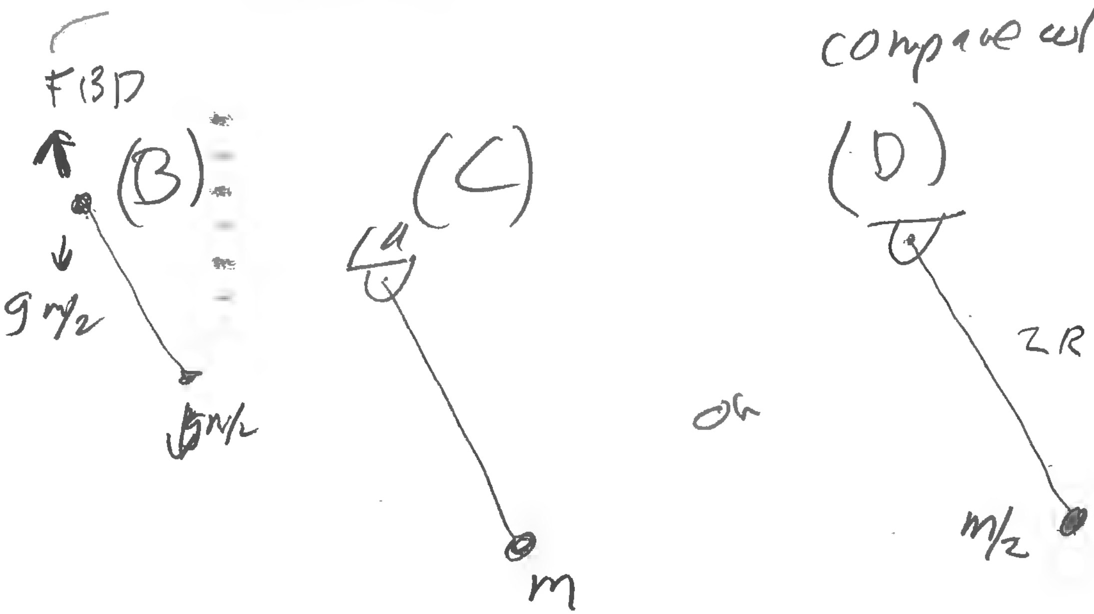
ex)



identical to hoop

same $\vec{r}_{G/O}$
same m
same I^G

ex



All 4 have same motion ← EoM

A & B have same reaction forces

end prelim example

Position Vel. & Accel.

* Cartesian (a) Lined up & b) crooked

* Polar

* Path

* Moving / Rotating Coords

You need to study and learn this

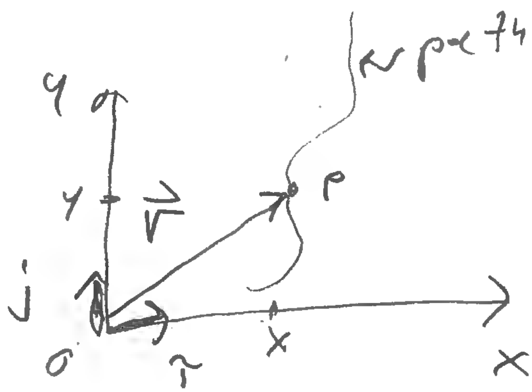
Recall: $\boxed{\vec{r} = \vec{r},\quad \vec{V} = \vec{v},\quad \vec{a} = \vec{a}}$

Cartesian

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

$$\vec{V} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

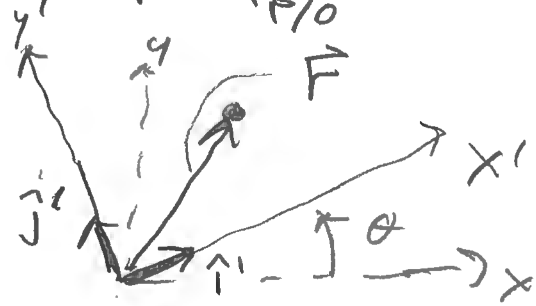
$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$



$\vec{r} = \vec{r}_{P/O}$

Crooked Cartesian

↳ but fixed



$$\vec{r} = x'\hat{\imath}' + y'\hat{\jmath}'$$

$$\vec{V} = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'$$

$$\vec{a} = \ddot{x}'\hat{\imath}' + \ddot{y}'\hat{\jmath}'$$

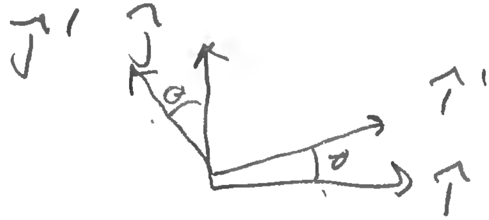
$\theta = \text{const}$

Recall

$$\boxed{\vec{r} = \vec{r}}$$

$$\{ x\hat{i} + y\hat{j} = x'\hat{i}' + y'\hat{j}' \}$$

$$\{\}\cdot\hat{i} \Rightarrow \quad x = x'\overbrace{\hat{i}'\cdot\hat{i}}^{\cos\theta} + y'\overbrace{\hat{j}'\cdot\hat{i}}^{-\sin\theta}$$



$$x = x'\cos\theta - y'\sin\theta$$

do again w/ $\hat{j}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

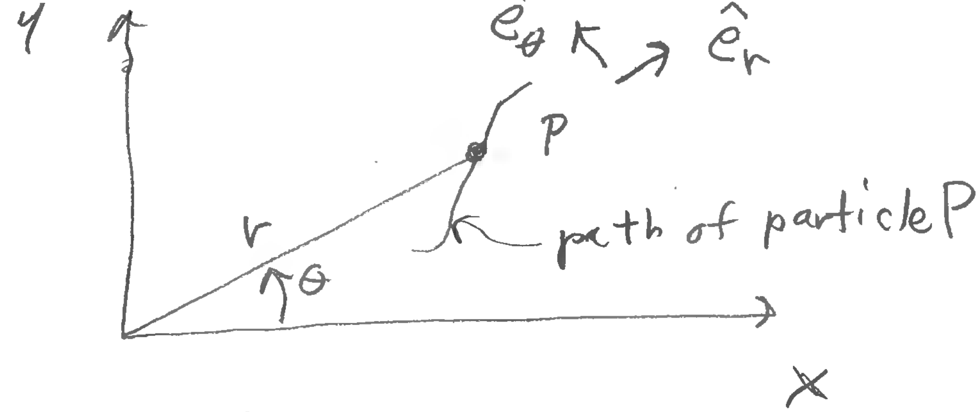
---

do again w/ $\hat{i}'$ & $\hat{j}'$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Polar Coord

a coord syst, based on P



$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

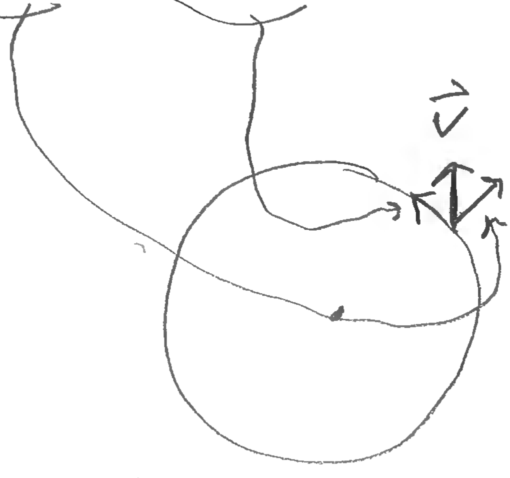
$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \frac{d}{dt}\vec{r}$$

$$= \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$$

$$\boxed{\dot{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta}$$

$$= \dot{r}\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$



$$\vec{a} = \frac{d}{dt}(\vec{v})$$

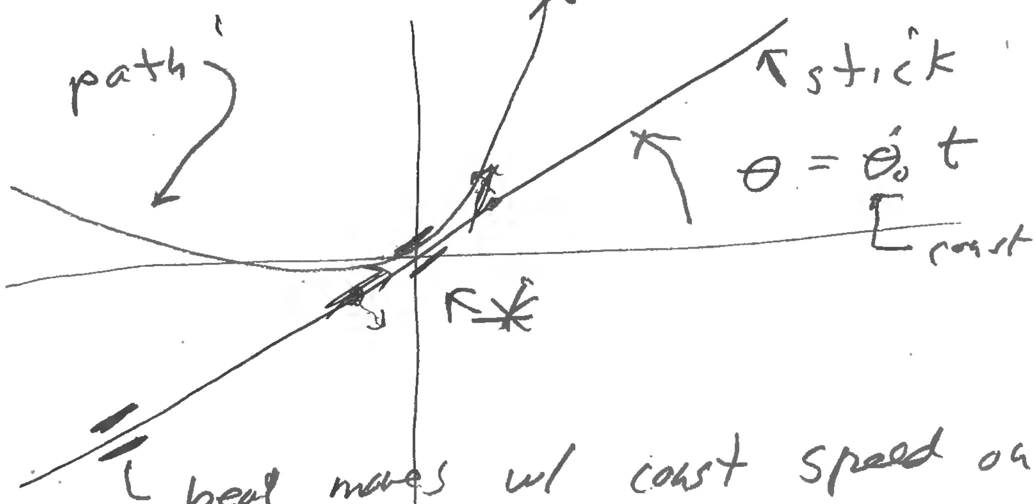
$$= \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right)$$

$$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$$

$$= \left(\ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r\right) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$$

$$\boxed{\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + \underbrace{2\dot{r}\dot{\theta}}_{\text{coriolis}}\right)\hat{e}_\theta}$$



L bead moves w/ const speed on stick $\Rightarrow$ const $\dot{r}$

stick ex) $\ddot{r} = 0$

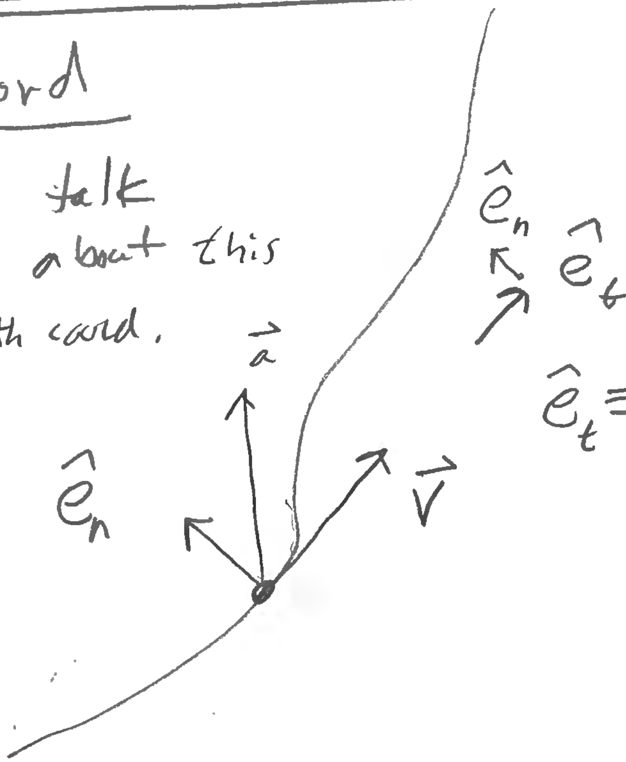
$\ddot{\theta} = 0$

part. at origin $\Rightarrow$ $r = 0$

$$\Rightarrow \boxed{\vec{a} = 2\dot{r}\dot{\theta}\hat{e}_\theta}$$

Path coord

$\vec{r} =$ don't talk about this w/ path coord.



$$\hat{e}_t \equiv \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{\vec{v} = v\hat{e}_t}$$

$$\vec{a} = \dot{v}\hat{e}_t + v\dot{\hat{e}}_t$$

$$\boxed{\vec{a} = \dot{v}\hat{e}_t + v\,(v\kappa)\,\hat{e}_n}$$

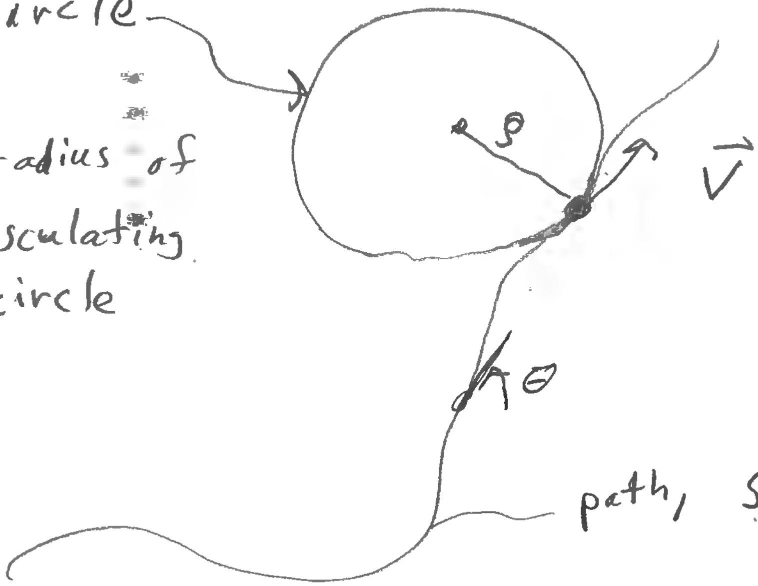
$$\vec{a} = \dot{v}\hat{e}_t + \kappa v^2 \hat{e}_n$$

