Your Name: ______________________

T&AM 203 Final Exam

Tuesday Dec 12, 2000 3:00 — 5:30 PM

Draft March 20, 2007

5 problems, 100 points, and 150 minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Ask for extra scrap paper if you need it.

b) Full credit if

- $\rightarrow$free body diagrams$\leftarrow$ are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- $\uparrow\rightarrow$ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- $\pm$ all signs and directions are well defined with sketches and/or words;
- $\rightarrow\!|$ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- $\gg$ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ `= 18`" instead of, say, "`theta7dot = 18`". Pick generic (not special) numerical values for constants not defined in the problem statement.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	____/20
Problem	2:	____/20
Problem	3:	____/20
Problem	4:	____/20
Problem	5:	____/20
TOTAL:		**____/100**

1)(20 pts) Spring mass.

a) (5 pts) Find the equation of motion, a differential equation, for the variable x in the system above. Your differential equation can contain x, its time derivatives, m, c, k, and ℓ_0 (Please read item (b) on the cover page.)

b) (5 pts) Assume $c = 0$, $x(t = 0) = d$, and $\dot{x}(t = 0) = 0$. What is $\dot{x}$ at time t (answer in terms of some or all of m, k, ℓ_0, d, and t.

c) (5 pts) Assume relatively large c ($c^2 > 4km$), $x(t = 0) = d$, and $\dot{x}(t = 0) = 0$. Find $x(t)$ (or write code that would find $x(t)$).

d) (5 pts) Whether or not you have succeeded at part (c) above, make a clear plot of x *vs* t for the conditions in part (c) above.

(work for problem 1, cont'd.)

2)(20 pts) Car on a ramp. A junior level engineering design course asks students to build a cart (mass $= m_c$) that rolls down a ramp with angle θ. A small weight (mass $m_w \ll m_c$) is placed on top of the cart on a surface tipped with respect to the cart (angle ϕ). Assume the small mass does not slide. Assume massless wheels with frictionless bearings

a) (5 pts) Find the acceleration of the cart. Answer in terms of some or all of $m_c, g, \hat{\mathbf{i}}, \theta$ and $\hat{\mathbf{j}}$. (In accordance with the directions on the front cover you may use other convenient coordinates if you like.).

b) (10 pts) What coefficient of friction μ is required (the smallest that will work) to keep the small mass from sliding as the cart rolls down the slope? Answer in terms of some or all of m_c, m_w, g, θ, and ϕ.

c) (5 pts) What angle ϕ will allow a small mass to ride on the cart with the smallest coefficient of friction? Answer in terms of some or all of m_c, m_w, g, and θ. (You get full credit for a correct answer to this question even if the answer to (b) is incorrect. Conversely, an answer based on incorrect work in part (b) is incorrect.)

(Work for problem 2, cont'd.)

3)(20 pts) A swinging disk. A uniform disk of mass m and radius R is hinged at one end and swings in its plane from a hinge on its circumference.

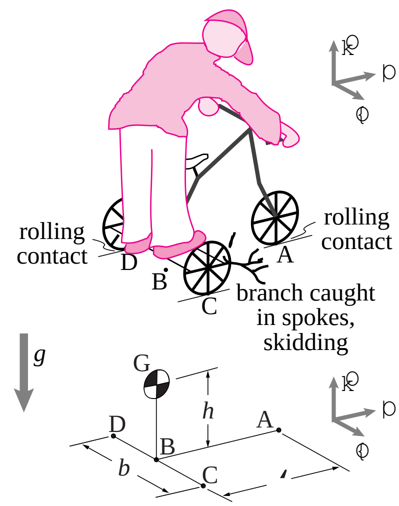
a) (10 pts) Find a differential equation that describes its motion. Describe the motion with an angle θ that is zero when the disk is hanging straight down. (Your equation should have in it some or all of θ, its time derivatives, m, g, and R.

b) (5 pts) What is the period t_p of small oscillations? Answer in terms of some or all of m, R and g.

c) (5 pts) If instead the disk was swinging in the perpendicular direction (with its center moving perpendicular to the plane of the disk) would the frequency of oscillation be higher, lower, or the same? (Correct guess earns one point.)

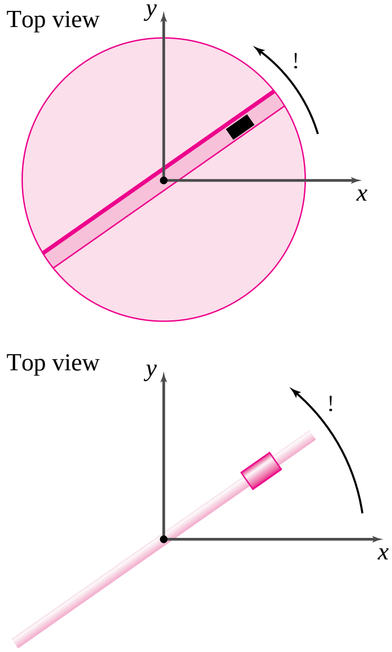
(work for problem 3, cont'd.)

4)(20 pts) Speeding tricycle gets a branch caught in the right rear wheel. A scared-stiff tricyclist riding on level ground gets a branch stuck in the right rear wheel so the wheel skids with friction coefficient μ. Assume that the center of mass of the tricycle-person system is directly above the rear axle. Assume that the left rear wheel and the front wheel have negligible mass, good bearings, and have sufficient friction that they roll in the $\hat{\mathbf{j}}$ direction without slip, thus constraining the overall motion of the tricycle. Dimensions are shown in the lower sketch. *Find the acceleration of the tricycle* (in terms of some or all of ℓ, h, b, m, $[I^{cm}]$, μ, g, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$).
[Hint: check your answer against special cases for which you might guess the answer, such as when $\mu = 0$ or when $h = 0$.]



(Work for problem 4, cont'd.)

5)(20 pts) **Mass on a lightly greased slotted turntable or spinning uniform rod.** Assume that the rod/turntable in the figure is massless and also free to rotate. Assume that at $t = 0$, the angular velocity of the rod/turntable is $1\,\mathrm{rad/s}$, that the radius of the bead is one meter, and that the radial velocity of the bead, dR/dt, is zero. The bead is free to slide on the rod. Where is the bead at $t = 5\,\mathrm{sec}$?



(Work for problem 5, cont'd.)

Your Name: ____________________

T&AM 203 Prelim 1

Tuesday Sept 26, 2000 7:30 — 9:00$^+$ PM

Draft September 26, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Two pages of formulas, from the front of the text, and a blank page for tentative scrap work are provided at the back. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`".

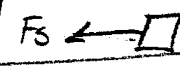
c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	____/40
Problem	2:	____/30
Problem	3:	____/30
TOTAL:		____**/100**

1) (40 pts)

1a) (15 pts) A mass m is connected to a spring k and launched from its static equilibrium position at a speed of v_0. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? (Answer in terms of some or all of m, k, and v_0.) [Neglect gravity and friction.] $\mapsto$ $x=0$ AT EQUILIBRIUM

F.B.D $\qquad F_s = kx$



$\underline{\dot{L}} = m\ddot{x} = -kx \Rightarrow \ddot{x} + k/m\, x = 0$ $\qquad$ I.C.'s $x(0)=0$, $\dot{x}(0) = V_0$

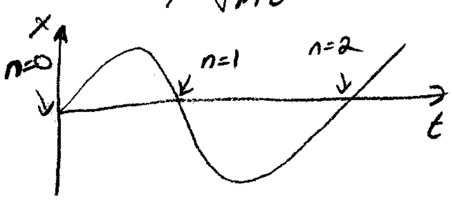
GENERAL SOLUTION $\Rightarrow x(t) = A\cos(\sqrt{\tfrac{k}{m}}t) + B\sin(\sqrt{\tfrac{k}{m}}t)$

$\dot{x}(t) = -\sqrt{\tfrac{k}{m}}A\sin(\sqrt{\tfrac{k}{m}}t) + \sqrt{\tfrac{k}{m}}B\cos(\sqrt{\tfrac{k}{m}}t)$

$$\left.\begin{aligned} x(0) &= A = 0 \Rightarrow A = 0 \\ \dot{x}(0) &= \sqrt{\tfrac{k}{m}}B = V_0 \Rightarrow B = V_0\sqrt{m/k} \end{aligned}\right\} \Rightarrow x(t) = V_0\sqrt{\tfrac{m}{k}}\sin(\sqrt{\tfrac{k}{m}}t)$$

NEED TO FIND WHEN $x(t) = 0 \Rightarrow V_0\sqrt{\tfrac{m}{k}}\sin(\sqrt{\tfrac{k}{m}}t) = 0 \Rightarrow \sin(\sqrt{\tfrac{k}{m}}t) = 0$

$\Rightarrow \sqrt{\tfrac{k}{m}}t = n\pi$

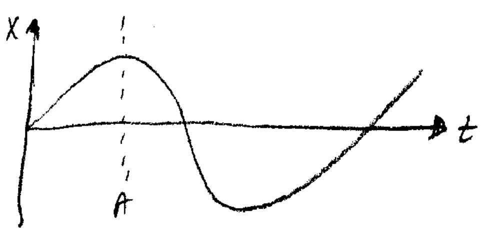


$\sqrt{\tfrac{k}{m}}\,t = 2\pi,\ \boxed{t = 2\pi\sqrt{\tfrac{m}{k}}}$

1b) (10 pts)

For the mass above, how far does the mass move from the launch position before it first reverses its velocity?

