Solution for Prelim 2 Problem 2 Fall 2020 MAE 4730/5730

Note: this problem is effectively the same as Fall 2014 Prelim 1 Problem 1, which can be found as Problem B.1 in Buche's solutions.

Start out by doing some calculus with the given shape of the wire and take some derivatives:

$$y = A\sin(cx), \qquad \dot{y} = Ac\dot{x}\cos(cx), \qquad \ddot{y} = Ac\left[\ddot{x}\cos(cx) - c\dot{x}^2\sin(cx)\right]. \tag{1}$$

We need to find $\ddot{x}$ in terms of $\dot{x}$, x, and the parameters m, g, A, and c to complete the problem. To do so, we consider linear momentum balance on the bead – a normal force $\mathbf{N}$ and gravitational force $-mg\hat{\mathbf{j}}$ act on the bead,

$$\mathbf{F} = \mathbf{N} - mg\hat{\mathbf{j}} = m\mathbf{a} = m\left(\ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}}\right).$$

We do not need to solve for the normal force and will therefore eliminate it from our problem. The normal force always acts orthogonal to the wire, which is then always orthogonal to the bead velocity, so $\mathbf{N}\cdot\mathbf{v} = 0$ (this force does no work). With $\mathbf{v} = \left(\dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}}\right)$, we then receive (after dividing out the mass as well)

$$-g\dot{y} = \dot{x}\ddot{x} + \dot{y}\ddot{y}. \tag{2}$$

Since the mass has dropped out here and was not present in Eq. (1), our final result for $\ddot{y}$ will also not contain the mass. Note that Eq. (2) can also be retrieved using conservation of energy (writing total energy, taking the time derivative, setting equal to zero), or using Lagrange/Hamiltons equations.[1] Moving on: combing Eqs. (1) and (2),

$$\ddot{y} = Ac\left[\ddot{x}\cos(cx) - c\dot{x}^2\sin(cx)\right] = -\left(\frac{\dot{x}}{\dot{y}}\,\ddot{x} - g\right) = -\frac{\ddot{x}}{Ac\cos(cx)} - g,$$

which is then algebraicly solved for

$$\ddot{x} = \frac{Ac\cos(cx)\left[Ac^2\dot{x}\sin(cx) - g\right]}{1 + \left[Ac\cos(cx)\right]^2}.$$

We could have alternatively solved for $\ddot{y}$ first. Substitute this into Eq. (1) and simplify for

$$\ddot{y} = -Ac^2\left\{\frac{\dot{x}^2\sin(cx) + Ag\cos^2(cx)}{1 + \left[Ac\cos(cx)\right]^2}\right\}.$$

[1] The form of Eq. (2) is actually quite universal – for an arbitrary wire shape and gravity acting in the $-y$ direction, we would combine Eq. (2) with time derivatives of the wire shape $f(x(t), y(t)) = 0$ to find the equations of motion.