$$\kappa = 1/\rho$$

ρ = radius of curvature

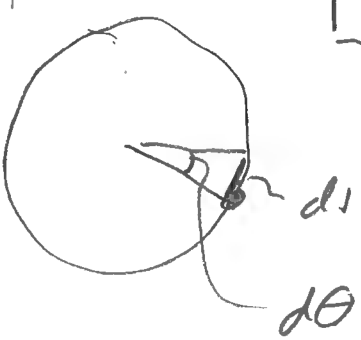
osculating ≡ kissing circle

ρ = radius of osculating circle



path, s = arc length = dist along path

$$\boxed{\frac{d\theta}{ds} = \frac{1}{\rho}}$$



$$ds = \rho\, d\theta$$

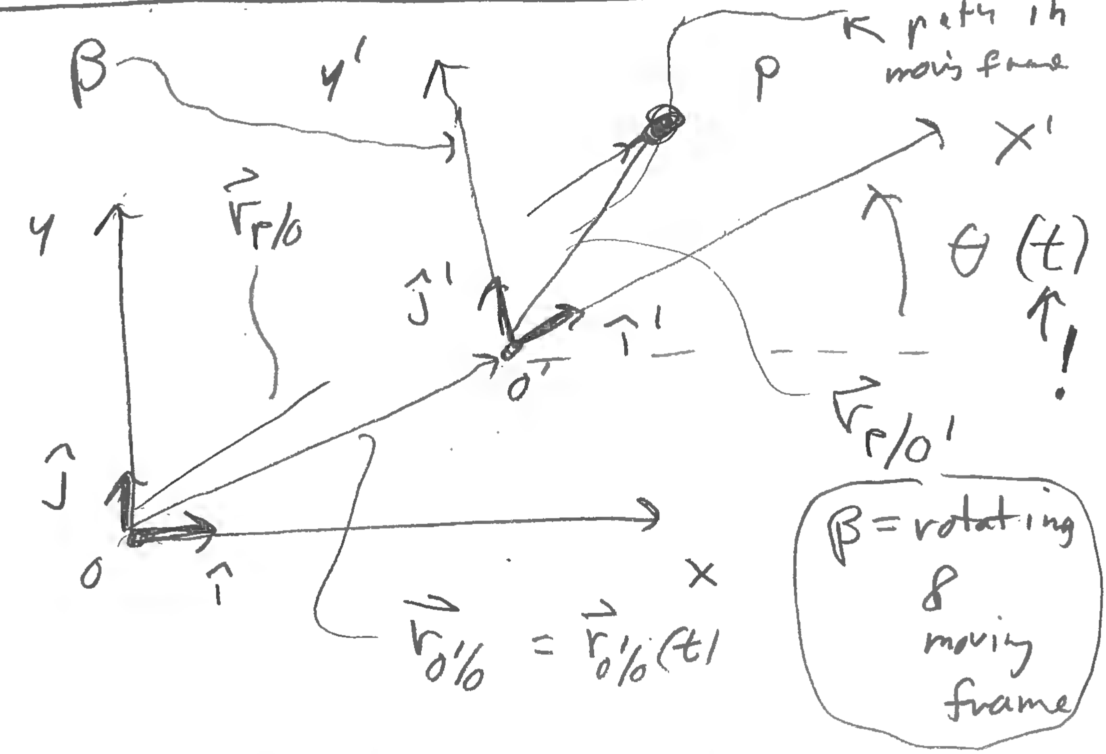
$$\boxed{\frac{d\theta}{ds} = 1/\rho} = \kappa$$

$\Rightarrow$ any part. in any motion has at any instant

$$\vec{a} = \underbrace{\text{speeding up}}_{\dot{v}\hat{e}_t} + \underbrace{\text{centrip. term}}_{\frac{v^2}{\rho}\hat{e}_n}$$

See MATH 1920 textbook

Rotating & Moving Frames



Position

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

$\vec{r}_{O'/O}$: $(x_0 \hat{i} + y_0 \hat{j})$

$\vec{r}_{P/O'}$: $(x'_{P/O'} \hat{i}' + y'_{P/O'} \hat{j}')$

$$\vec{V} = \vec{V}_{O'/O} + {}^{F}\frac{d}{dt}\left(\vec{r}_{P/O'}\right)$$

${}^{F}\frac{d}{dt}$ = deriv. in fixed frame

$${}^{F}\frac{d}{dt}\left(\vec{r}_{P/O'}\right) = \frac{d}{dt}\left[x'_{P/O'} \hat{i}' + y'_{P/O'} \hat{j}' \right]$$

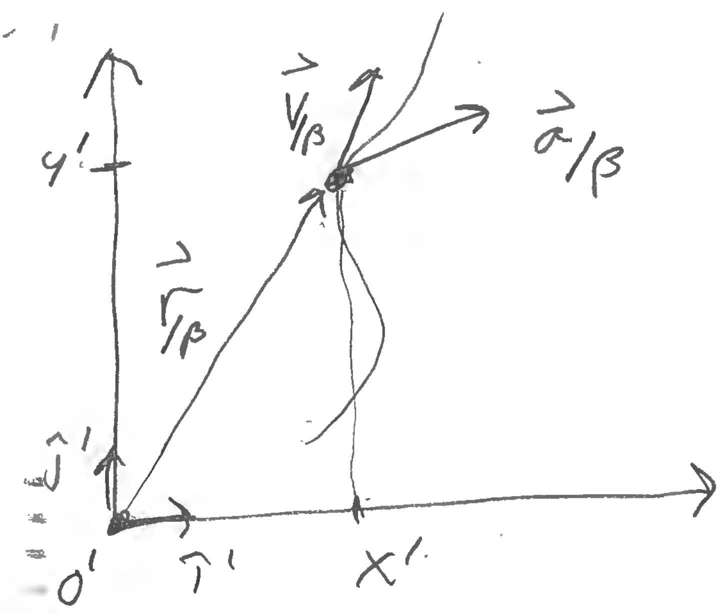
$$= \left(\dot{x}'_{P/O'} \hat{i}' + \dot{y}'_{P/O'} \hat{j}'\right)$$

$$+ \underbrace{\left(x'_{P/O'} \dot{\hat{i}}' + y'_{P/O'} \dot{\hat{j}}'\right)}$$

$$= \underbrace{{}^{B}\frac{d}{dt}\vec{r}_{P/O'}} + (\;\cdot\;)$$

vel. observed by crazy frame

This paper is $\mathcal{B}$



Imagine this paper moving and rotating

$$
{}^{\beta}\frac{d\vec{r}_{P/O'}}{dt} = \dot{x}'_{P/O'}\,\hat{\imath}' + \dot{y}'_{P/O'}\,\hat{\jmath}'
$$

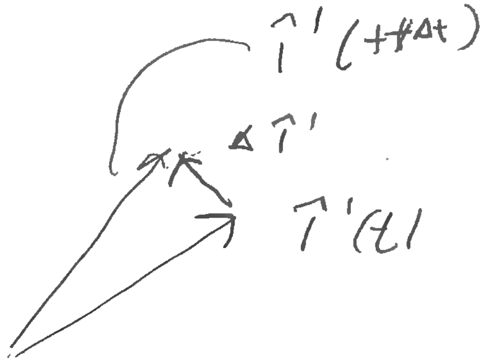
$$
= \vec{V}_{P/\beta} \neq \vec{V}_{P/\mathcal{F}}
$$

because paper is moving and rotating rel. to $\mathcal{F}$ ← fixed frame

Aside

$$\dot{\hat{\imath}}' = \dot{\theta}\,\hat{\jmath}' = \dot{\theta}\,\hat{k} \times \hat{\imath}'$$

$$\dot{\hat{\jmath}}' = -\dot{\theta}\,\hat{\imath}' = \dot{\theta}\,\hat{k} \times \hat{\jmath}'$$



$$\boxed{\dot{\hat{\imath}}' = \vec{\omega} \times \hat{\imath}' \qquad \vec{\omega} = \dot{\theta}\,\hat{k} \qquad \dot{\hat{\jmath}}' = \vec{\omega} \times \hat{\jmath}'}$$

$$\Rightarrow \quad x'_{P/O'}\,\dot{\hat{\imath}}' + y'_{P/O'}\,\dot{\hat{\jmath}}'$$

$$= \vec{\omega} \times \vec{r}_{P/O'}$$

$$\boxed{\vec{V}_{P/O} = \vec{V}_{O'/O} + \underbrace{\frac{{}^{B}d\,\vec{r}_{P/O'}}{dt}}_{\text{"}\vec{V}_{rel}\text{"} = \vec{V}_{P/B}} + \vec{\omega} \times \vec{r}_{P/O'}}$$

Now, acceleration

$$\vec{a}_{P/O} = \frac{d}{dt}\left(\vec{V}_{P/O}\right)$$

$$= \frac{d}{dt}\Bigg[\vec{V}_{O'/O} + \underbrace{\frac{{}^{B}d\vec{r}_{P/O'}}{dt}}_{\vec{V}_{P/B}} + \vec{\omega} \times \vec{r}_{P/O'}\Bigg] \quad (1)$$

For any vector $\vec{Q}$

"Q-dot formula"

$$\boxed{\frac{{}^{F}d\vec{Q}}{dt} = \frac{{}^{B}d\vec{Q}}{dt} + \vec{\omega} \times \vec{Q}}$$

deriv. just like deriv. of $\vec{V}_{P/O'}$

See textbook

1 ⇒

$$\vec{a}_{P/O'} = \overset{1}{\vec{a}_{O'/O}}$$

$$+ \left(\overset{2}{\vec{a}_{P/\beta}} + \overset{3}{\vec{\omega} \times \vec{V}_{P/\beta}}\right)$$

$$\underset{4}{\dot{\vec{\omega}} \times \vec{r}_{P/O'}} + \vec{\omega} \times \dot{\vec{r}}_{P/O'}$$

$$\Big[\overset{5}{{}^{\beta}\dot{\vec{r}}_{P/O'}} + \underset{6}{\vec{\omega} \times \vec{r}_{P/O'}}\Big]$$

2 of ~~three~~ 6 terms are the same

Note!

$$\vec{a}_{P/\beta} = \frac{{}^{\beta}d\left(\vec{V}_{P/\beta}\right)}{dt}$$

$$\vec{V}_{P/\beta} = \frac{{}^{\beta}d\,\vec{r}_{P/O'}}{dt}$$

⇒

$$\boxed{\vec{a}_{P/O} = \vec{a}_{O'/O} + \vec{a}_{P/\beta} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'}) + \dot{\vec{\omega}} \times \vec{r}_{P/O'} + \underbrace{2\vec{\omega} \times \vec{V}_{P/\beta}}_{\text{coliolis}}}$$

5 term accel. formula

$$\vec{a}_{P/O} = \vec{a}_{O'/O}$$ ← accel. of ref. pt. on moving frame

$$+ \vec{a}_{P/B}$$ ← acc. as observed by friend who moves on moving frame $\equiv \ddot{x}'_{P/O'}\hat{\imath}' + \ddot{y}'_{P/O}\hat{\jmath}'$

$$+ \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'})$$ ← centripetal term $= -\omega^2 \vec{r}_{P/O'}$ (in 2D)

$$+ \dot{\vec{\omega}} \times \vec{r}_{P/O'}$$ ← accel. due to frame rot. rate changing

$$+ 2\vec{\omega} \times \vec{v}_{P/B}$$ ← coriolis term (takes thought)

$$\vec{v}_{P/B} \equiv \dot{x}'_{P/O'}\hat{\imath}' + \dot{y}'_{P/O'}\hat{\jmath}'$$

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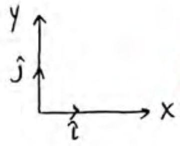
4/22/19

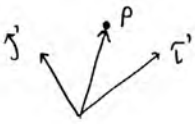
TODAY: RELATIVE MOTION, ETC.

- Fixed Frame



$\boxed{V_{P/F} = \dot{\vec{r}}_{P/O}}$

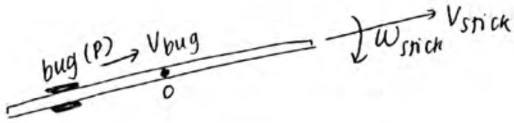
Body Frame:



$\vec{V}_{P/B} = \frac{d}{dt}\vec{r}_{P/O'}$

$\boxed{\vec{V}_{P/B} = \dot{x}'\hat{i}' + \dot{y}'\hat{j}'}$

- 3 term velocity formula:



$$\boxed{\vec{V}_{P/F} = \vec{V}_{O'/O} + \vec{V}_{P/B} + \vec{\omega}\times\vec{r}_{P/O'}}$$

$\vec{V}_{P/B} = \dot{x}'_p\hat{i}' + \dot{y}'_p\hat{j}'$ $\quad \vec{\omega} = \vec{\omega}_{P/F}$

dependent on time

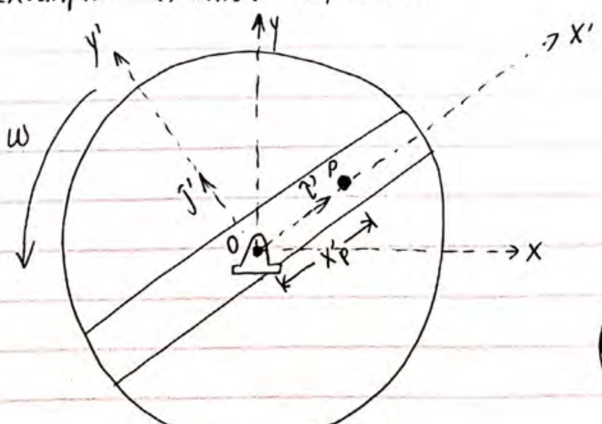
$$\vec{V}_{P/F} = \frac{d}{dt}\left[x_{O'/O}\hat{i} + y_{O'/O}\hat{j} + x'_{P/O'}\hat{i}' + y'_{P/O'}\hat{j}'\right]$$

- 5 term accel. formula:

$$\boxed{\vec{a}_{P/F} = \vec{a}_{O'/F} + \vec{a}_{P/B} - \omega^2\vec{r}_{P/O'} + \dot{\vec{\omega}}\times\vec{r}_{P/O'} + 2\vec{\omega}\times\vec{V}_{P/B}}$$

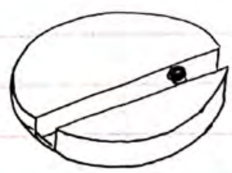
($-\omega^2\vec{r}_{P/O'}$ comes from $\vec{\omega}\times(\vec{\omega}\times\vec{r}_{P/O'})$)

- Example: (turntable w/ a slot)

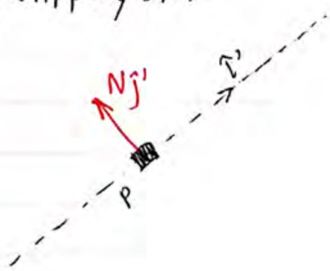


(GIVEN)

P has mass m.; slippery slot



(FBD)



$x' = x'_p$ in this problem, O' and O are overlapping

(LMB) $\Sigma \vec{F} = m\vec{a}$ ($\vec{a} \to \vec{a}_{P/F}$)

① $N\hat{j}' = m\left[\vec{a}_{O'/O} + \vec{a}_{P/B} - \omega^2\vec{r}_{P/O'} + \dot{\vec{\omega}}\times\vec{r}_{P/O'} + 2\vec{\omega}\times\vec{V}_{P/B}\right]$

$\vec{a}_{O'/O} = \vec{0}$

$\dot{\vec{\omega}} = 0$ b/c ω is const.