FROM 1a), $x(t) = V_0\sqrt{\tfrac{m}{k}}\sin(\sqrt{\tfrac{k}{m}}t)$



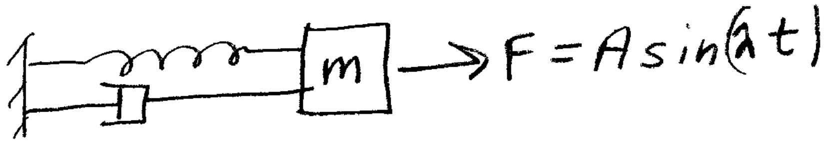
REVERSES VELOCITY AT $t=A$, ITS POSITION CORRESPONDS TO THE AMPLITUDE OF THE MOTION

THE AMPLITUDE OF $x(t)$ IS $V_0\sqrt{m/k}$,

SO $\boxed{d = V_0\sqrt{\tfrac{m}{k}}}$

1c) (15 pts)

A mass $m = 1\,\mathrm{kg}$ is held in place by a spring $k = 1\,\mathrm{N/m}$ and dashpot $c = 1\,\mathrm{N/(m/s)}$. An oscillating force is applied of $F = A\sin(\lambda t)$, with $A = 1\,\mathrm{N}$ and $\lambda = 1/\mathrm{s}$. After any initial transients have died down, how far does the mass go back and forth (the distance from one extreme to the other)?



$\rightarrow \underline{i}$

$\rightarrow x$

relaxed position of spring

FBD: $T_S \leftarrow \square \rightarrow F$, $T_D \leftarrow$

LMB: $\{\sum \underline{F} = \dot{\underline{L}}\}$

$$-T_S - T_D + F = m\ddot{x}$$

$$\{\}\cdot\underline{i} \Rightarrow -kx - c\dot{x} + A\sin\lambda t = m\ddot{x}$$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = A\sin\lambda t$$

$$\Rightarrow \ddot{x} + \dot{x} + x = \sin(t) \quad \text{[in consistant m, kg, s units]}$$

Guess steady soln. of form: $x = B\cos t + C\sin t$

[why? Because it works]

$$\ddot{(B\cos t + C\sin t)} + \dot{(B\cos t + C\sin t)} + (B\cos t + C\sin t) = \sin t$$

$$(-B\cos t - C\sin t) + (-B\sin t + C\cos t) + (B\cos t + C\sin t) = \sin t$$

Collect sine & cosine terms [eg, $\int_0^{2\pi}\{\ \}\cos(t)\,dt$ plucks out cosine]

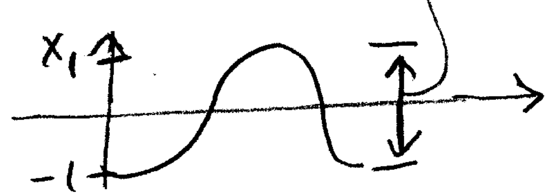
$$-B + C + B = 0 \quad \text{(cos terms)}$$

$$\Rightarrow C = 0$$

$$-C - B + C = 1 \quad \text{(sin terms)}$$

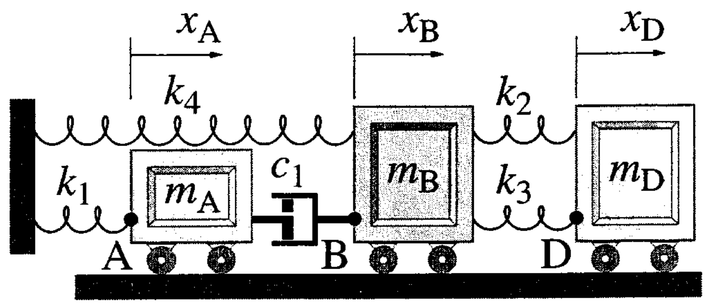
$$\Rightarrow B = -1$$

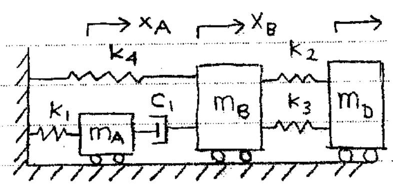
$$\Rightarrow x = -\cos(t)$$

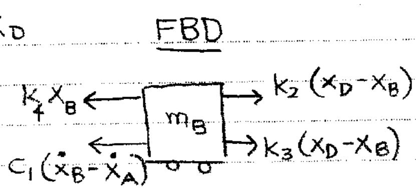


$\boxed{X_{dist} = 2\,\mathrm{m}}$

2)(30 pts) A system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$. Given k_1, k_2, k_3, k_4, c_1, m_A, m_B, m_D, x_A, x_B, x_D, $\dot{x}_A$, $\dot{x}_B$, and $\dot{x}_D$, find the acceleration of mass B, $\underline{\mathbf{a}}_B = \ddot{x}_B\,\hat{\mathbf{i}}$.







LMB: $\{\Sigma \underline{F} = m_B \underline{a}_B\} \cdot \hat{i}$

$$\Rightarrow -k_4 x_B - c_1(\dot{x}_B - \dot{x}_A) + (k_2 + k_3)(x_D - x_B) = m_B \ddot{x}_B$$

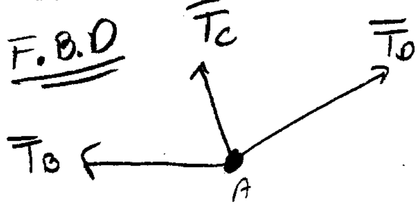
$$\therefore \ddot{x}_B = \frac{1}{m_B}\left[c_1 \dot{x}_A - c_1 \dot{x}_B - (k_2 + k_3 + k_4) x_B + (k_2 + k_3) x_D\right]$$

$$\underline{a}_B = \ddot{x}_B \hat{i}$$

3) (30 pts)

In three-dimensional space with no gravity a particle with $m = 3\,\mathrm{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\underline{\mathbf{a}} = (a\,\hat{\mathbf{i}})\,\mathrm{m/s^2}$ where a is not yet known. The tension in AB is 4 N The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the code to find a. The last line should read `a = ...` with `a` being assigned to the acceleration in the $\hat{\mathbf{i}}$ direction.

F.B.D



$\Sigma = \vec{T}_B + \vec{T}_C + \vec{T}_D = m\vec{a} = 3a\vec{i}$ m/s

$|\vec{T}_B| = T_B = 4N$

DEFINE UNIT VECTORS $\vec{T}_B = T_B\vec{\lambda}_B$, $\vec{T}_C = T_C\vec{\lambda}_C$, $\vec{T}_D = T_D\vec{\lambda}_D$

DEFINE $\vec{b}$ S.T. $m\vec{a} = a\vec{b}$ $\{\vec{b} = (m\hat{i})\}$

THEN $T_B\vec{\lambda}_B + T_C\vec{\lambda}_C + T_D\vec{\lambda}_D = a\vec{b}$ → NOW PUT ALL UNKNOWNS ON LEFT KNOWNS ON RIGHT

$$\Rightarrow T_C\vec{\lambda}_C + T_D\vec{\lambda}_D - a\vec{b} = -T_B\vec{\lambda}_B$$

$$\Downarrow$$

$$\left[\vec{\lambda}_C \middle| \vec{\lambda}_D \middle| -\hat{b}\right]\begin{bmatrix} T_C \\ T_D \\ a \end{bmatrix} = -T_B\left[\vec{\lambda}_B\right]$$

```
% a MATLAB script file to find 3 tensions
m   = 3;
a   = [ 1 2 3]';

rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';

uAB = rAB/norm(rAB); % norm gives vector magnitude
% You write the code below (however many lines you need).
% Don't copy any of the numbers above.
% Don't do any arithmetic on the side.
uAC=rAC/norm(rAC);
uAD=rAD/norm(rAD);
b=[m 0 0]';
Tb=4;
r=-Tb*uAB;
A=[uAC uAD -b];
res=A\r;
a=res(3);
```

"Solutions"

Your Name: ANDY RUINA

T&AM 203 Prelim 2

Tuesday Oct 24, 2000 7:30 — 9:00$^+$ PM

Draft October 16, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
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- work is I.) neat,
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- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot\theta_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/40
Problem	2:	/30
Problem	3:	/30
TOTAL:		**/100**

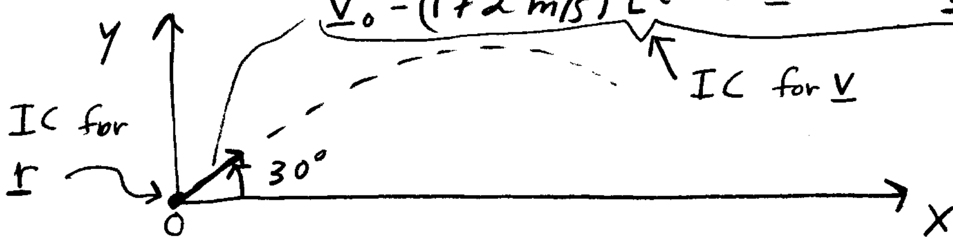
1)(40 pts) **Projectile motion.** Someone in the mideast shot a projectile at someone else. The basic facts:

- Launched from the origin.
- Projectile mass = 1 kg.
- Launch angle 30° above horizontal.
- Launch speed 172 m/s.
- Drag proportional to cv^2 with $c = .01$ kg / m.
- Gravity $g = 10$ m/s.

