Solution for Prelim 1 Problem 1 Fall 2020 MAE 4730/5730

The force on either mass is $F_G = -GmM/d^2$, negative being toward the opposing mass. The acceleration is (see polar coordinate formula) $a = -r\omega^2$, where r is the distance between each mass and the center of mass and ω is the (constant) angular velocity of either mass about the center of mass. We will utilize linear momentum balance which for each mass is

$$-\frac{GmM}{d^2} = -mr_{\mathrm{m/G}}\omega^2 = -Mr_{\mathrm{M/G}}\omega^2$$

These equations, combined with the constraint $r_{\mathrm{m/G}} + r_{\mathrm{M/G}} = d$, allows us to solve for the distances as well as the angular velocity,

$$r_{\mathrm{m/G}} = \frac{M}{m+M}\,d \qquad r_{\mathrm{M/G}} = \frac{m}{m+M}\,d \qquad \omega = \sqrt{\frac{G(m+M)}{d^3}}$$

We will consider the angular momentum about the center of mass

$$\mathbf{H}_{/\mathrm{G}} = \sum_{i=\mathrm{m,M}} m_i \mathbf{r}_{i/\mathrm{G}} \times \mathbf{v}_{i/\mathrm{G}}$$

which in our 2D case has $\mathbf{r}_{i/\mathrm{G}} \times \mathbf{v}_{i/\mathrm{G}} = r_{i/\mathrm{G}}^2 \omega \hat{\mathbf{k}}$, so

$$\mathbf{H}_{/\mathrm{G}} = \left(mr_{\mathrm{m/G}}^2 + Mr_{\mathrm{M/G}}^2\right)\omega\hat{\mathbf{k}}$$

which, with our earlier relations for the distances, may be simplified to

$$\mathbf{H}_{/\mathrm{G}} = \left(\frac{mM}{m+M}\,\omega d^2\right)\hat{\mathbf{k}} = \left(mM\sqrt{\frac{Gd}{m+M}}\right)\hat{\mathbf{k}}$$

Note: two-body problems are often easier to handle using the reduced mass $mM/(m+M)$ and its corresponding one-body problem.