$\vec{r}_{P/O'} = x'\hat{i}'$, $\vec{V}_{P/O'} = \dot{x}'\hat{j}'$, $\vec{a}_{P/O'} = \ddot{x}'\hat{j}'$

$\vec{a}_P = \vec{0} + \ddot{x}'\hat{i}' - \omega^2 x'\hat{i}' + \vec{0} + 2\omega\hat{k}\times\dot{x}'\hat{i}'$ ⊛

> Plug ⊛ into ①: & dot w/ $\hat{i}'$

$$0 = m[\ddot{x}' - \omega^2 x'_p] \Rightarrow \boxed{\ddot{x}' = \omega^2 x'}$$

> Guess $x = Ce^{\lambda t}$: $C\lambda^2 e^{\lambda t} = C\omega^2 e^{\lambda t}$

$\lambda = \omega$ (good guess) , $\lambda = -\omega$

$$\Downarrow$$

$$x = C_1 e^{\omega t} + C_2 e^{-\omega t}$$

> Find C_1 & C_2 from IC's

> Solve DEQ on MATLAB

Wed. April 25, 2019

[The missing 2 pages by (great) student are posted online]

TODAY

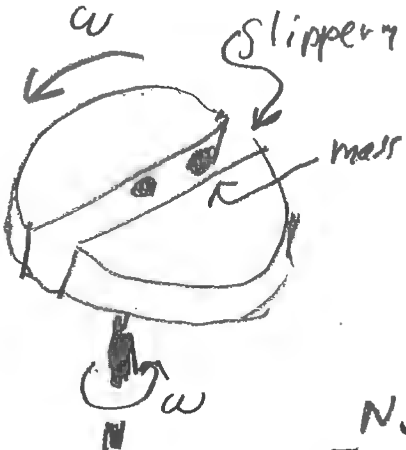
1) bead in slot (cont'd)

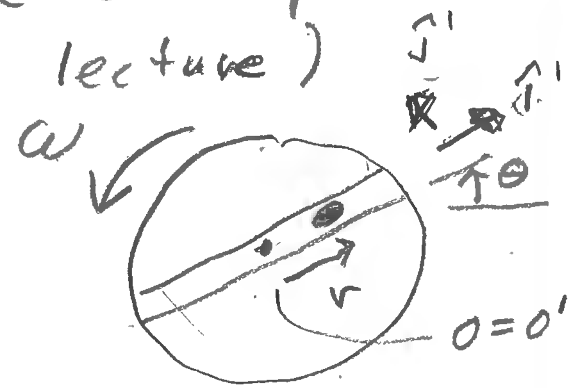
2) ~~bead on wire~~ Vibrating base pendulum

(~~slider crank mechanism~~)

NO MORE NEW TOOLS!

Bead in slot (see also prev. lecture)





ω = const.

FBD

To find $\vec{a}$, see prev. lecture

LMB, $\sum \vec{F} = m\vec{a}$

$$N\hat{\jmath}' = (\ddot{r} - r\omega^2)\hat{\imath}' + 2\dot{r}\omega\hat{\jmath}'$$

$\{\,\}\cdot\hat{\imath}' \Rightarrow$ $\boxed{\ddot{r} = \omega^2 r}$ $\Rightarrow r \sim e^{\omega t}$

Look at path

chain rule, twice

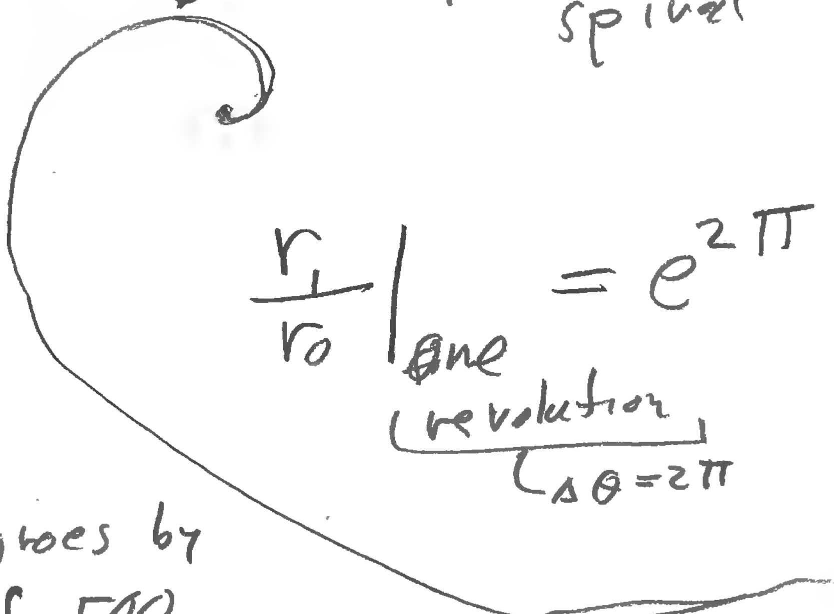
$$\frac{d^2 r}{d\theta^2} = \frac{d^2 r}{dt^2}\left(\frac{dt}{d\theta}\right)^2 = \frac{1}{\omega^2}\frac{d^2 r}{dt^2}$$

no. coeff.

$$\Rightarrow \boxed{\frac{d^2 r}{d\theta^2} = r}$$

$$\Rightarrow r = \underbrace{c_1 e^{\theta}}_{\text{dominates}} + c_2 e^{-\theta}$$

exponential spiral



$$\left.\frac{r_1}{r_0}\right|_{\substack{\text{one}\\ \text{revolution}\\ \Delta\theta = 2\pi}} = e^{2\pi}$$

radius groes by factor of 500 each revolution

≈ 500

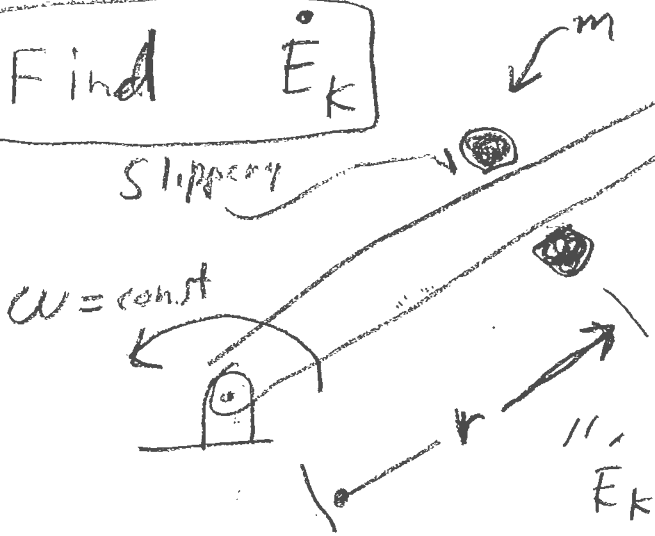
(actually 535....)

QUIZ

Given $\omega = \text{const}$, r, $\dot{r}$, m

$\boxed{\text{Find } \dot{E}_K}$

No gravity



Student guesses

"$\dot{E}_K = m r \omega^2 \dot{r}$" X

$E_K = \frac{1}{2} m r^2 \omega^2$ X

$I^O \omega^2/2$ X

"$\dot{E}_K = 2 m \omega \dot{r}^2$
$= P$
$= \vec{F} \cdot \vec{V}$
$= \dots$" X

a) ? got it

b) ? maybe

c) ? not yet

d) ? having trouble

e) ? No way

$\vec{\text{vel.}} = \vec{V} = \dot{r}\,\hat{\imath}' + \omega r\,\hat{\jmath}'$

$$\vec{V} = \dot{r}\,\hat{\imath}' + r\omega\,\hat{\jmath}'$$

$$\vec{F} = m\vec{a}$$

$$\Rightarrow \left\{ N\hat{\jmath}' = \left((\ddot{r} - r\omega^2)\,\hat{\imath}' + 2\dot{r}\omega\,\hat{\jmath}'\right) m \right\}$$

$$\{\ \}\cdot\hat{\jmath}' = \qquad N = 2\dot{r}\omega m$$

$$\vec{F} = 2\dot{r}\omega\,\hat{\jmath}'\, m$$

$$\dot{E}_K = \vec{F}\cdot\vec{V}$$

$$= (2\dot{r}\omega\,\hat{\jmath}')\cdot(\dot{r}\,\hat{\imath}' + r\omega\,\hat{\jmath}')$$

$$\boxed{\dot{E}_K = 2\dot{r}r\omega^2 m}$$

Alternative: way to find $\dot{E}_K$

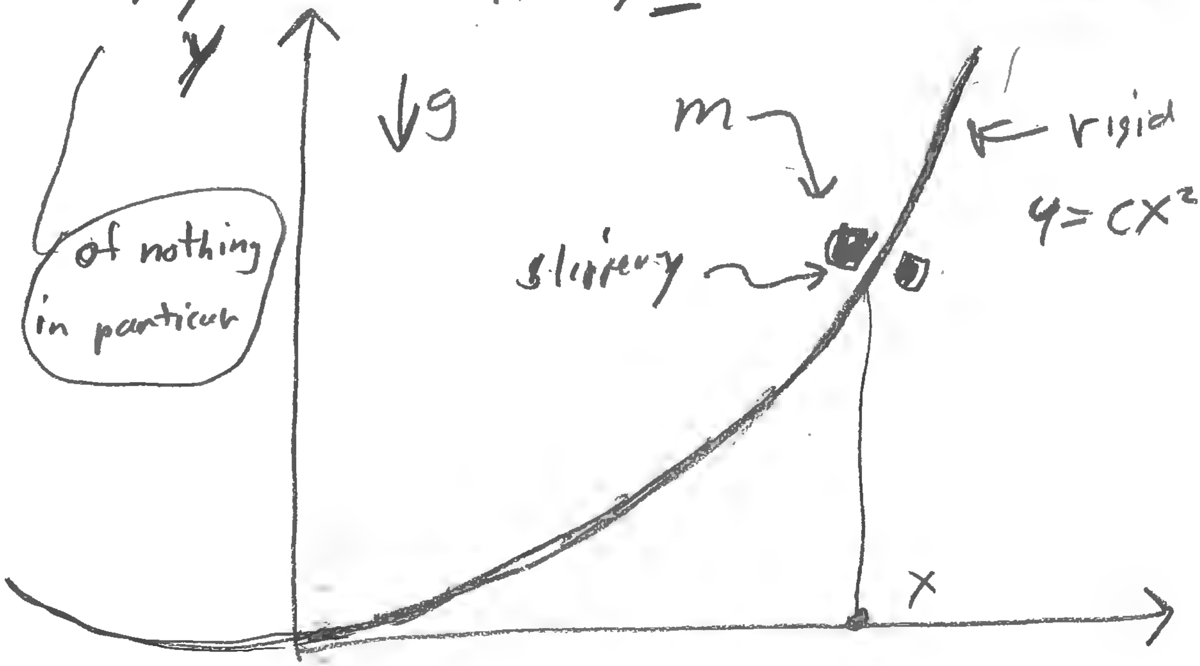
$$E_K = \frac{1}{2} m\,\vec{V}\cdot\vec{V}$$

$$= \frac{1}{2} m\left(\dot{r}^2 + r^2\omega^2\right)$$

then differentiate and use

$$\ddot{r} = 2\omega\dot{r}$$

ex) Bead on wire
slippery, in gravity



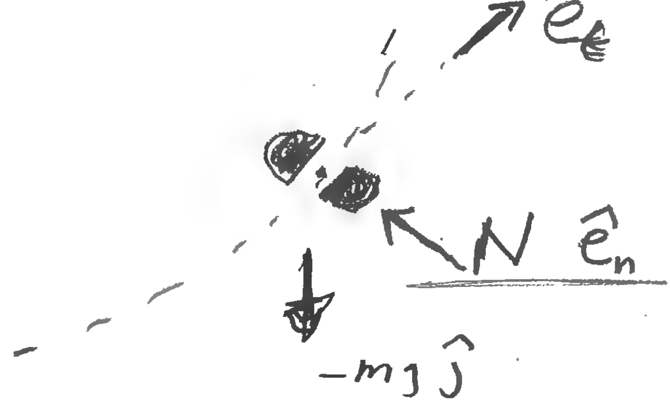
1 DoF want 1 EoM $\ddot{x}$

Given State: $x, \dot{x} \Rightarrow y, \dot{y}$

Parameters : m, c, g

Find $\ddot{x}, \ddot{y}$, Reaction Force

FBD!