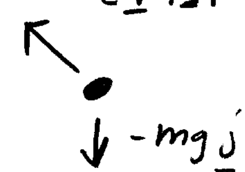
a) (25 pts) Write MATLAB code to find the height at $t = 1$ s. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, num setup, find height at 1 s].

b) (15 pts) Estimate the height at $t = 1$ s using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the first second of your solution.]

$$\underbrace{\underline{V}_0 = (172\ m/s)\,[\cos 30°\,\underline{i} + \sin 30°\,\underline{j}]}_{\text{IC for } \underline{V}}$$



FBD:



$V = |\underline{V}| = (\dot{x}^2 + \dot{y}^2)^{1/2}$

LMB: $\sum \underline{F} = m\underline{a}$

$$\{-c\underline{V}\,V - mg\underline{j} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j})\}$$

$$\left.\begin{array}{ll} \{\,\}\cdot\underline{i} \Rightarrow \dot{V}_x = -cV_xV/m, & \dot{x} = V_x \\ \{\,\}\cdot\underline{j} \Rightarrow \dot{V}_y = -cV_yV/m - g, & \dot{y} = V_y \end{array}\right\}\text{ODEs}$$

driver

```
x0=0; y0=0;
vx0 = 172*cos(pi*30/180);
vy0 = 172*sin(pi*30/180);
z0=[x0 y0 vx0 vy0]';
tspan=[0 1];
[t z]=ode45('sadam', tspan, z0);
height= z(end,2)
```

(gives ≈ 46.5 m)

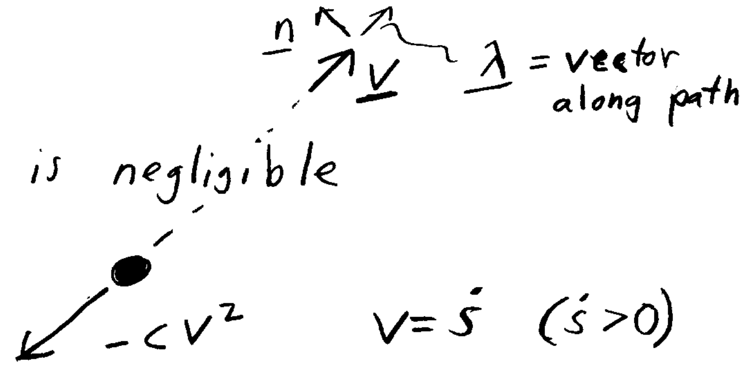
in file sadam.m

```
function zdot= sadam(t,z)
c=.01; m=1; g=10;
x=z(1); y=z(2);
vx=z(3); vy=z(4);
xdot=vx; ydot=vy;
v=(vx^2+vy^2)^.5;
vxdot=-c*vx*v/m;
vydot=-c*vx*v/m - g;
zdot=[xdot ydot...
      vxdot vydot]';
```

(work for problem 1, cont'd.)

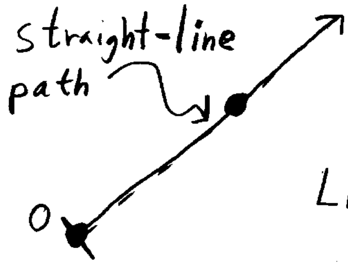
b) Assume gravity is negligible

$\Rightarrow$ FBD:



$V = \dot{s} \quad (\dot{s} > 0)$

No force in $\underline{n}$ dir. ($\perp$ to path) $\Rightarrow$ straight line motion



s = arclength in dir. of motion

LMB: $m\dot{V} = -cV^2$ ①

First solve ①: $\frac{dV}{dt} = \frac{-c}{m}V^2 \Rightarrow \frac{dV}{V^2} = \frac{-c}{m}dt$

$\Rightarrow \quad +V^{-1} = \frac{+c}{m}t + C$

IC: $V(t=0) = V_0 \Rightarrow C = \frac{1}{V_0} \Rightarrow V = \frac{1}{\frac{1}{V_0} + \frac{c}{m}t} = V_0 \frac{1}{1 + \frac{cV_0}{m}t}$

$\Rightarrow \quad \dot{s} = V_0 \frac{1}{1 + \frac{cV_0}{m}t} \Rightarrow ds = V_0 \frac{dt}{1 + \frac{cV_0}{m}t}$

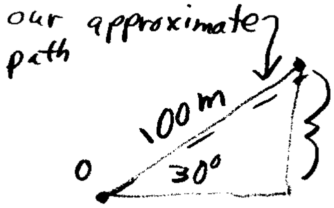
$\Rightarrow \quad s = \frac{V_0 m}{cV_0} \ln\left(1 + \frac{cV_0}{m}t\right) + C$

IC: $s(t=0) = 0 \Rightarrow C = 0 \Rightarrow \boxed{s = \frac{m}{c}\ln\left(1 + \frac{cV_0}{m}t\right)}$

Plug in #s: $s = \frac{1\text{kg}}{.01\text{ kg/m}} \ln\left(1 + \frac{(.01\text{ kg/m})(172\text{ m/s})}{1\text{kg}}(1\text{s})\right)$

$= 100\text{m} \cdot \ln(1 + 1.72) = 100\text{m} \ln(2.72)$

$\approx 100\text{m} \cdot \ln(e) = 100\text{m}$



actual path

"exact" Matlab ans. is 46.5m

$h \approx 100\text{m} \sin(30^\circ) = \boxed{50\text{ m}}$

at end ($t = 1$s) $|\dot{V}| = cV^2 = .01\left(\frac{172}{2.72}\right)^2 = \left(\frac{1.72}{2.72}\right)^2 172$

(in m/s²) $|\dot{V}| >> 10 = g$ (no big error to neglect g)

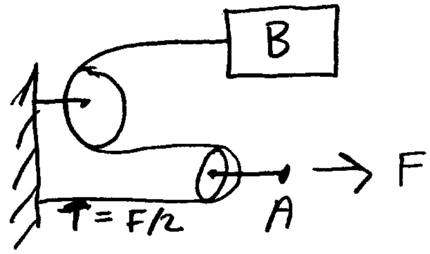
2)(30 pts) Design a pulley system. You are to design a pulley system to move a mass. There is no gravity. Point A has a force $\underline{\mathbf{F}} = F\hat{\mathbf{i}}$ pulling it to the right. Mass B has mass m_B. You can connect the point A to the mass with any number of ideal strings and ideal pulleys. You can make use of rigid walls or supports anywhere you like (say, to the right or left of the mass). You must design the system so that the mass B accelerates to the left with $\frac{F}{2m_B}$ (i.e., $\underline{\mathbf{a}}_B = -\frac{F}{2m_B}\hat{\mathbf{i}}$).

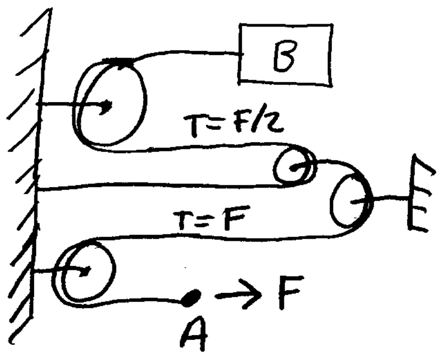
a) (25 pts) Draw the system clearly. Justify your answer with enough words or equations so a reasonable person, say a grader, can tell that you understand your solution.

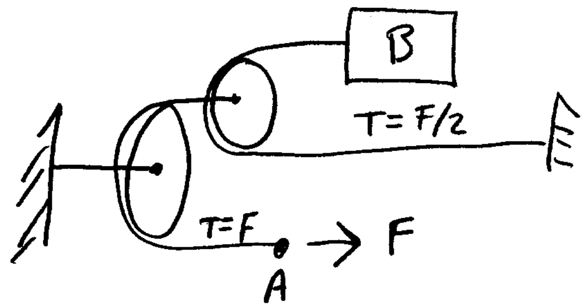
b) (5 pts) Find the acceleration of point A.

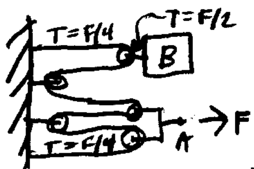
$\rightarrow \underline{\hat{i}}$

a) Some solns:









etc...

In all cases m_B pulled to left with tension $= F/2 \Rightarrow a_B = -\frac{F}{2m_B}\hat{\underline{i}}$

b) Power balance ⇒ $T_B(-V_B) = T_A V_A$

V_B = vel. of B to the right

$$\left\{\frac{F}{2}(-V_B) = F V_A\right\}$$

$\frac{d}{dt}(*) \Rightarrow$

$$\frac{F}{2}(-a_B) = F a_A$$

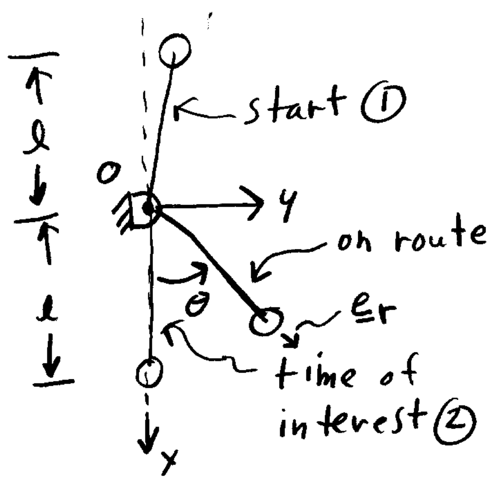
$$a_A = \frac{-a_B}{2} = \frac{-(-F/2m_B)}{2}$$

$$\boxed{a_A = \frac{F}{4m_B}}$$

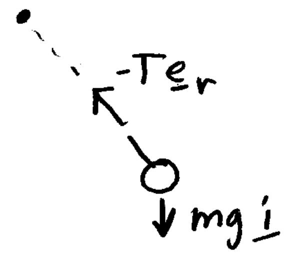
3)(30 pts) Tension in pendulum. 2D. A simple pendulum consists of a point mass m connected by a rigid massless rod with length ℓ t. a frictionless hinge at O. The only applied force is from gravity. It is released from a vertical orientation with the mass directly *above* the hinge. It is pushed very slightly to the right (with a velocity that you can assume is arbitrarily small) and thus slowly at first falls, then quickly swings through the vertically down orientation and then back up on the left side.

At the instant when the mass passes through the vertically down position (mass directly below hinge) what is the tension in the rod? (i.e., find T in terms of m, ℓ and g).

