# Solution for Prelim 2 Problem 1 Fall 2020 MAE 4730/5730

Note: This problem is related to Fall 2017 Homework Problem 30 which can be found in Buche's solutions. There, the goal was instead to find the equations of motion, not the force. The type of pendulum (point mass on a massless stick versus the pendulum here) also differs between the problems.

The force $\mathbf{F}_\mathrm{C}$ acting on the pendulum from it being connected to the cart is an internal force: we write it as $+\mathbf{F}_\mathrm{C}$ in the free body diagram for the pendulum and $-\mathbf{F}_\mathrm{C}$ in that for the cart (see Fig. 1). The linear momentum balance equations for the cart are

$$-F_{\mathrm{C},x} = m_\mathrm{C}\ddot{x}_\mathrm{C}, \qquad N - m_\mathrm{C} g - F_{\mathrm{C},y} = 0,$$

and those for the pendulum are

$$F_{\mathrm{C},x} = m_\mathrm{p} a_{\mathrm{G},x}, \qquad F_{\mathrm{C},y} - m_\mathrm{p} g = m_\mathrm{p} a_{\mathrm{G},y}.$$

We expand the acceleration of G as

$$\mathbf{a}_\mathrm{G} = \mathbf{a}_\mathrm{C} + \mathbf{a}_{\mathrm{G/C}} = \mathbf{a}_\mathrm{C} + d\left(-\dot{\theta}^2\hat{\mathbf{e}}_r + \ddot{\theta}\hat{\mathbf{e}}_\theta\right),$$

in order to write the components of $\mathbf{a}_\mathrm{G}$ as

$$a_{\mathrm{G},x} = \ddot{x}_\mathrm{C} - d\dot{\theta}^2\sin\theta + d\ddot{\theta}\cos\theta, \qquad a_{\mathrm{G},y} = d\dot{\theta}^2\cos\theta + d\ddot{\theta}\sin\theta.$$

The angular momentum balance equation about G is

$$\mathbf{r}_{\mathrm{C/G}} \times \mathbf{F}_\mathrm{C} = -\left(F_{\mathrm{C},x} d\cos\theta + F_{\mathrm{C},y} d\sin\theta\right)\hat{\mathbf{k}} = I^\mathrm{G}\ddot{\theta}\hat{\mathbf{k}}$$

We now have system that is linear in our unknowns with mass matrix $\mathbf{M}$:

$$\underbrace{\begin{pmatrix} m_\mathrm{C} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ m_\mathrm{p} & m_\mathrm{p} d\cos\theta & 0 & -1 & 0 \\ 0 & m_\mathrm{p} d\sin\theta & 0 & 0 & -1 \\ 0 & I^\mathrm{G} & 0 & d\cos\theta & d\sin\theta \end{pmatrix}}_{\mathbf{M}} \cdot \begin{pmatrix} \ddot{x}_\mathrm{C} \\ \ddot{\theta} \\ N \\ F_{\mathrm{C},x} \\ F_{\mathrm{C},y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ m_\mathrm{C} g \\ m_\mathrm{p} d\dot{\theta}^2\sin\theta \\ -m_\mathrm{p}\left[d\dot{\theta}^2\cos\theta + g\right] \\ 0 \end{pmatrix}}_{\mathbf{b}}.$$

Since $\theta$ and $\dot{\theta}$ are known, the mass matrix $\mathbf{M}$ and right-hand side $\mathbf{b}$ are known, so we may invert $\mathbf{M}$ to find our unknown vector at the current time and thus the two components of the force $F_{\mathrm{C},x}$ and $F_{\mathrm{C},y}$. The `MATLAB` script below accomplishes this:

```
% Code somewhere above specifies d, m_C, m_P, g, I_G, theta, and theta_dot
b = [0; m_C*g; m_P*d*theta_dot^2*sin(theta); ...
        -m_P*(d*theta_dot^2*cos(theta) - g; 0];
M = [m_C 0 0 1 0; 0 0 1 0 -1; m_P m_P*d*cos(theta) 0 -1 0; ...
        0 m_P*d*sin(theta) 0 0 -1; 0 I_G 0 d*cos(theta) d*sin(theta)];
x_dot = M\b;
F_C = [x_dot(4); x_dot(5)];
```

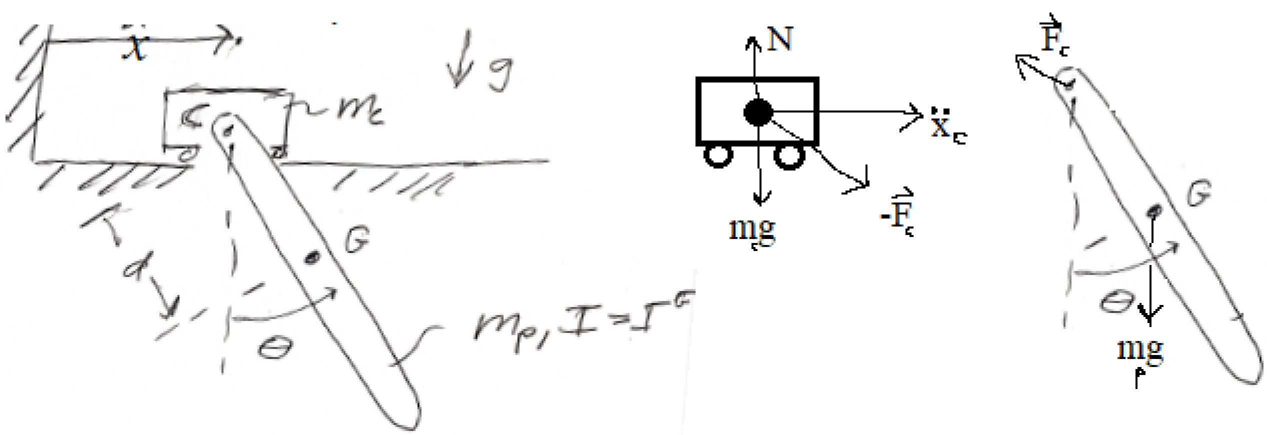


Figure 1: Original and free body diagrams for the cart and pendulum.