LMB:

$$\Sigma \vec{F} = m\vec{a}$$

$$\underbrace{\Lambda}_{-mg\hat{J}} \; N\hat{e}_n = m\left[\ddot{x}\hat{\imath} + \ddot{y}\hat{J}\right] \quad *$$

$\hat{n}$

Want to get rid of $y, \dot{y}, \ddot{y}$

Kinematics & geometry

curve $\Rightarrow y = cx^2$

$$\Rightarrow \dot{y} = 2cx\dot{x}$$

$$\Rightarrow \ddot{y} = 2c\left[\dot{x}^2 + x\ddot{x}\right] \quad \ddot{y}$$

$$\vec{a} = \ddot{x}\hat{\imath} + 2c(\dot{x}^2 + x\ddot{x})\hat{J}$$

find this

$$\vec{V} = \dot{x}\hat{\imath} + 2cx\dot{x}\hat{J}$$

$$\hat{e}_t = \vec{V}/|\vec{V}|, \quad \hat{e}_n = \hat{k} \times \hat{e}_t$$

Back to LMB $\quad \checkmark$ = known

$$\left\{ -mg\hat{J} + N\hat{e}_n = m\left[\ddot{x}\hat{\imath} + 2c(\dot{x}^2 + x\ddot{x})\hat{J}\right] \right\}$$

unknown

2 eqs in 2 unknowns

1 2D vec. eqn.

$$\{\quad\}\cdot\hat{e}_t \quad \text{or} \quad \{\quad\}\cdot\vec{v}$$

$\vec{v}$ is $\perp$ to $\hat{e}_n$

$\hat{e}_n\cdot\hat{e}_t=0$

$\hat{e}_t=\vec{v}/|\vec{v}|$

$$\left\{-\cancel{m}g\hat{j}+\cancel{N\hat{e}_n}=\cancel{m}\left[\ddot{x}\hat{i}+2c(\dot{x}^2+x\ddot{x})\hat{j}\right]\right\}\cdot\underbrace{(\dot{x}\hat{i}+2cx\dot{x}\hat{j})}_{\vec{v}} \quad *$$

Some algebra

$\boxed{\text{EoM}}$

$** \Rightarrow \boxed{\ddot{x} = \text{a mess}}$

involving $x, \dot{x}, c, \cancel{m}, g$ goes away

Now, How to find N?

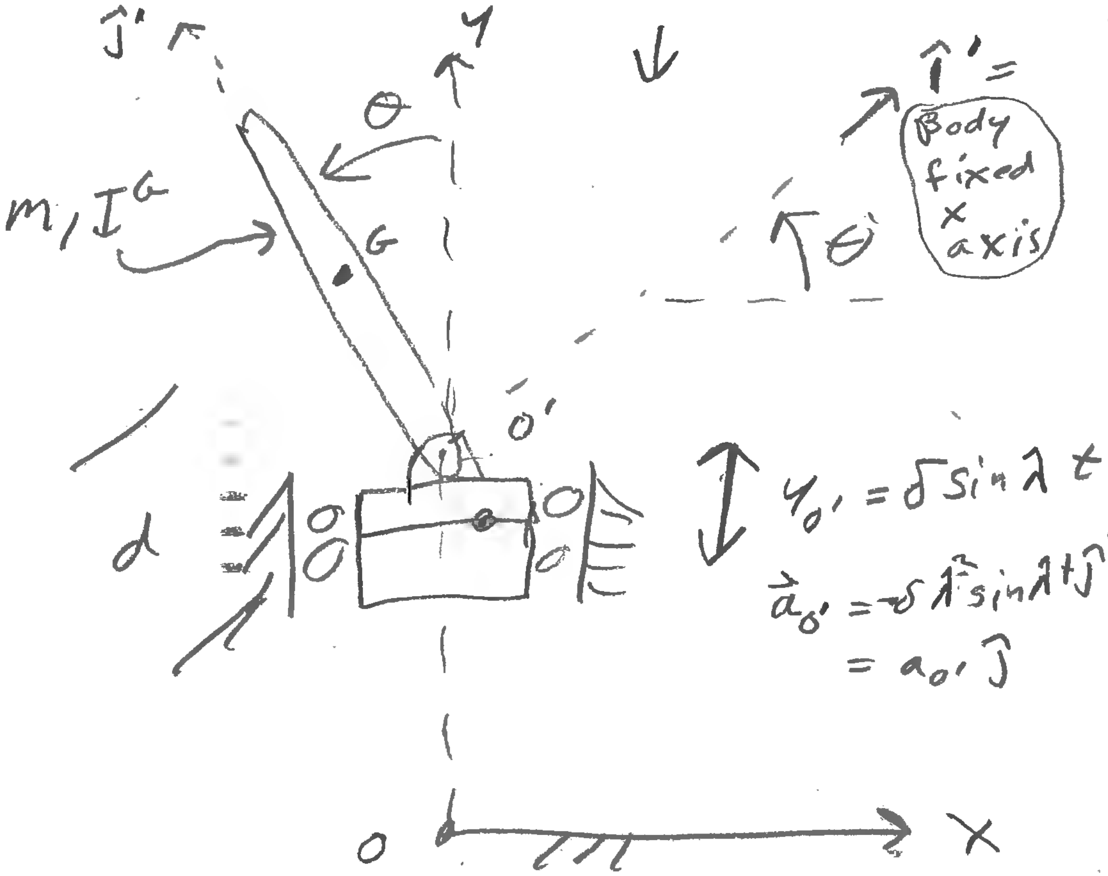
$\{\text{use } *, \text{ sub. in } **, \}\cdot\hat{e}_n$

$$\Rightarrow \boxed{N = \text{another mess}}$$

ex) Shaken inverted pendulum

(Recall demo: or google or old Ruina lectures

$\lambda \neq \omega = \dot{\theta}$

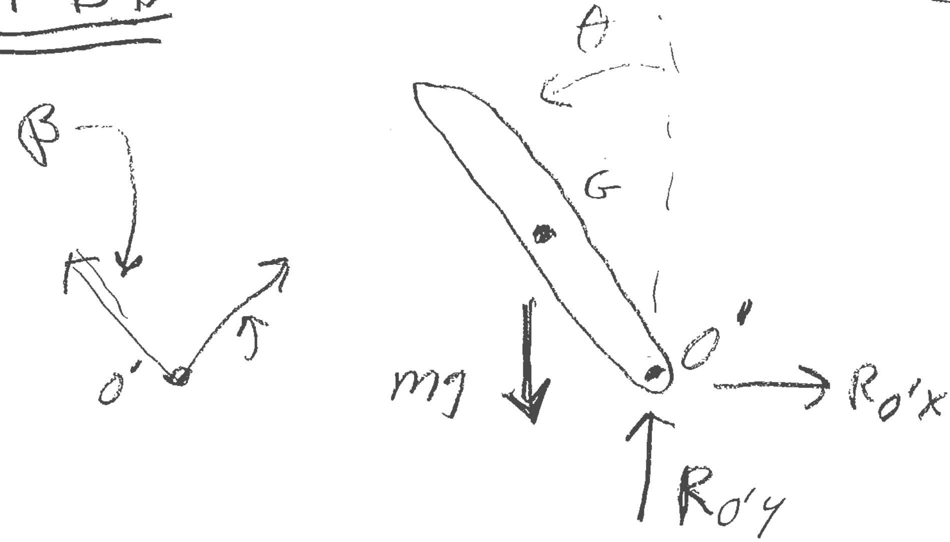


Given: $\delta, \lambda, g, m, I^G,$

Find EoM $\Rightarrow$ find $\ddot{\theta}$,

That is, given parameters & θ & $\dot{\theta}$

FBD



AMB $/O'$:

$$\sum \vec{M}_{/O'} = \text{"}\dot{\vec{H}}_{/O'}\text{"}$$

challenge

$$\underbrace{\vec{r}_{G/O'}}_{d\hat{J}'} \times (-mg\,\hat{J}) = \vec{r}_{G/O'} \times m\,\vec{a}_{G/\mathcal{F}} + I^{G}\underbrace{\alpha}_{\ddot{\theta}}\,\hat{K}$$

$$\vec{a}_{G/\mathcal{F}} = \underbrace{\vec{a}_{O'/\mathcal{F}}}_{-\delta\lambda^2 \sin\lambda t\,\hat{J}} + \cancelto{\vec{0}}{\vec{a}_{G/\beta}} + -\overbrace{\omega}^{\dot{\theta}\hat{K}}{}^2\,\underbrace{\vec{r}_{G/O'}}_{d\hat{J}'}$$

$$+ \underbrace{\dot{\vec{\omega}}_{\beta/\mathcal{F}}}_{\ddot{\theta}\hat{K}} \times \vec{r}_{G/O'} + 2\,\vec{\omega}_{\beta/\mathcal{F}} \times \cancelto{\vec{0}}{\vec{v}_{P/\beta}}$$

TO BE CONTINUED

Lecture 25

MON April 29, 2019

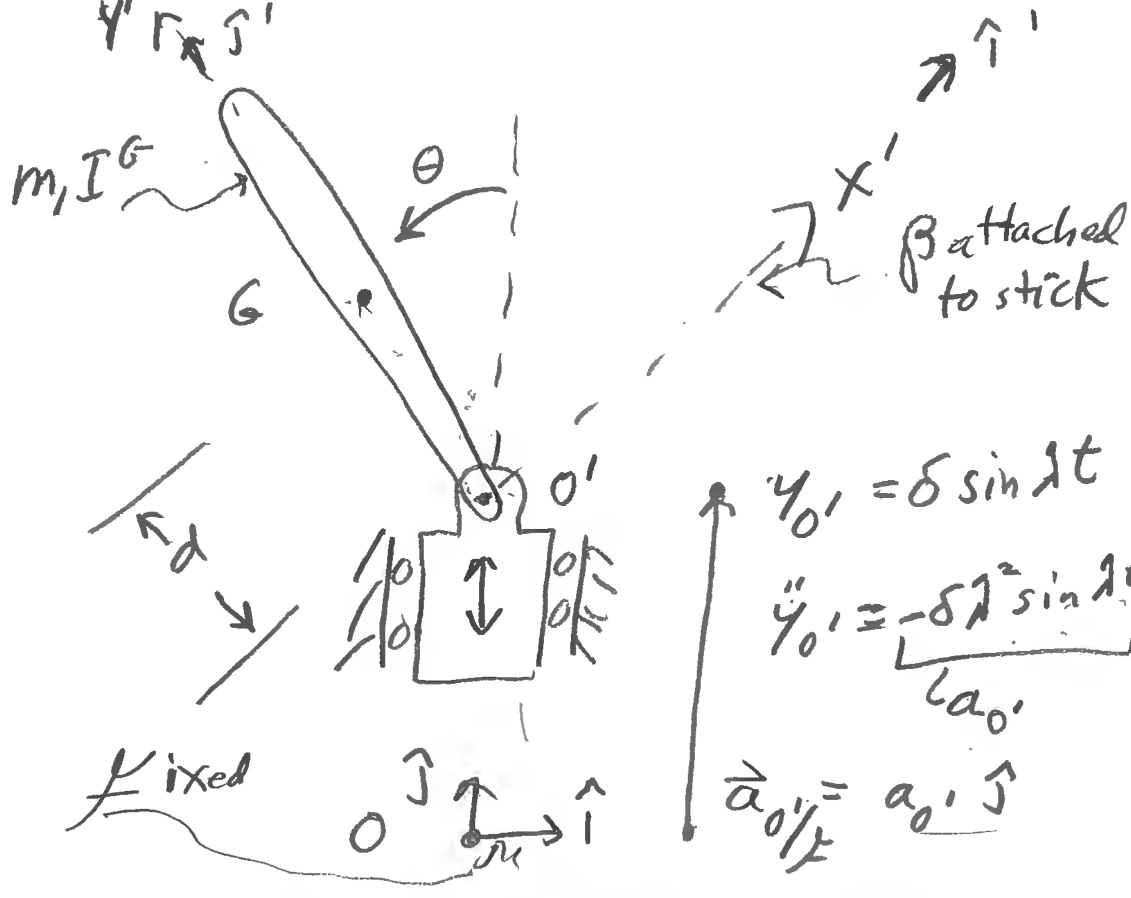
Today

1) Vibrating support (cont'd) for Pendulum

2) 2 DoF table in slot

3) 2 rods

Vibrating Pendulum (cont'd)