If you choose a MATLAB solution instead of pencil and paper (not required, just an option) use $m = 3\,\mathrm{kg}$, $\ell = 2\,\mathrm{m}$ and $g = 10\mathrm{m/s}^2$



FBD:



[just like example in lecture]

$$\underbrace{\Delta E_K}_{\text{increase in kin. energy from state ① to state ②}} = \overset{\text{work of gravity force}}{W}$$

(tension does not work because $T\underline{e}_r \perp \underline{v} = \dot\theta r \underline{e}_\theta$)

$$\Rightarrow \underbrace{\tfrac{1}{2} m v_2^2}_{\text{K.E. in state 2}} = 2mg\ell$$

$$\boxed{v_2^2/\ell = 4g} \; *$$

Instead of using energy one could do this:

$$\ddot\theta + (g/\ell)\sin\theta = 0$$
$$\ddot\theta\dot\theta + (g/\ell)\dot\theta\sin\theta = 0$$
$$\frac{d}{dt}(\dot\theta^2/2) + \frac{g}{\ell}\frac{d}{dt}(-\cos\theta) = 0$$
$$\dot\theta^2/2 - \frac{g}{\ell}\cos\theta = \text{Const}$$
$$\left.\begin{matrix}\text{at } \theta = \pi \\ \dot\theta = 0\end{matrix}\right] \Rightarrow \dot\theta^2/2 - (g/\ell)\cos\theta = g/\ell$$

LMB in state ②:

$$\sum \underline{F} = m\underline{a}$$

$$\left\{ -T\underline{i} + mg\underline{i} = m\left[\frac{v_2^2}{\ell}(-\underline{i}) + \ddot\theta\ell\underline{j}\right] \right\}$$

$$\{\ \}\cdot\underline{i} \Rightarrow T = mg + \frac{mv_2^2}{\ell} = mg + 4mg = \boxed{T = 5mg}$$

Tension is 5 times the weight

(Work for problem 3, cont'd.)

Alternative Matlab soln. to 3

(Use FBD from before)

AMB/o: $\sum \underline{M}_{/0} = \dot{\underline{H}}_{/0} \Rightarrow \ell\underline{e}_r \times (mg\underline{i}) = \ell\underline{e}_r \times m[\ell\ddot{\theta}\underline{e}_\theta - \dot{\theta}^2\ell\underline{e}_r]$

$$\Rightarrow -\ell mg\sin\theta = \ell^2 m\ddot{\theta}$$

$$\Rightarrow \ddot{\theta} = -(g/\ell)\sin\theta$$

$$\Rightarrow \left.\begin{array}{ll} 2) & \dot{\omega} = -(g/\ell)\sin\theta \\ 1) & \dot{\theta} = \omega \end{array}\right] \text{2 first order ODEs}$$

```
m= 3; l = 2; g=10;
theta0 = -pi;
omega0 = -.001;
z0 =[theta0   omega0]';
tspan= [0:.001:10]; % long enough, lots of pts.
[t z]= ode23('pendrhs', tspan, z0);
omegamax = max(z(:,2));
Tmax = m*(g + l*omegamax^2)
```

solve ODEs for long enough time

max $\dot{\omega}$ will be near bottom

calculate tension

pendrhs.m

```
function  zdot= pendrhs (t,z)
m=3; l=2; g=10
 theta= z(1); omega= z(2);
 thetadot= omega;
 omegadot= -(g/l)*sin(theta);
 zdot= [thetadot  omegadot]';
```

pend ODEs.

[This gives an answer of 149.7, close to 5·3·10]

"SOLUTIONS" Your Name: ANDY RUINA

T&AM 203 Final Exam

Friday December 17, 2004

Draft December 17, 2004

5 problems, 150 minutes (no extra time).

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems **poorly defined**;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot\theta_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors. If you cannot answer a problem with pencil and paper, you can get partial credit for a good Matlab solution.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

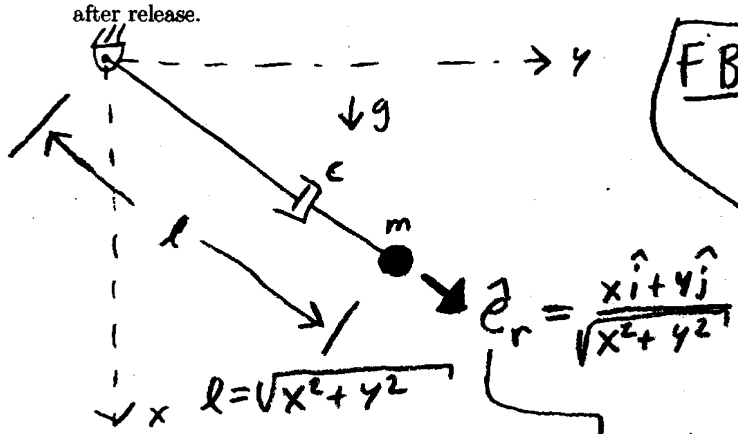
d) Even if not asked for, you can get partial credit by showing a Matlab solution to a problem you can't solve with pencil and paper.

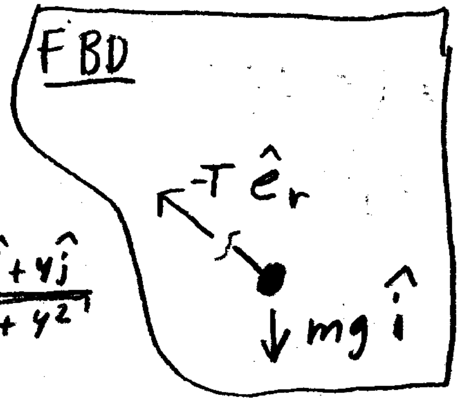
Problem 1:	/20
Problem 2:	/20
Problem 3:	/20
Problem 4:	/20
Problem 5:	/20
TOTAL:	**/100**

1) **(20 pt)** In the almost new sport of spongy jumping a spring is replaced by a dashpot c. Assume $m = 3\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, and $c = 7\,\mathrm{kg/s}$. The mass is released from rest at $x = 2\,\mathrm{m}$, $y = 3\,\mathrm{m}$.

a) (15 points) What are the equations of motion for this system (differential equations involving x and y and their derivatives)? (For this part of the problem please use m, c and g rather than their numerical values.

b) (5 points) *This part will only be graded if part (a) is almost entirely correct.* Write Matlab code that would give the arc-length of the center of mass trajectory over the first 5 seconds after release.





LMB: $\sum \underline{F}_i = m\underline{a} \Rightarrow -T\hat{e}_r + mg\hat{j} = m\underline{a}$, $\quad \underline{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$

$$T = c\dot{\ell} = c\left(\frac{\partial \ell}{\partial x}\dot{x} + \frac{\partial \ell}{\partial y}\dot{y}\right) = c\frac{1}{\sqrt{x^2+y^2}}(x\dot{x}+y\dot{y})$$

$$\Rightarrow \left\{\left(-c\frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}\right)\frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} + mg\hat{i} = (\ddot{x}\hat{i} + \ddot{y}\hat{j})m\right\}$$

$$\{\ \}\cdot\hat{i} \Rightarrow \quad \ddot{x} = \frac{-cx}{m}\frac{(x\dot{x}+y\dot{y})}{(x^2+y^2)} + g$$

$$\{\ \}\cdot\hat{j} \Rightarrow \quad \ddot{y} = \frac{-cy}{m}\frac{(x\dot{x}+y\dot{y})}{(x^2+y^2)}$$

$\dot{s} = |\underline{v}|$

$\dot{v}_x = \ddot{x}$ (from above)

$\dot{v}_y = \ddot{y}$ (from above)

$\dot{x} = v_x$

$\dot{y} = v_y$

$z_1 = x$
$z_2 = y$
$z_3 = v_x$
$z_4 = v_y$
$z_5 = s$

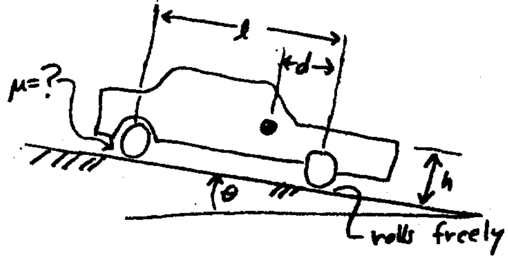
```
        x0  y0   v0   s0
z0 = [2  3  0 0  0]'.
tspan = [0  5];

[t z] = ode23('spongy', tspan, z0);
arclength = z(end, 5);
```

```
function   zdot = spongy(t, z)
m = 3;  c = 7;  g = 10;
x = z(1);  y = z(2);
vx = z(3);  vy = z(4);

xdot = vx;
ydot = vy;
  L2 = x^2 + y^2;
  D  =   c*(x*vx + y*vy)/(m*L2);
vxdot = -x*D + g;
vydot = -y*D;
sdot = sqrt(vx^2 + vy^2);
zdot = [xdot, ydot, vxdot, vydot, sdot]';
```

2) (20 pt) A car going down a hill of slope θ (measured from the horizontal) puts on its rear brakes, causing the rear wheels to skid. The negligible-mass front wheels roll freely. The car moment of inertia about its center of mass G is I and its mass is m. The wheels are a distance ℓ apart (front to back distance). On level ground G is a height h above the ground and a distance d behind the front wheel. In terms of some or all of m, I, g, d, h, ℓ and θ find the minimum coefficient of friction μ needed to slow the car down.