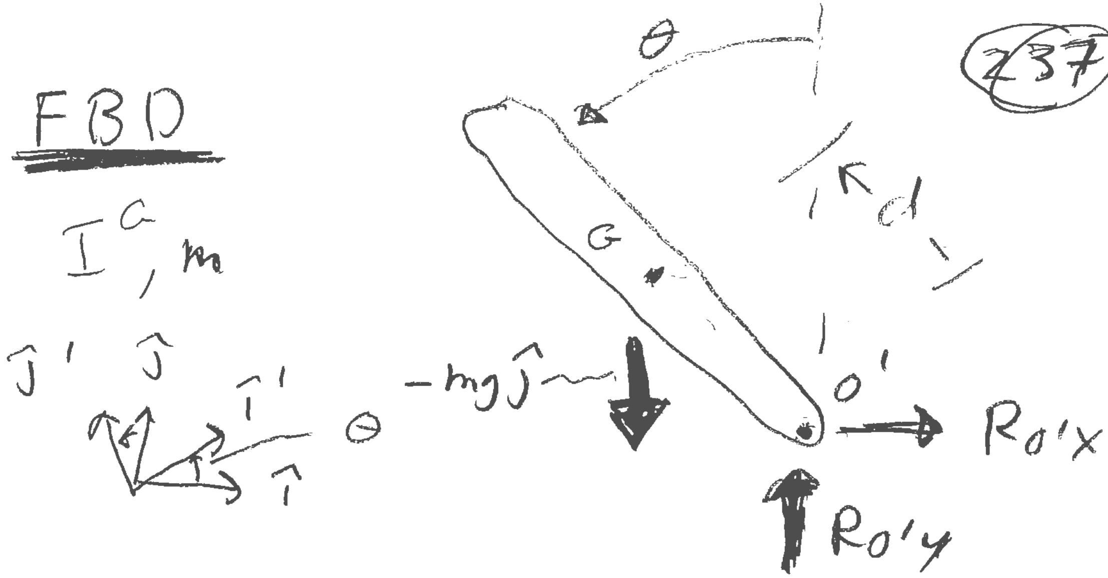
$y_{O'} = \delta \sin \lambda t$

$\ddot{y}_{O'} = \underbrace{-\delta \lambda^2 \sin \lambda t}_{a_{O'}}$

$\vec{a}_{O'/F} = a_{O'} \hat{J}$

FBD

I^G, m



$AMB_{/O'}$:

$$\sum \vec{M}_{/O'} = \dot{\vec{H}}_{/O'}$$

$$= \underbrace{\vec{r}_{G/O'}}_{d\hat{j}'} \times m\vec{a}_{G/F} + I^G \ddot{\theta}\hat{k}$$

$$\sum \vec{M}_{/O'} = \underbrace{\vec{r}_{G/O'}}_{d\hat{j}'} \times -mg\hat{j}$$

$$\omega_{B/F} = \omega = \dot{\theta}$$

$$\vec{a}_{G/F} = \overset{a_{O'}\hat{j}}{\cancel{\vec{a}_{O'/F}}} + \overset{\vec{0}}{\cancel{\vec{a}_{G/B}}} + -\omega^2 \vec{r}_{G/O'}$$

$$+ \underbrace{\dot{\omega}}_{\ddot{\theta}}\hat{k} \times \vec{r}_{G/O'} + 2\omega\hat{k} \times \overset{\vec{0}}{\cancel{\vec{v}_{G/B}}}$$

Do all dot & cross prods. □

$$\Rightarrow \boxed{\ddot{\theta} = \frac{(g+a_0)dm}{I^G + d^2m} \sin\theta} \quad *$$

□ = make sure you can do them

$$a_0 = -\lambda^2 \delta \sin \lambda t$$

Note: EoM in moving frame is like having altered gravity. (a thm. in mechanics)

Note: $I^G = 0$ (special case)

$$* \Rightarrow \boxed{\ddot{\theta} = \frac{(g+a_0)}{d} \sin\theta}$$

$$a_0 = 0 \Rightarrow \boxed{\ddot{\theta} = \frac{g}{d} \sin\theta}$$

(simple inverted pendulum) ✓

ON computer

define ω -

$$\boxed{\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \frac{g + -\delta\lambda^2\sin(\lambda t)}{I^G + md^2}\sin\theta\end{aligned}}$$

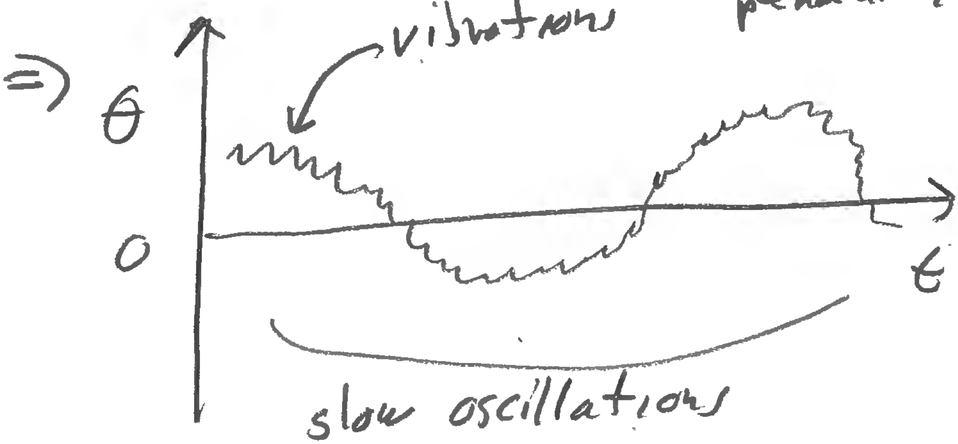
$$z = \begin{bmatrix}\theta \\ \omega\end{bmatrix}$$

$\Rightarrow$ ODE45

For stability choose $\delta \ll d$

$\lambda^2\delta \gg g$

google: vibrating inverted pendulum



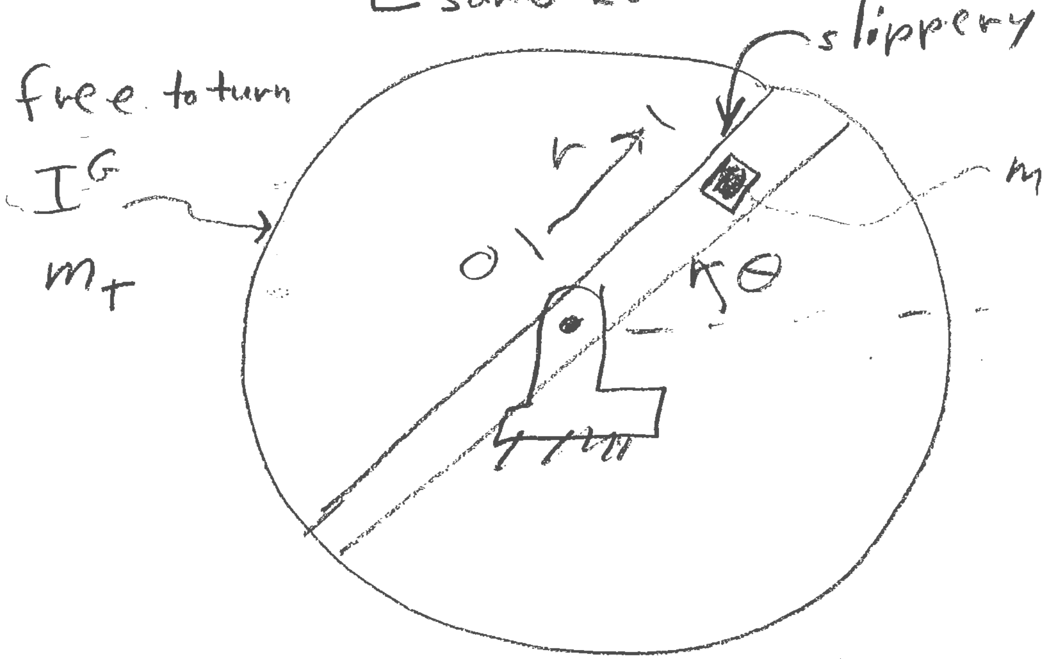
ex) 2 DoF

mass in slot

turntable is free (to turn)
(otherwise held in place)

(no gravity)

[Also $\equiv$ bead on stick]
same EoM



EoM = ?

DoF : r, θ

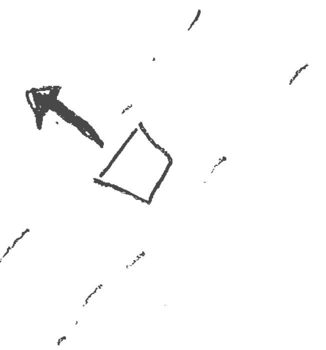
DoF
a) 1
b) 2 ✓
c) 3
d) 4
e) None of above

FBDs

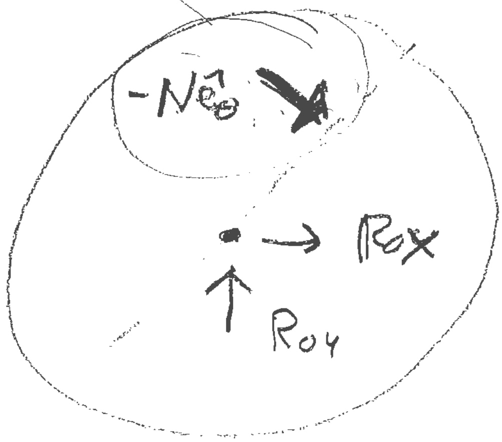
$= \searrow N$

<u>mass</u>

$N\hat{e}_\theta$

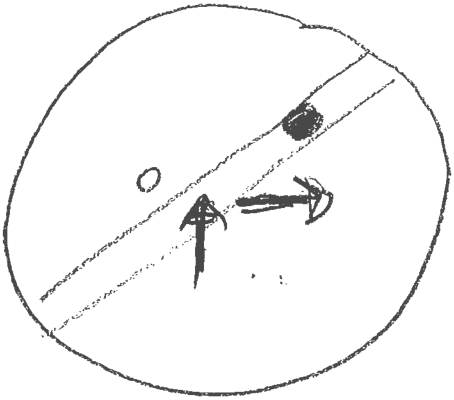


<u>turntable</u>



<u>System</u>

any 2 FBDs
$\Rightarrow$ 3rd



1) <u>AMB system</u> $/O$: ✓ (next page)

2) (LMB for mass) $\cdot\hat{e}_r$ ✓ (")

[These eliminate $\underbrace{\ }_{\text{unknown}}$ reaction forces.]

AMB syst /o

$$\sum \vec{M}_{/o} = \dot{\vec{H}}_{/o}$$

$$\vec{0} = \underbrace{\vec{r}_{P/o}}_{r\hat{e}_r} \times m\vec{a}_P + I^G \ddot{\theta} \hat{k}$$

$$\vec{a}_P = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\Rightarrow \quad \vec{0} = \underbrace{\left[m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) + I^G\ddot{\theta}\right]}_{=0} \hat{k} \quad *$$

(LMB for mass) $\cdot \hat{e}_r$

$$\left\{ \sum \vec{F} = m\vec{a} \right\} \cdot \hat{e}_r$$

$$\left\{ N\hat{e}_\theta = m\left[(\ddot{r} - r\dot{\theta}^2) + (\ldots)\hat{e}_\theta\right] \right\} \cdot \hat{e}_r$$

$$\Rightarrow \boxed{\ddot{r} - r\dot{\theta}^2 = 0} \quad **$$

$\Rightarrow$ 2 2nd order ODES

$$\boxed{\begin{aligned} ** \Rightarrow \quad \ddot{r} &= + r\dot{\theta}^2 \\ * \Rightarrow \quad \ddot{\theta} &= \frac{-2mr\dot{r}\dot{\theta}}{I^G + mr^2} \end{aligned}}$$

flies out

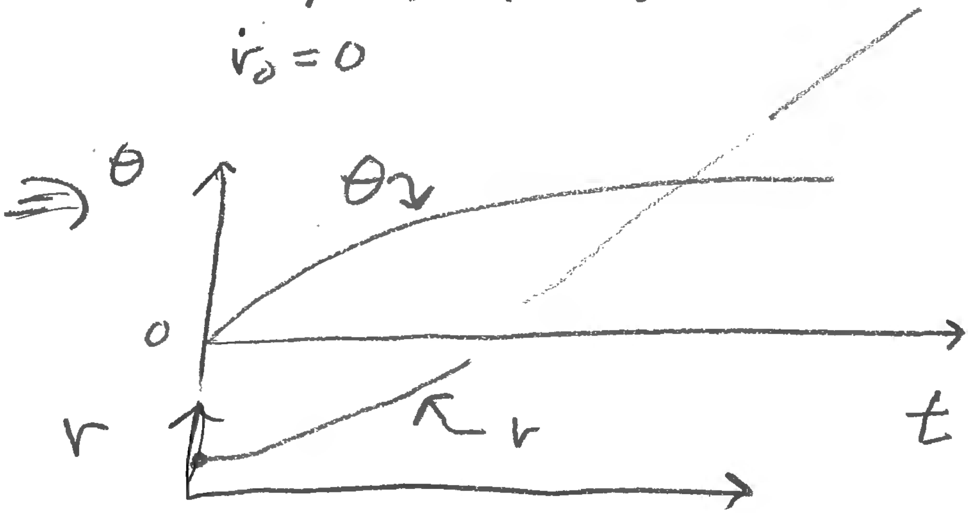
slows down

$$\Rightarrow \quad \dot{z} = f(z) \Rightarrow \text{matlab}$$

$$z = \begin{bmatrix} r \\ v_r \\ \theta \\ \omega \end{bmatrix}$$

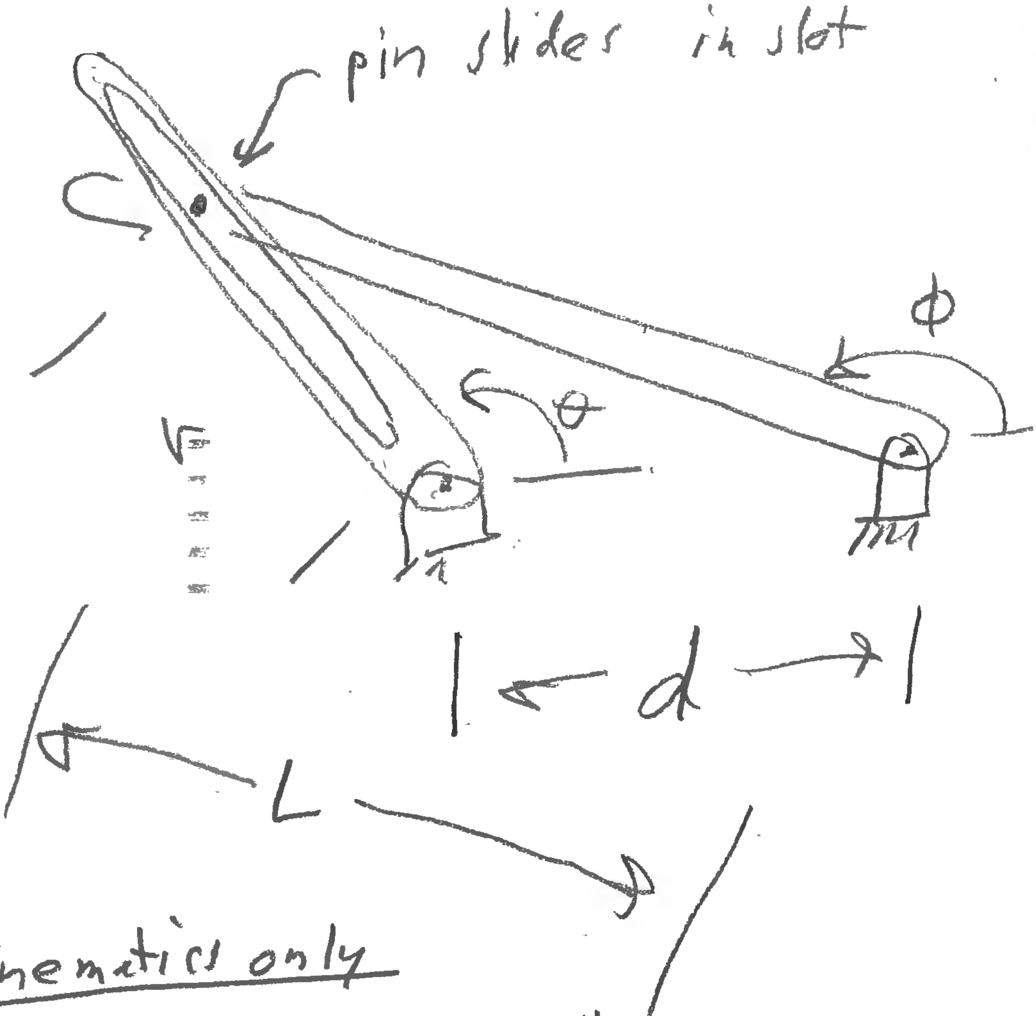
Given: r_0, $\theta_0 = 0$, $\dot{\theta}_0 \neq 0$
$\dot{r}_0 = 0$