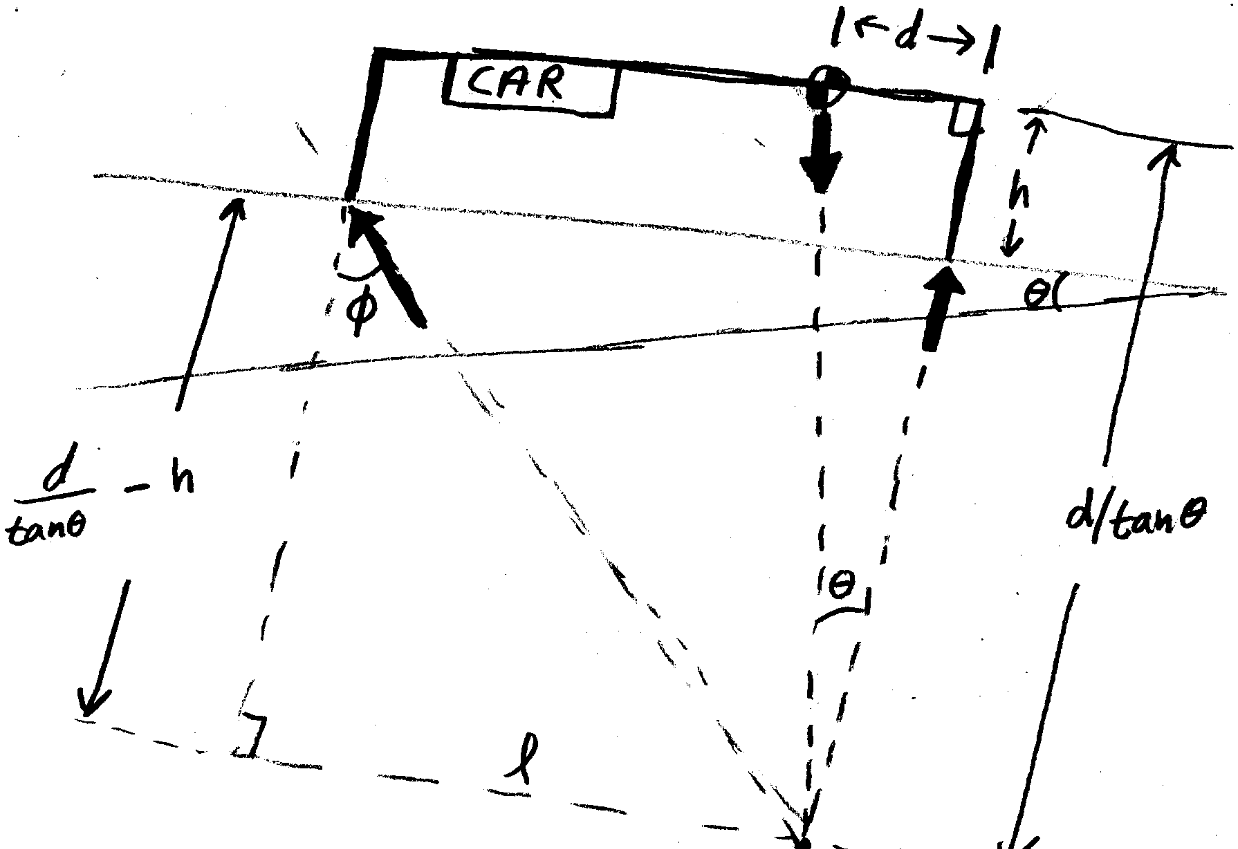


at critical μ

$\underline{a=0} \Rightarrow \underline{\text{statics}}$

3-force body
$\Rightarrow$ all forces intersect at one point

FBD



$$\tan\phi = \boxed{\mu = \frac{\ell}{\frac{d}{\tan\theta} - h}}$$

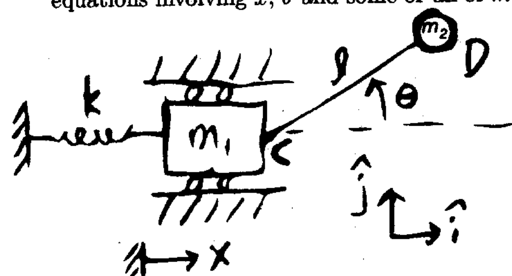
Sanity checks

$\theta \to 0 \Rightarrow \mu \to 0$ ✓

$h=0,\ d=\ell \Rightarrow \mu = \tan\theta$ ✓

$d \to (\tan\theta) h \Rightarrow \mu \to \infty$ ✓

3) (20 pt) m_1 slides horizontally with x measuring the stretch of the spring k from its unstretched length. Point mass m_2 is at the end of a massless rod of length ℓ the other end of which is hinged on m_1. Neglect gravity. Find differential equations that govern the motion of the two masses (differential equations involving x, θ and some or all of m_1, m_2 and ℓ, k.

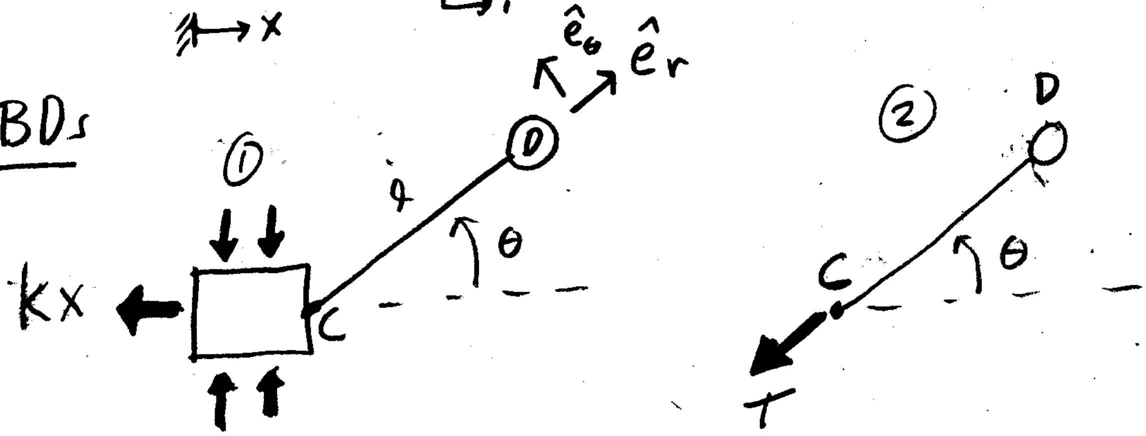


$\hat{e}_\theta \cdot \hat{i} = -\sin\theta$

$\hat{e}_r \cdot \hat{i} = \cos\theta$

$\hat{e}_r \times \hat{i} = -\sin\theta$

FBDs



FBD 1 $\quad \{\sum \underline{F}_i = m\underline{a}_i\} \cdot \hat{i}$

$$-kx = m_1\ddot{x} + m_2\left(\ddot{x} + (\ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_r)\cdot\hat{i}\right)$$

$$\boxed{-kx = (m_1+m_2)\ddot{x} - \ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta} \quad ①$$

FBD 2 $\quad \underline{AMB}_{/C} \Rightarrow$

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$$

$$= \underline{a}_C + \underline{a}_{D/C}$$

$$\underline{0} = \underline{r}_{D/C} \times (m\underline{a}_D)$$

$$= \ell\hat{e}_r \times m\left[(\ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_r) + \ddot{x}\hat{i}\right]$$

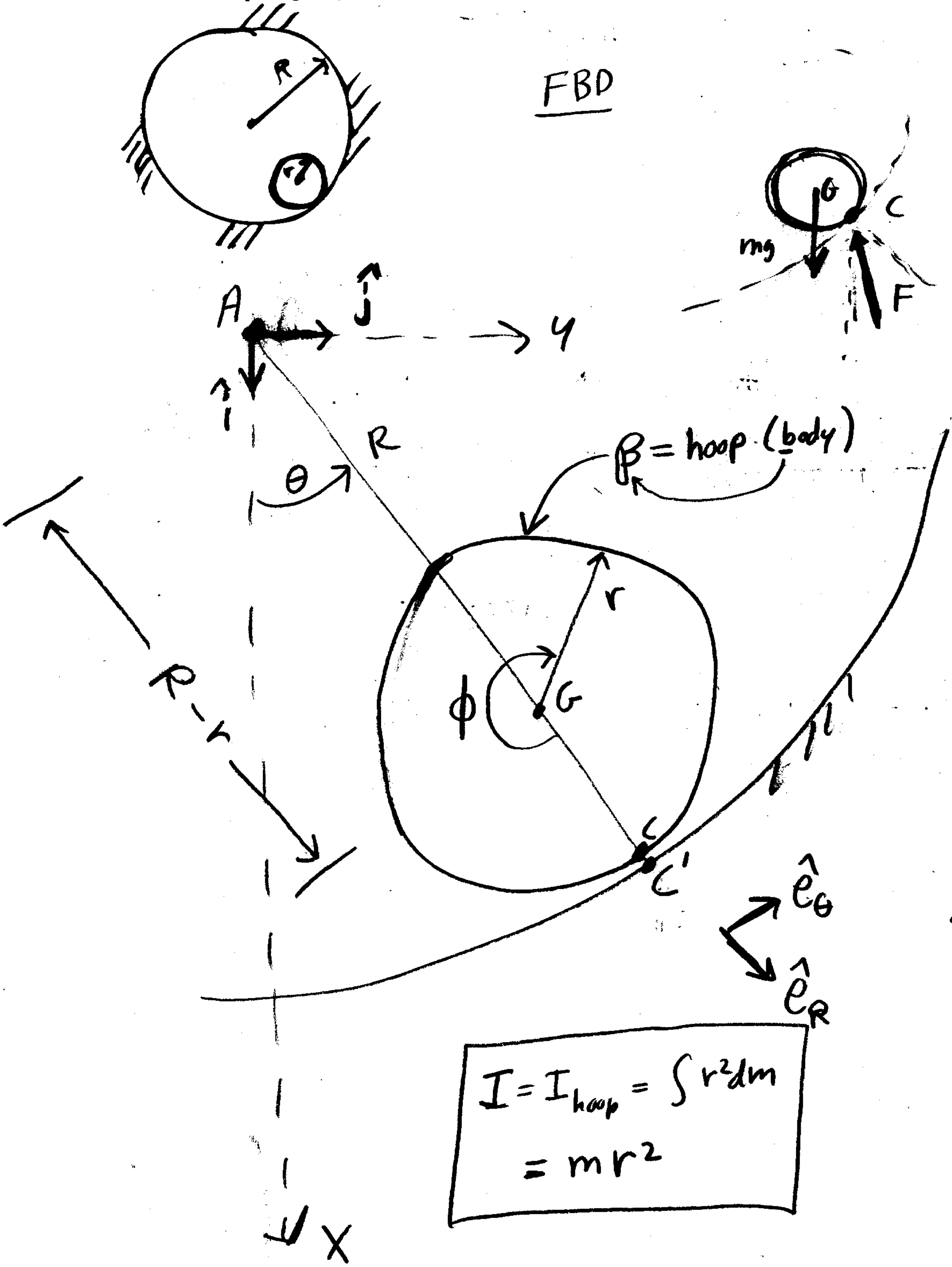
$$\left\{ = \ell m\ddot{x}(\sin\theta)\hat{k} + \ell^2 m\ddot{\theta}\hat{k} \right\}$$

$\{ \ \} \cdot \hat{k} \Rightarrow \quad \boxed{\ddot{\theta} - \frac{\ddot{x}}{\ell} \sin\theta = 0}$ (2)

[looks just like the pendulum eqn. but with $-\ddot{x}$ instead of g. Why? pendulum can't distinguish g from an accelerating frame.]