$\Rightarrow$



ex)

2 sticks



Kinematics only

Given $L, d, \theta, \dot{\theta}, \ddot{\theta}$

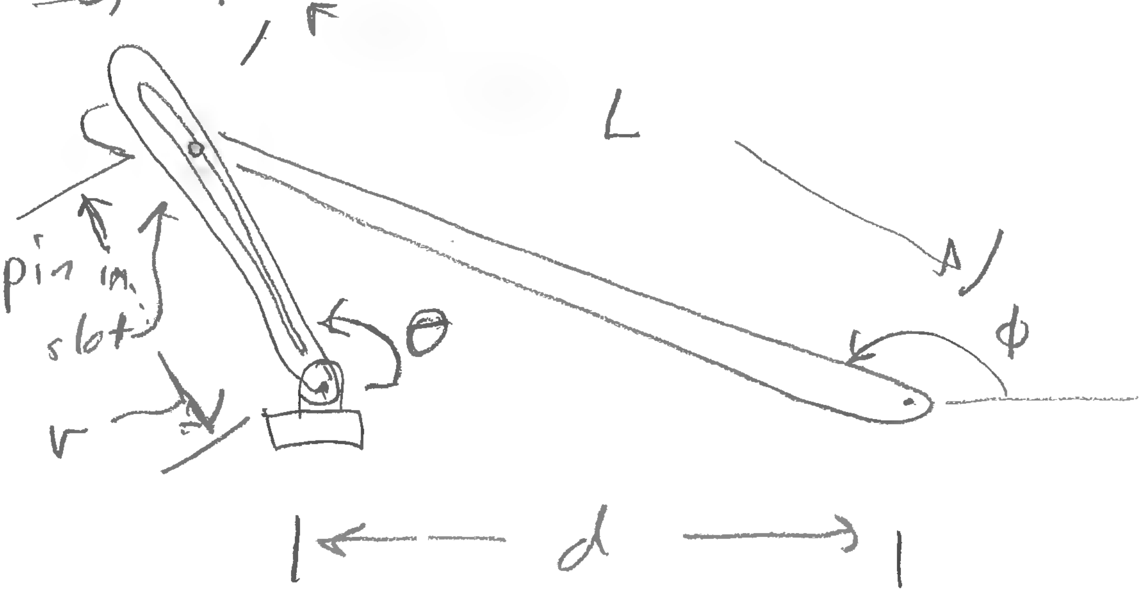
find $\phi, \dot{\phi}, \ddot{\phi}$?

$r, \dot{r}, \ddot{r}$

Wed May 1, 2019
Lecture 26

TODAY

1) 2 sticks (cont'd)
2) 2 balls
3) ~~ladder~~



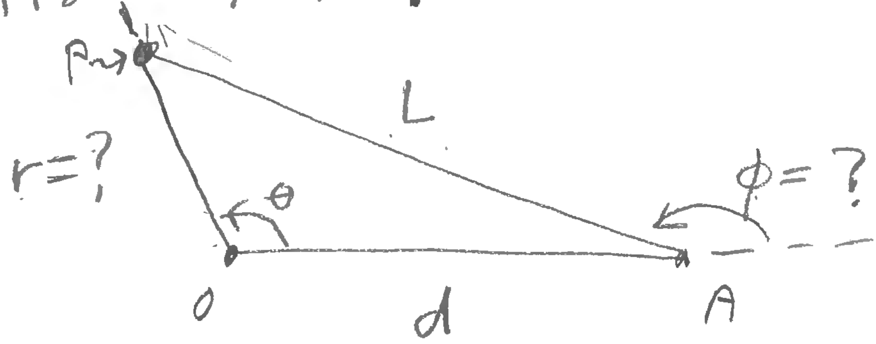
Given: $d, L, \theta, \dot{\theta}, \ddot{\theta}$

Find: $r, \dot{r}, \ddot{r}, \phi, \dot{\phi}, \ddot{\phi}$?

1. Positions : r, ϕ

given : L, d, θ

find : r, ϕ ?



Solve triangle:

$$\vec{r}_{P/O} = \vec{r}_{P/O}$$

$$(r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}) = (d + L\cos\phi)\,\hat{i} + L\sin\phi\,\hat{j} \quad *$$

$*$ is 2 eqs for ϕ, r :

2 eqs

$\{\ \}\cdot\hat{i} \Rightarrow$ $\boxed{r\cos\theta = d + L\cos\phi}$

$\{\ \}\cdot\hat{j} \Rightarrow$ $\boxed{r\sin\theta = L\sin\phi}$

then solve eqs.

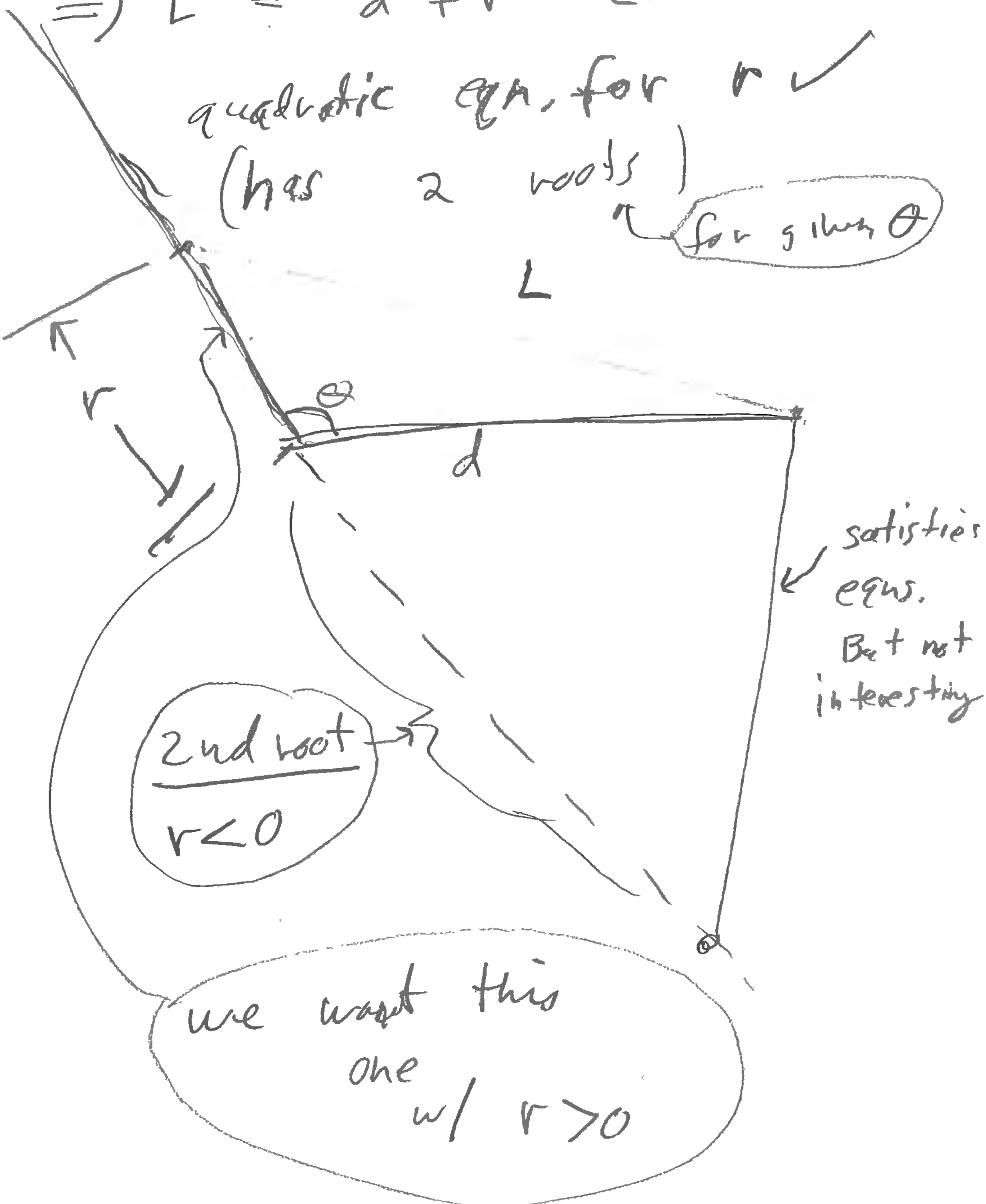
<u>Instead</u>, use law of cosines

$$\text{"} c^2 = a^2 + b^2 - 2ab\cos\theta \text{"}$$

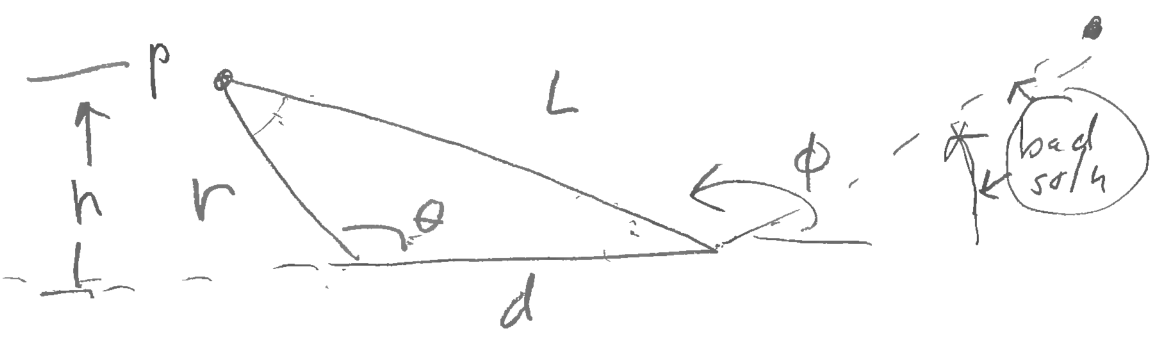
only one unknown

$$\Rightarrow L^2 = d^2 + r^2 - 2dr\cos\theta$$

quadratic eqn. for r ✓
(has 2 roots) for given θ



Need to find ϕ



$$h = h$$

$$r \sin\theta = L \sin\phi$$

$$\phi = \operatorname{asin}\left(\frac{r}{L}\sin\theta\right) \checkmark$$

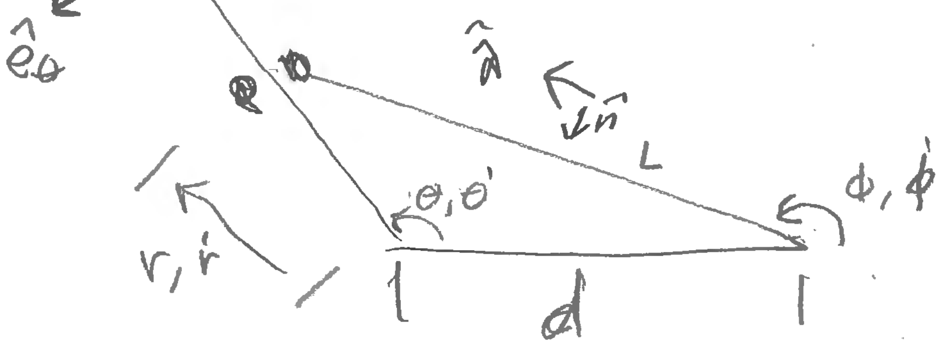
larger of 2 solns. is good

Now, have $\boxed{r \;\&\; \phi}$

---

2) Find $\dot{r}$ & $\dot{\phi}$

$$\vec{V}_p = \vec{V}_p \;*$$



$$* \Rightarrow \dot{\phi} L \hat{n} = (\dot{r} \qquad) \hat{e}_r$$

$$+ (r\dot{\theta} \qquad) \hat{e}_\theta$$

2 eqs for $\dot{\phi}$ & $\dot{r}$

linear

! $\Rightarrow$ easy

we know these

$$\hat{\lambda} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\hat{n} = \hat{k} \times \hat{\lambda}$$

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = \hat{k} \times \hat{e}_r$$

Solve, then

$\Rightarrow$ $\boxed{\dot{r} \;\&\; \dot{\phi}}$

3. **Accel.**

$$\vec{a}_p = \vec{a}_p$$

$$-L\dot{\phi}^2\hat{a} + L\ddot{\phi}\hat{n} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

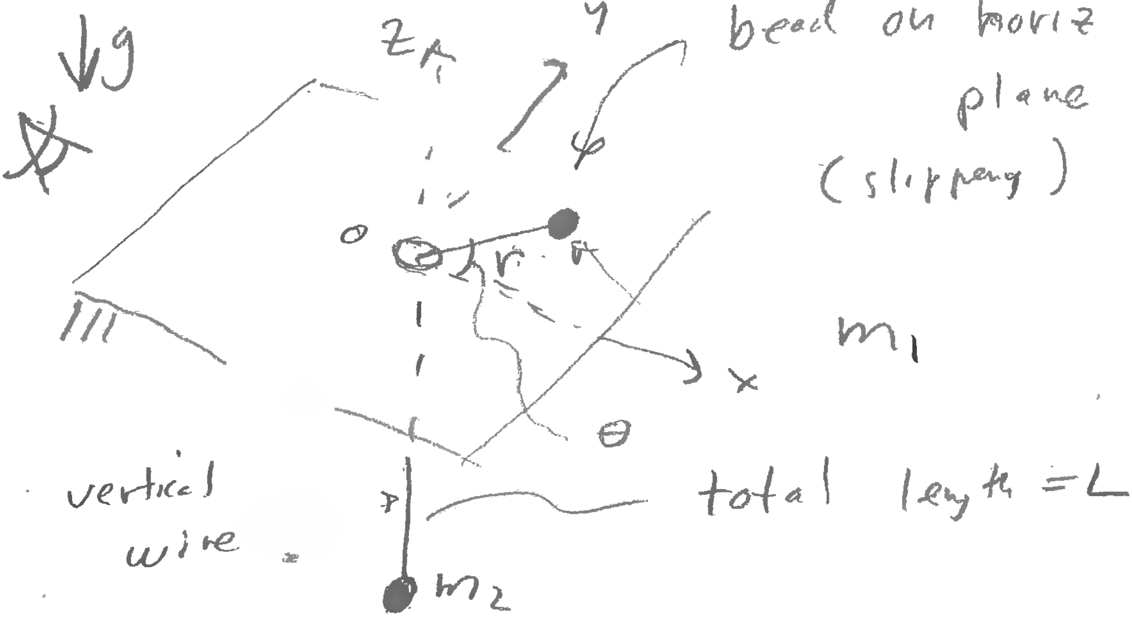
2 eqs for 2 unknowns
$\ddot{r}$ & $\ddot{\phi}$

Linear eqs. $\Rightarrow$ $\boxed{\text{easy}}$

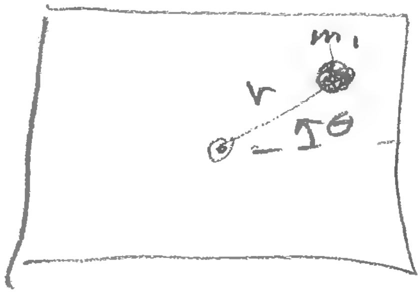
$$\boxed{\text{See Matlab Symbolic Soln.}}$$

ex) 2 beads & string (250)

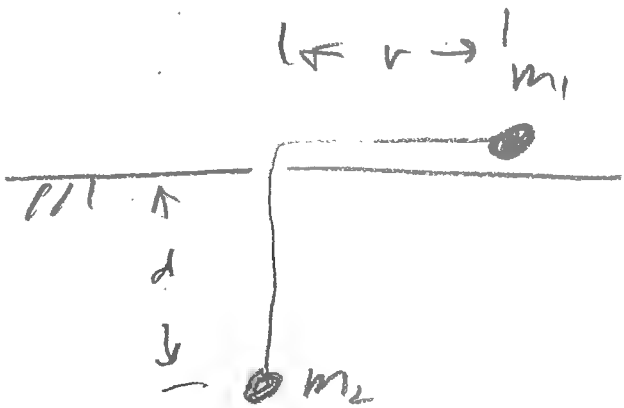
EoM = ?



looking down



side view



$L = d + r$

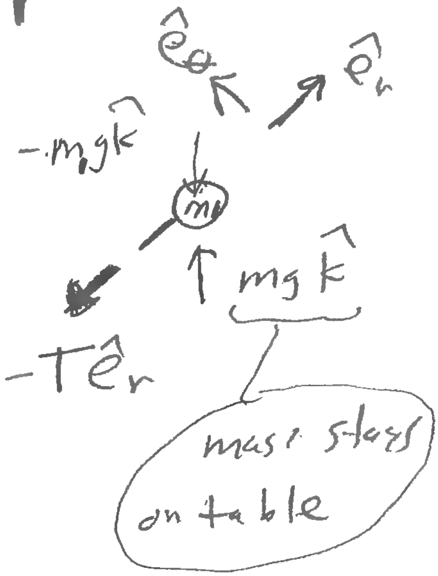
How many DoF?

Ignore swinging DoFs
(lower mass just goes up & down)

2 DoF: θ, r

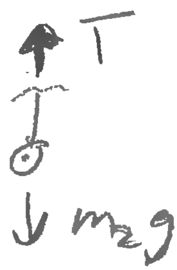
FBDs

(basically 2D, looking down)



hanging mass

assume no swinging



AMB of upper mass/O

(my inserted above "upper" with a caret)

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\vec{0} = \vec{r} \times m\vec{a}$$

$$= r\hat{e}_r \times$$

(1) $$m\left[(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + (\ldots)\hat{e}_r\right]$$

$\Rightarrow$ 1 eqn for $\ddot{r}$ & $\ddot{\theta}$

2nd Eqn.: $\boxed{\dot{E}_{tot} = 0}$

$\Rightarrow$ 2nd eqn.

Alt. approach

deal w/ T (don't kill it at start)

Lower mass

$LMB \cdot \hat{k}$

$$\boxed{\ddot{r} m_2 = T - mg}$$

$\ddot{d} = -\ddot{r}$

$L = \text{const}, \Rightarrow$

$T = \ddot{r} + mg$

upper mass

$LMB \cdot \hat{e}_r$

$$-T = m(\ddot{r} - r\dot{\theta}^2) \quad ②$$

one more eqn w/ $\ddot{r}$

① & ② are EoM

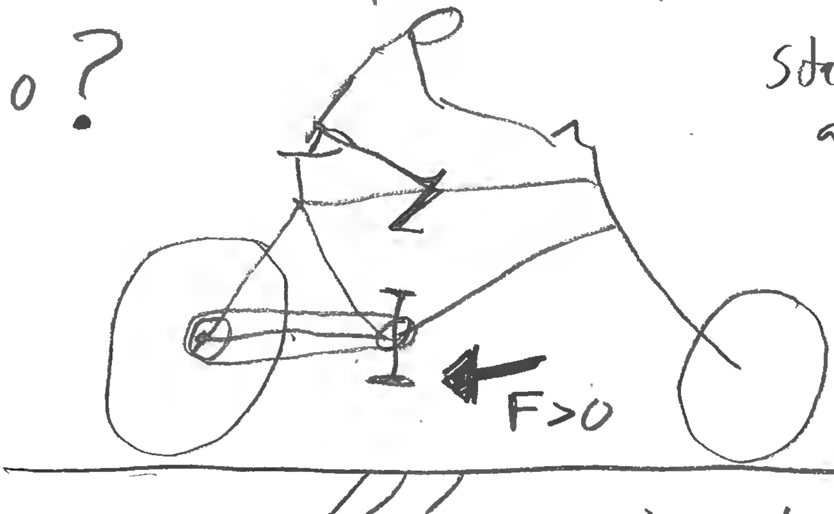
MONDAY MAY 6, 2019

LAST LECTURE

TODAY

1) Bike pedal puzzle
2) Falling ladder & instantaneous center
3) Intro. to 3D

Q) Which way does the bike go?



stationary at start

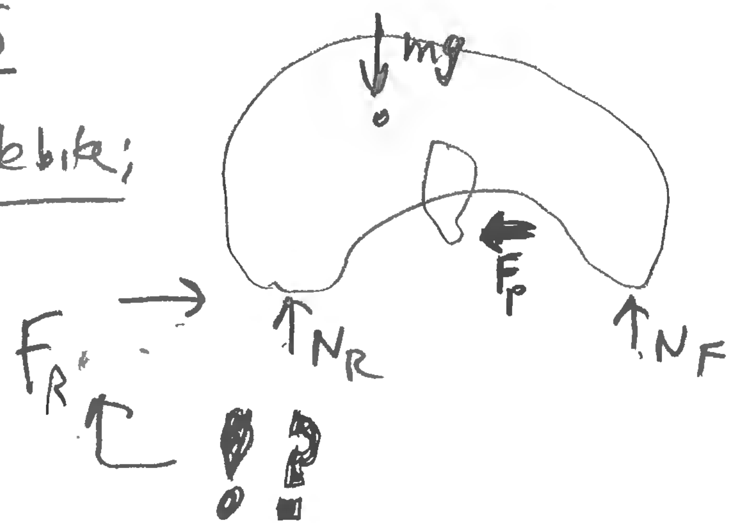
F = small ish

rear wheel does not skid

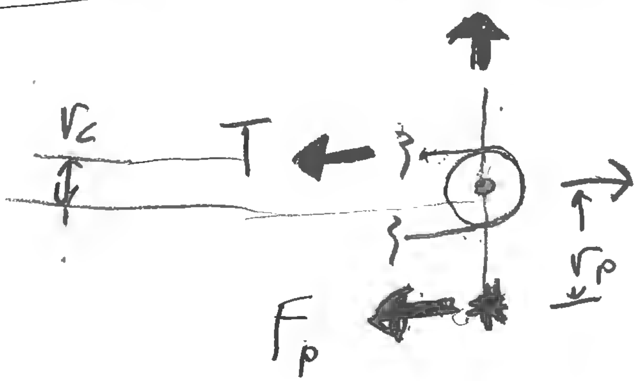
a) doesn't move
b) left
c) right
d) depends on info. not given
e) Not well posed.

FBDs

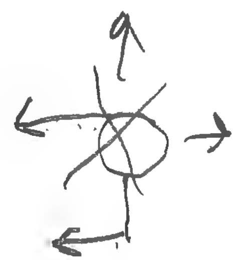
Whole bike:



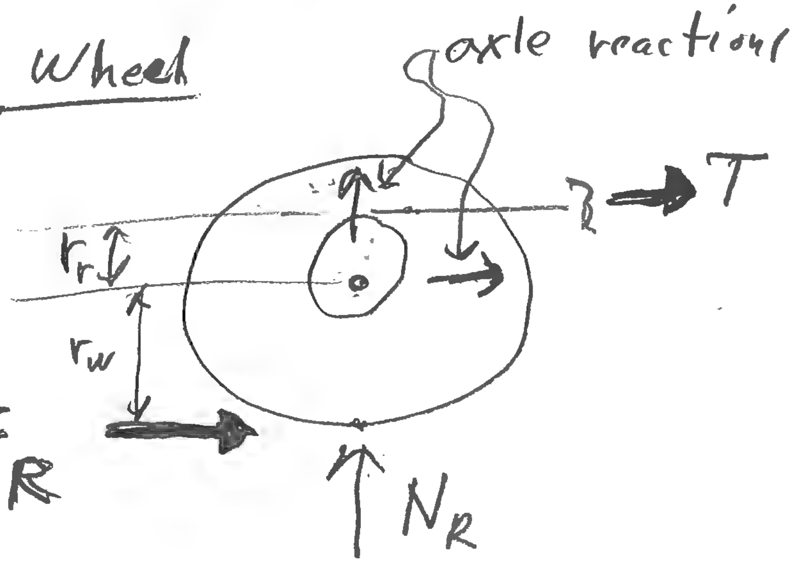
Front Sprocket & Crank



bad FBD



Rear Wheel



4 unknown reactions

& $\vec{a}_b = a_b \hat{i}$

unknown

LMB Whole bike

use statics for internal parts.

$$\boxed{F_R - F_p = m a_B}$$

Is F_R bigger or smaller than F_p?