(1) & (2) are 2 coupled 2nd order ODEs for θ, x.

4) (20 pt) Consider a hollow thin-walled pipe with mass m and radius r. The friction coefficient μ is large enough so the pipe rolls without slip inside a rigid stationary hollow pipe with radius R. Find the period of small oscillations (near the bottom) in terms of some or all of m, r, R, μ and the downwards-point gravity constant g.



$$\boxed{I = I_{hoop} = \int r^2 dm = mr^2}$$

Kinematics:

rolling contact $\Rightarrow$ $r\phi = R\theta$ ①

$$\underline{\omega} = \underline{\omega}_{hoop} = \underline{\omega}_{\beta} = (\dot{\theta} - \dot{\phi})\hat{k} = \left(\dot{\theta} - \frac{R}{r}\dot{\theta}\right)\hat{k}$$

$$\underline{\omega}_{\beta} = (1 - R/r)\dot{\theta}\hat{k}$$

$$\underline{a}_G = \ddot{\theta}(R-r)\hat{e}_\theta - \dot{\theta}^2(R-r)\hat{e}_R$$

AMB$_{/C}$:

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$$

$$\underline{r}_{G/C} \times mg\hat{i} = \underline{r}_{G/C} \times m\underline{a}_G + I\dot{\underline{\omega}}_\beta$$

$$r\sin\theta\, mg\hat{k} = (-r\hat{e}_R) \times \left[\ddot{\theta}(R-r)\hat{e}_\theta - \dot{\theta}^2(R-r)\hat{e}_R\right] + I\ddot{\theta}(1-R/r)\hat{k}$$

$$\{ \quad = -r(R-r)\ddot{\theta}\hat{k}m + I\ddot{\theta}(1-R/r)\hat{k} \}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad rmg\sin\theta = -\left[r(R-r)m + \left(\frac{R}{r} - 1\right)I\right]\ddot{\theta}$$

$$= -\underbrace{(I + mr^2)}_{I = mr^2}\left(\frac{R}{r} - 1\right)\ddot{\theta}$$

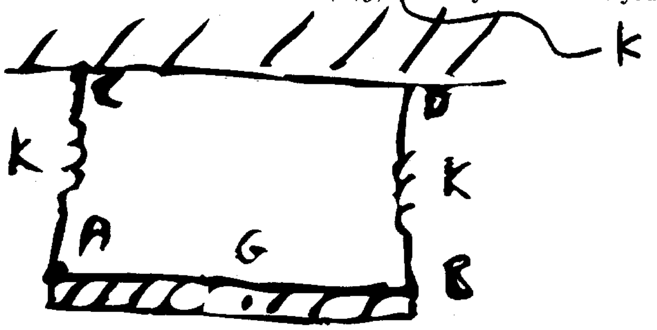
$$\Rightarrow \quad \ddot{\theta} + \frac{\cancel{r}\cancel{m}g}{2\cancel{m}r^{\cancel{2}}(R/r - 1)}\sin\theta = 0$$

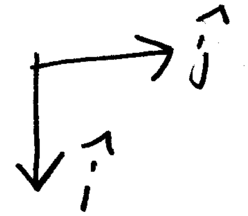
$$\underset{\theta \to 0}{\Rightarrow} \quad \ddot{\theta} + \underbrace{\left[\frac{g}{2(R-r)}\right]}_{\lambda^2}\theta = 0$$

$$\Rightarrow \theta = A\sin(\lambda t) + B\cos\lambda t$$

$$\lambda t_p = 2\pi \Rightarrow \boxed{t_p = \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{2(R-r)}{g}}}$$

5) (20 pt) A uniform bar AB with length ℓ and mass m is hanging (gravity constant $= g$) in stationary equilibrium from two identical springs k. Suddenly but gently spring A~~B~~D is cut by a laser beam. Immediately after the cut, what is the acceleration of the rod center at G. Answer in terms of some or all of m, k, g, ℓ and any base vectors you define with clear sketches.

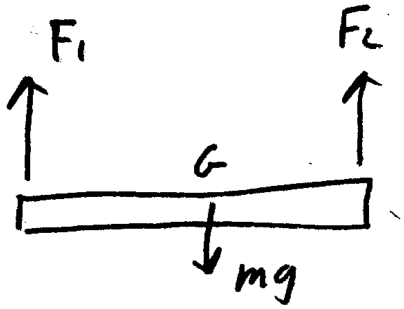




FBDs

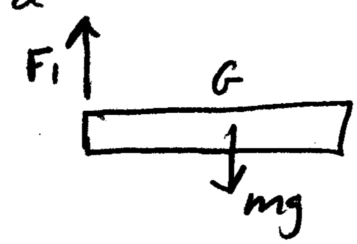
before

(statics)



$\Rightarrow F_1 = mg/2$

after



$\sum \underline{F}_i = m\underline{a}_G$

$-F_1\hat{i} + mg\hat{i} = m\underline{a}_G$

$\left(-\frac{mg}{2} + mg\right)\hat{i} = m\underline{a}_G$

$\boxed{\underline{a}_G = \frac{g}{2}\hat{i}}$

Your Name: ____________________

T&AM 203 Makeup Prelim

Monday December 6, 2004 noon - 1:30$^+$ PM

Draft December 6, 2004. From the Fall 1996 Final exam

Do any 3 problems.

Please follow these directions to ease grading and to maximize your score.

a) No ca1cu1at0rs allowed. Ask for extra scrap paper if you need it.

b) Do any 3 problems.

c) Full credit if

- work is
  - I.) neat,
  - II.) clear, and
  - III.) well organized;

$\pm$ all signs and directions are well defined with sketches and/or words;

$\uparrow\rightarrow$ any dimensions, coordinates, variables and base vectors that you add are clearly defined;

- correct units and correct vector notation are used, when appropriate;
- to the extent that a problem seems ambiguous or not perfectly defined, you clearly state any reasonable assumptions that you make;

$\rightarrow\!|$ reasonable justification, enough to distinguish an informed answer from a guess, is given;

- $\longrightarrow$free body diagrams$\longleftarrow$ (FBD's) are drawn when appropriate;
- your answers are $\boxed{\text{boxed}}$ in; and
- your answers are TIDILY REDUCED.

d) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

$\gg$ MATLAB commands which would generate the desired answer count as a correct answer for all problems. *Some problems may only be only practically solvable with a computer.* You must be clear about how to interpret the MATLAB output as the answer to the question. If the problem statement is in terms of variables instead of numbers, MATLAB we will assume that the variables have been assigned values prior to the MATLAB commands you write.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25
Problem	4:	/25

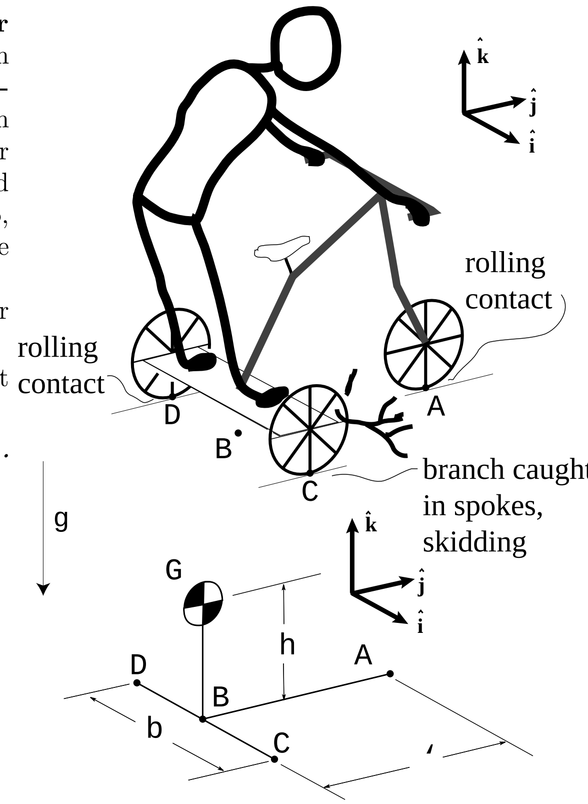
1) (25 pt) Speeding tricycle gets a branch caught in the right rear wheel. A scared-stiff tricyclist riding on level ground gets a branch stuck in the right rear wheel so the wheel skids with friction coefficient μ. Assume that the center of mass of the tricycle-person system is directly above the rear axle. Assume that the left rear wheel and the front wheel have negligible mass, good bearings, and have sufficient friction that they roll in the $\hat{\mathbf{j}}$ direction without slip, thus constraining the overall motion of the tricycle. Dimensions are shown in the lower sketch.

Find the accereration of the tricycle (in terms of some or all of ℓ, h, b, m, $[I^{cm}]$, μ, g, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$).

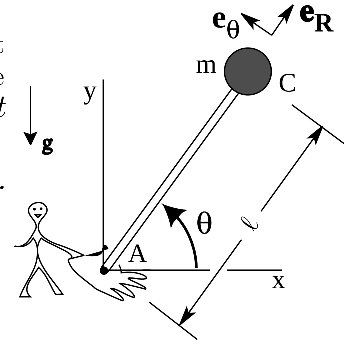
[Hint: check your answer against special cases for which you might guess the answer, such as when $\mu = 0$ or when $h = 0$.]