AMB of crank / hinge,

$$\boxed{T = \frac{r_p}{r_c} F_p}$$

Rear Wheel

$$F_R = \frac{r_r}{r_w} T$$

sabutitute

$$F_R = \underbrace{\frac{r_r}{r_w} \frac{r_p}{r_c}}_{\text{usually} < 1} F_p$$

Force increase because crank > front sprocket

force decrease because rear wheel > rear Sprocket

Energy approach

Work of $F_p > 0$

for bike to gain Energy

$\Rightarrow$ pedal goes backwards

Kinematics analysis

$\Rightarrow$ pedal moves same dir. of bike. Happens if

A few calcs. w/ $\vec{\omega} \times \vec{F}$ $\Rightarrow$ $\Rightarrow$ $\dfrac{r_r}{r_w} \dfrac{r_p}{r_c} < 1$

$\Rightarrow$ bike goes backwards

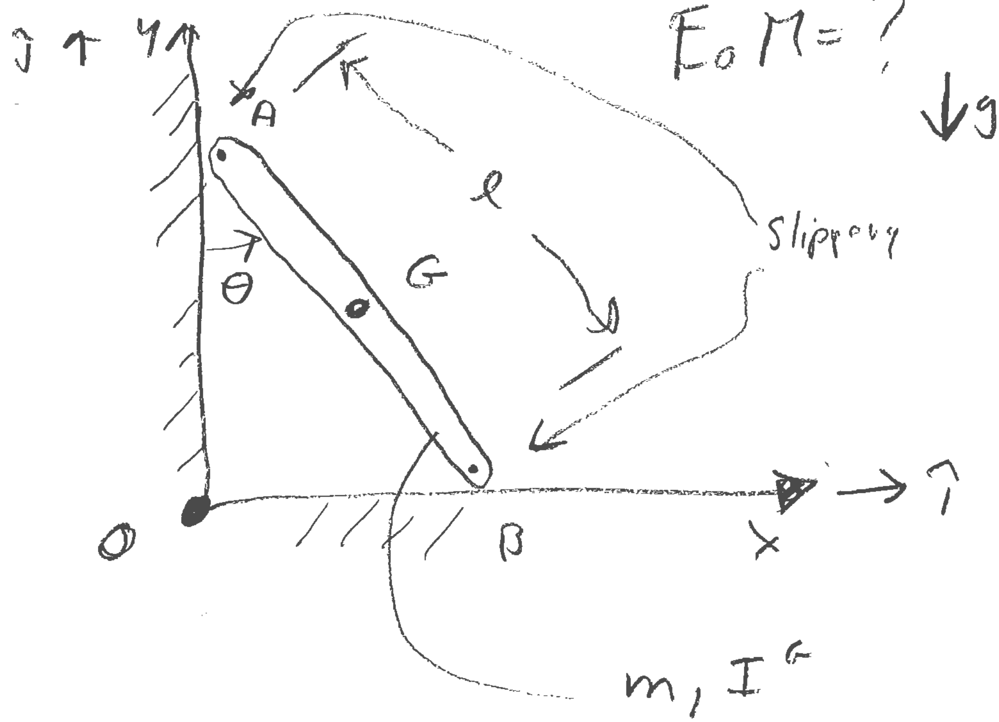
See veritasium video

google it

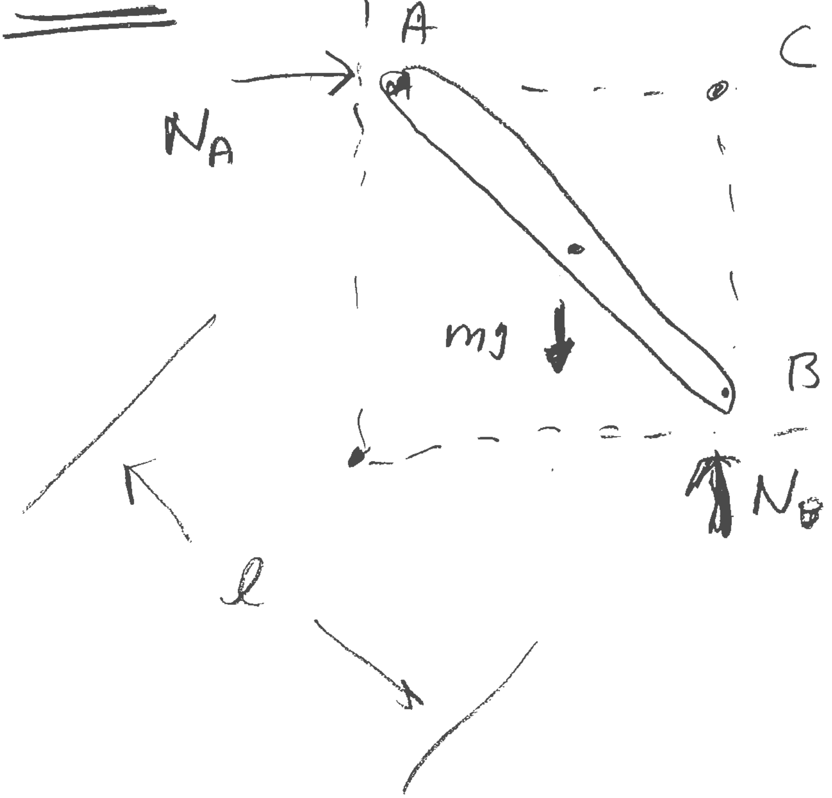
(259)

ex) Falling ladder

EoM = ?



FBD



Kinematics

\# DoF ?

ANS: 1 : θ

$\uparrow$ good one

Need: $\vec{a}_G$, $\dot{\vec{\omega}}$ from

θ, $\dot{\theta}$, $\ddot{\theta}$

Approach 2:

$$\vec{r}_G = \vec{r}_G$$

$$x_G \hat{\imath} + y_G \hat{\jmath} = y_A \hat{\jmath} + \vec{r}_{G/A}$$

$$\vec{r}_{G/A} = \frac{\ell}{2}(\sin\theta \hat{\imath} - \cos\theta \hat{\jmath})$$

also

$$= x_B \hat{\imath} + \vec{r}_{G/B}$$

etc.

$\Rightarrow$ x_G, y_G in terms of θ

<u>vel:</u> $\vec{V}_G = \vec{V}_G$ two ways

acc: $\vec{a}_G = \vec{a}_G$ two ways

$\vec{\omega} = \ddot{\theta}\hat{k}$

$\Rightarrow$ $\vec{a}_G$, $\dot{\vec{\omega}}$ in terms of $\theta, \dot{\theta}, \ddot{\theta}$

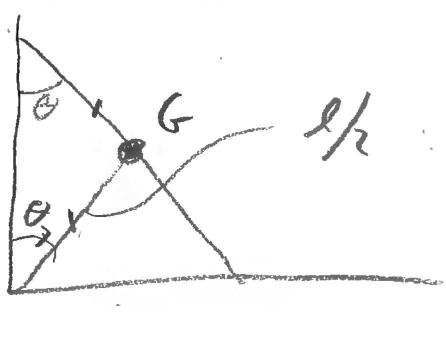
<u>this works (1).</u> Generally easier for <u>hard</u> problems

<u>Approach 1</u>

$$\boxed{\vec{a}_{G/f} = \frac{d^2}{dt^2}(\vec{r}_{G/O})}$$

Always, for all probs

Generally easier for easy problems



G moves in circle w/ radius $l/2$

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$$\vec{r}_{G/O} = \frac{l}{2}\left(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right)$$

$$\Rightarrow \vec{v}_{G/O} = - - - - -$$

$$\Rightarrow \vec{a}_{G/O} = -\dot{\theta}^2 \frac{l}{2}\left(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right)$$

$$+ \ddot{\theta}\frac{l}{2}\left(\cos\theta\,\hat{\imath} + -\sin\theta\,\hat{\jmath}\right)$$

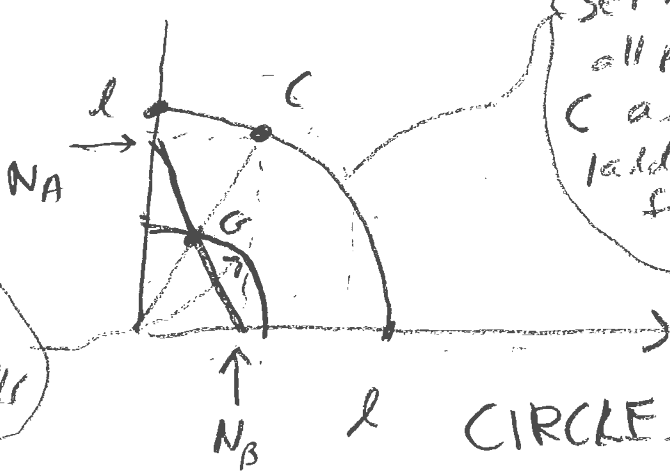
AMB $/C$ $\qquad \sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$

"kills N_A & N_B"

$$\left\{ \vec{r}_{G/C} \times (-mg\hat{\jmath}) = \vec{r}_{G/C} \times m\vec{a}_G + I^G \ddot{\theta}\,\hat{k} \right\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \boxed{\text{E.o.M}}$$

Curious facts

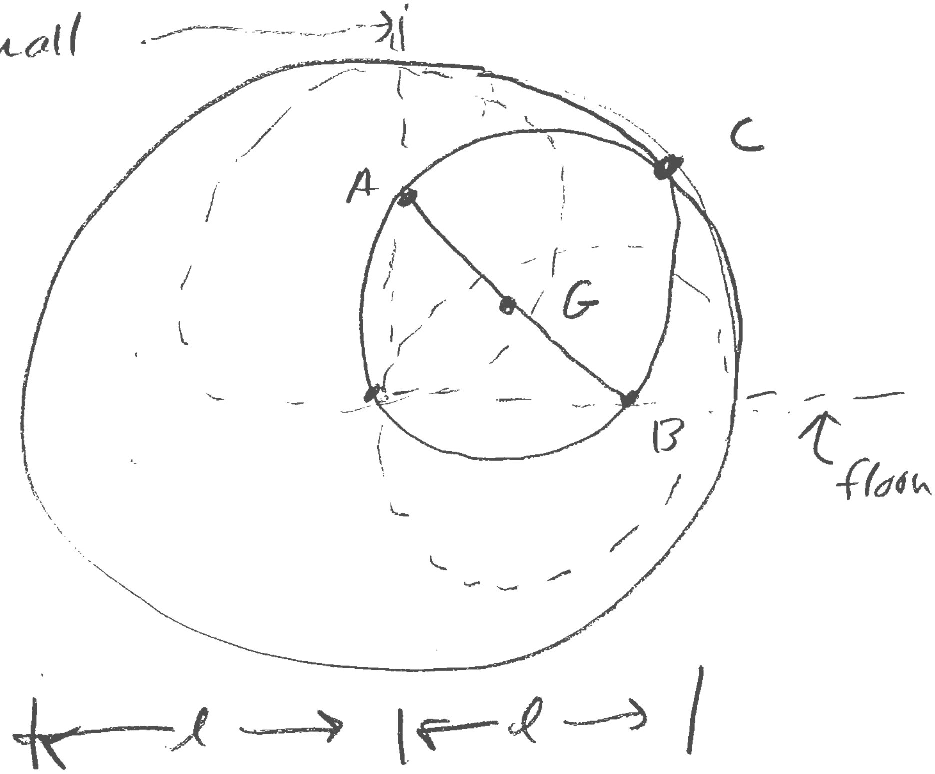


Set of all pts G as ladder falls

Set of all pts C as ladder falls

CIRCLES!

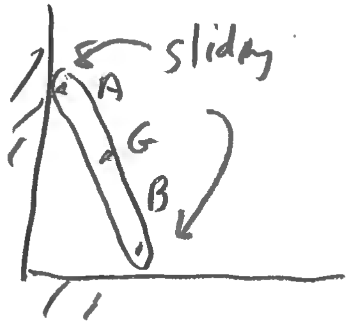
Consider disk w/ diameter l rolling inside circle w/ radius l



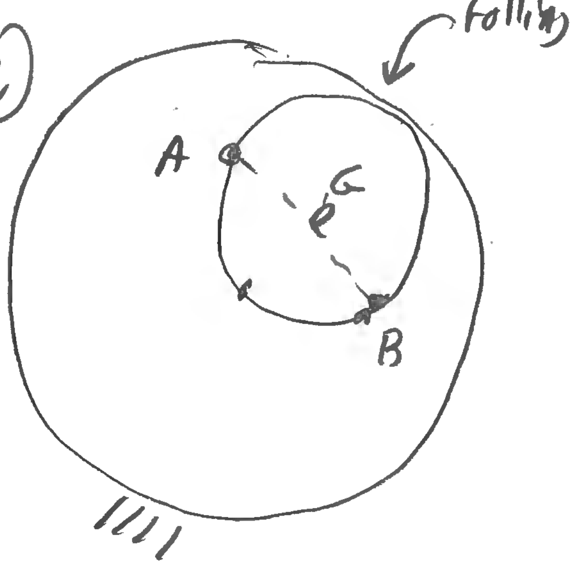
Motion of ladder is motion of diameter of inner disk.

These 3 problems
are kind of the same

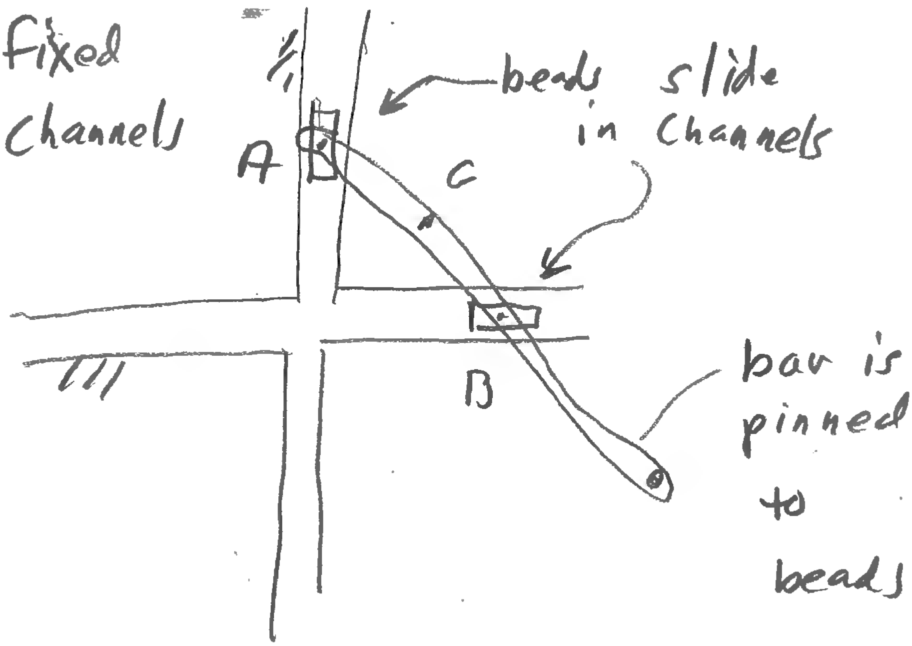
①



②



③



A & B & G have same
motions for all 3

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Advertisement for Intermediate Dynamics 4730

*3260 is <u>not</u> a pre-req.

Whats in it?

I. harder 2D problems (many moving parts)

Diff. Alg. Eqs & Lagrange Eqs

II. 3D

Main thing:

$$\vec{H}_{/G} = \begin{cases} \dot{\theta} I^G \hat{k} & \text{2D} \\ \underline{\underline{I}}^G \cdot \dot{\vec{\omega}} + \vec{\omega} \times (\underline{\underline{I}}^G \cdot \vec{\omega}) & \text{3D} \end{cases}$$

3D vector

moment of inertia tensor