$\Leftarrow$ *Please put scrap work for problem 1 on the page to the left* $\Leftarrow$.

$\Downarrow$ *Put neat work to be graded for problem 1 below* $\Downarrow$.

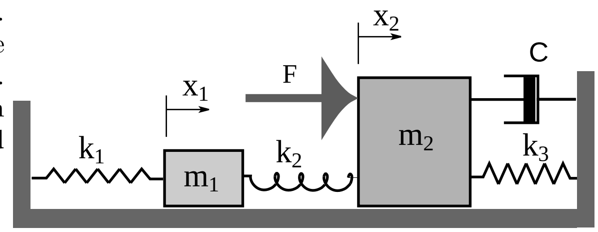


2) (25 pt) Balancing a broom. Assume the hand is accelerating to the right with acceleration $\mathbf{a} = a\,\hat{\mathbf{i}}$. What is the force of the hand on the broom in terms of m, ℓ, θ, $\dot{\theta}$, a, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and g? *(You may not have any $\hat{\mathbf{e}}_R$ or $\hat{\mathbf{e}}_\theta$ in your answer.)*



$\Leftarrow$ *Please put scrap work for problem 2 on the page to the left* $\Leftarrow$.
$\Downarrow$ *Put neat work to be graded for problem 2 below* $\Downarrow$.

3) (25 pt) Equations of motion. Two masses are connected to fixed supports and each other with the three springs and dashpot shown. The force F acts on mass 2. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when the springs are unstretched. The ground is frictionless. The governing equations for the system shown can be writen in first order form if we define $v_1 \equiv \dot{x}_1$ and $v_2 \equiv \dot{x}_2$.



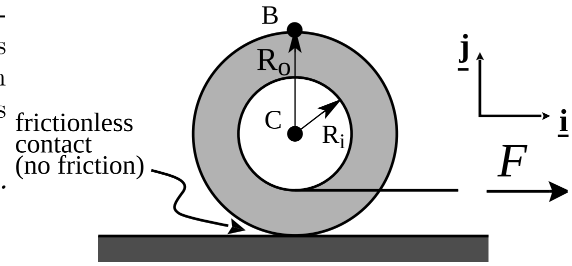
a) (10 points) Fill in the 16 terms of the 4×4 matrix below and the 4 terms of the blank column vector so that the equations are the correct equations for the system shown. Your answer should be in terms of any or all of the constants m_1, m_2, k_1, k_2,k_3, C, the constant force F, and t. Getting the signs right is important.

b) (10 points) Write MATLAB commands in appropriate functions and script files to find and plot $v_1(t)$ for 10 units of time. Make up appropriate initial conditions. If you need to use the big matrix you have defined at the bottom of the page indicate its place in your code, you need not copy it in for MATLAB term by term.

$\Leftarrow$ *Please put MATLAB code for problem 3 on the page to the left* $\Leftarrow$.
$\Downarrow$ *Put other neat work to be graded for problem 3 on this page* $\Rightarrow$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

4) (25 pt) **The film**, *Heat Treatment of Aluminum*, is placed on a very slippery table. Assume that the film and reel (together) have mass distributed the same as a uniform disk of radius R_i. What, in terms of R_i, R_o, m, g, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and F are the accelerations of points C and B at the instant shown (the start of motion)?



$\Leftarrow$ *Please put scrap work for problem 4 on the page to the left* $\Leftarrow$.

$\Downarrow$ *Put neat work to be graded for problem 4 below* $\Downarrow$.

"SOLUTIONS"

Your Name: ANDY RUINA

Section day and time: ____________

T&AM 203 Prelim 1

Tuesday Sept 28, 2004

Draft September 27, 2004

3 problems, 25 points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

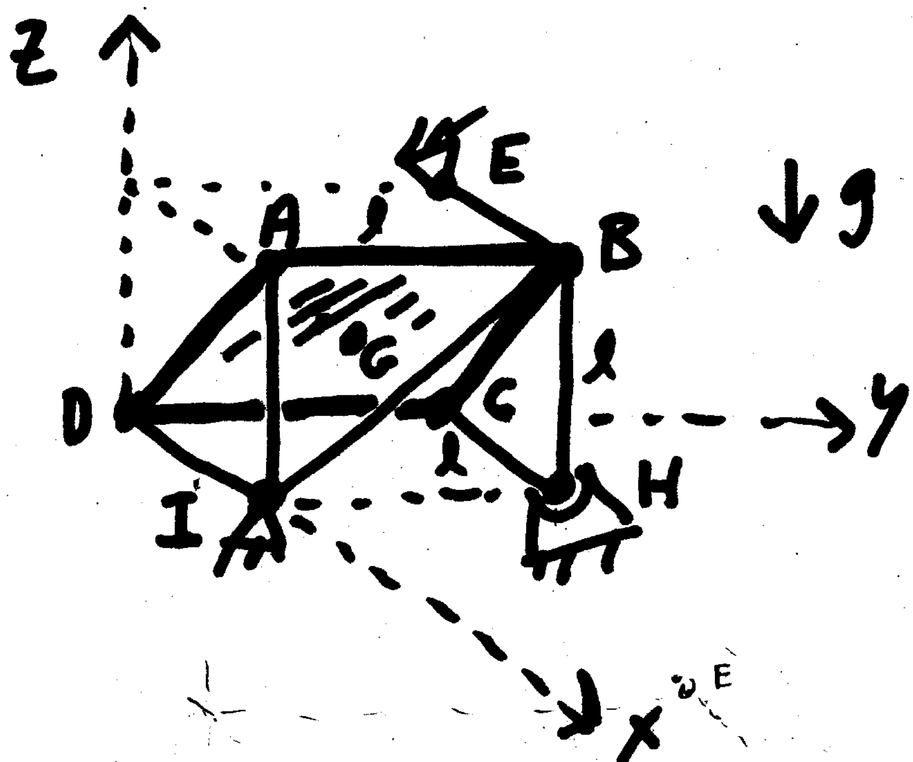
b) Full credit if:

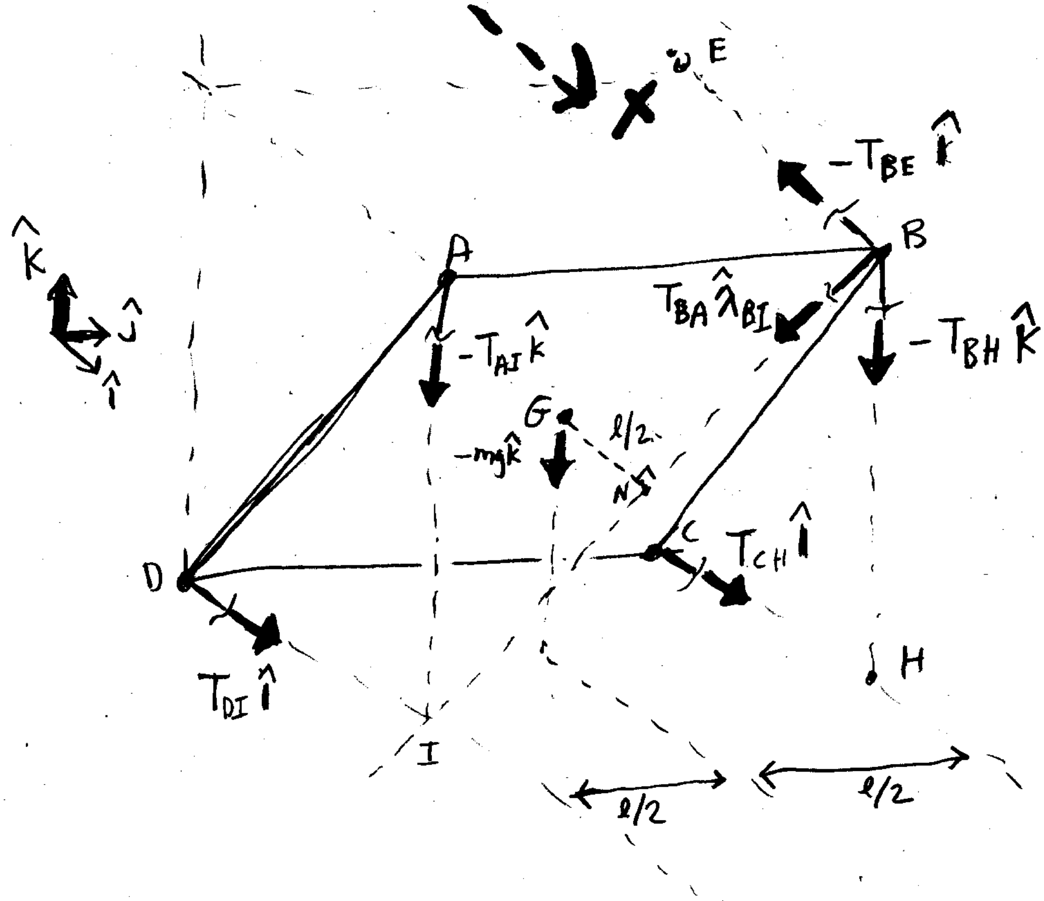
- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25

1) (25 pt) Statics. The uniform plate ABCD with mass m is held up by 6 bars (EB, HB, HC, IB, IA & ID). Find the tension in *any three* of these bars. Answer in terms of some or all of m, g, and ℓ.





$$\{\sum \underline{F}_i = \underline{0}\} \cdot \hat{j} \Rightarrow \quad T_{BI}\, \hat{\lambda}_{BI} \cdot \hat{j} = 0 \Rightarrow \boxed{T_{BI} = 0}$$

(no other forces have $\hat{j}$ components)

$$\underset{\uparrow \text{axis}}{\sum M_{/DI}} = 0 \quad \Rightarrow \quad -mg\, \ell/2 + (-T_{BH})\,\ell = 0 \Rightarrow \boxed{T_{BH} = -mg/2}$$

(all tensions besides T_{BH} either intersect axis DI or are parallel to it)

$$\sum M_{/EB} = 0 \quad \Rightarrow \quad -mg\, \ell/2 + (-T_{AI})\,\ell = 0 \Rightarrow \boxed{T_{AI} = -mg/2}$$

(all other tensions don't contribute)

$$\sum M_{/IB} = 0 \Rightarrow \quad \underbrace{\left(\frac{2}{\sqrt{2}}\, mg\right)}_{\text{force} \perp \text{to axis}} \underbrace{\frac{\ell}{2}}_{\text{lever arm}} + T_{CH} \overbrace{\frac{\sqrt{2}\,\ell}{2}}^{d_{NH}} = 0 \Rightarrow \boxed{T_{CH} = -mg/2}$$

lever arm

force $\perp$ to axis & lever arm

(No other tensions contribute)

$$\sum M_{/IC} = 0 \Rightarrow \quad \underbrace{\left(T_{BE}\frac{\sqrt{2}}{2}\right)\ell}_{\text{slide } T_{BE} \text{ to E to see this}} - \underbrace{\overset{-mg/2}{T_{BH}}\left(\frac{\sqrt{2}}{2}\,\ell\right)}_{\text{slide } T_{BH} \text{ to H to see this}} = 0 \Rightarrow \overset{*}{\boxed{T_{BE} = -mg/2}}$$

(gravity & other tensions drop out)

$$\sum M_{/BH} = 0 \quad \Rightarrow \quad T_{DI}\,\ell = 0 \quad \Rightarrow \boxed{T_{DI} = 0}$$

* $\sum M_{/IH} = 0$ gives T_{BE} in one shot.

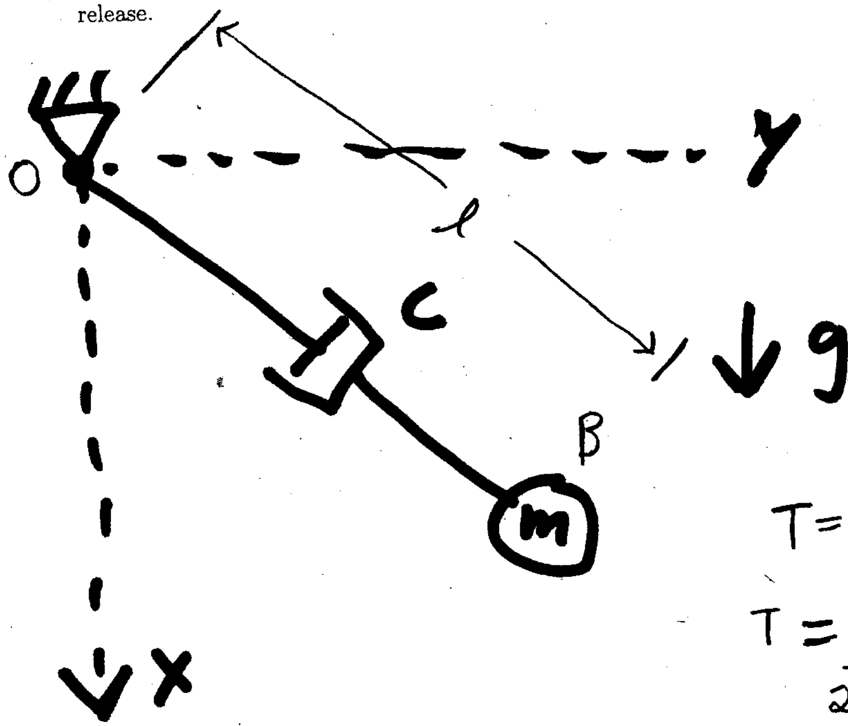
<u>Note</u>: all 6 tensions can be found one at a time, never using values of other tensions.

2) **(25 pt)** In the new sport of spongy jumping a spring is replaced by a dashpot c. Assume $m = 7\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, and $c = 13\,\mathrm{kg/s}$. The mass is released from rest at $x = 0\,\mathrm{m}$, $y = 5\,\mathrm{m}$.

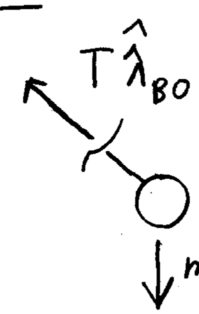
a) Just after (one milli-micro second after) release what is the acceleration of the mass?

b) What are the equations of motion for this system (differential equations involving x and y and their derivatives)?

c) Write Matlab code that would give the x coordinate of the center of mass 16 seconds after release.



FBD



$$T = c\,\dot{\ell}_{OB} = c\frac{d}{dt}\sqrt{x^2+y^2}$$

$$T = \frac{c\,(2x\dot{x}+2y\dot{y})}{2\sqrt{x^2+y^2}}$$

$$T = c\,(x\dot{x}+y\dot{y})/\ell$$

$$\hat{\lambda}_{MB} = \underline{r}_{BO}/|\underline{r}_{BO}| = (-x\,\hat{\imath} - y\,\hat{\jmath})/\ell$$

$$T\hat{\lambda}_{BO} = \frac{-c(x\dot{x}+y\dot{y})}{\ell^2}(x\hat{\imath}+y\hat{\jmath})$$

LMB $\quad \sum \underline{F}_i = m\underline{a}$

$$\left\{ mg\,\hat{\imath} - \frac{c(x\dot{x}+y\dot{y})}{\ell^2}(x\hat{\imath}+y\hat{\jmath}) = m(\ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath}) \right\}$$

$$\{\ \}\cdot\hat{\imath} \Rightarrow \quad \boxed{mg - \frac{c(x^2\dot{x}+xy\dot{y})}{\ell^2} = m\ddot{x}} \quad (a)$$

$$\{\ \}\cdot\hat{\jmath} \Rightarrow \quad \boxed{-\frac{c\,xy\dot{x}+y^2\dot{y}}{\ell^2} = m\ddot{y}}$$

define $V_x = \dot{x}$, $V_y = \dot{y}$

$$\Rightarrow \begin{array}{l} \dot{z}_1 = \\ \dot{z}_2 = \\ \dot{z}_3 = \\ \dot{z}_4 = \end{array} \boxed{\begin{array}{l} \dot{x} = V_x \\ \dot{y} = V_y \\ \dot{V}_x = g - c(x^2 V_x + xy V_y)/(\ell^2 m) \\ \dot{V}_y = -c(xy V_x + y^2 V_y)/(\ell^2 m) \end{array}} \ ^*$$

(a) in first order form

(b) at $t = 0^+$ $V_x = 0$, $V_y = 0 \Rightarrow \dot{V}_x = g$, $\dot{V}_y = 0$

eqn *

$$\Rightarrow \boxed{\underline{a}(0^+) = g\hat{\imath}} \ \text{(b)} \quad (= 10\,\text{m/s}^2\,\hat{\imath})$$

```
tspan=[0 16];
z0 = [0 5 0 0]';
[t z]=ode23('sponge', tspan, z0);
Xe = z(end,1)
```

useful Matlab command

(c)

run this after saving sponge.m

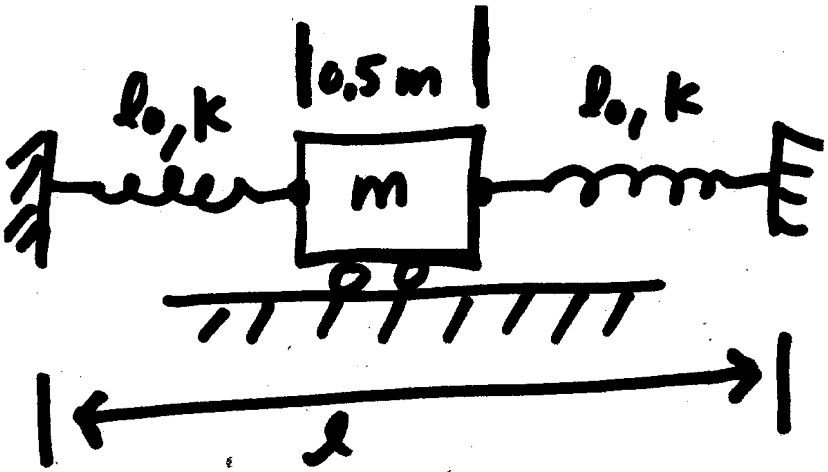
sponge.m

```
function zdot= sponge(t,z)
m=7; g=10; c=13
x= z(1); y= z(2); Vx= z(3); Vy= z(4);
xdot= Vx ;
ydot= Vy;
l2=     x^2 + y^2 ;
Vxdot= g - c*(x^2*Vx + x*y*Vy)/(l2*m);
Vydot= -c*(x*y*Vx + y^2*Vy)/(l2*m);
zdot= [xdot ydot Vxdot Vydot]';
```

3) **(25 pt)** A mass $m = 7\,\mathrm{kg}$ is held in place by two equal springs with $k = 5\,\mathrm{N/m}$ and $\ell_0 = 3\,\mathrm{m}$.

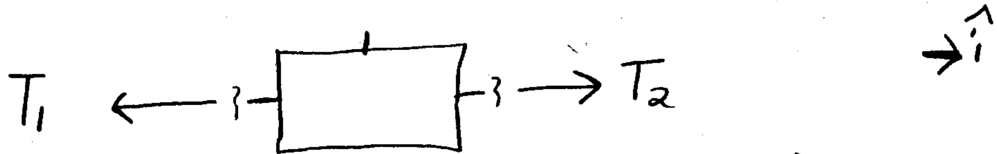
a) How long does one oscillation take if $\ell = 6.5\,\mathrm{m}$?

b) How long does one oscillation take if $\ell = 12.5\,\mathrm{m}$?



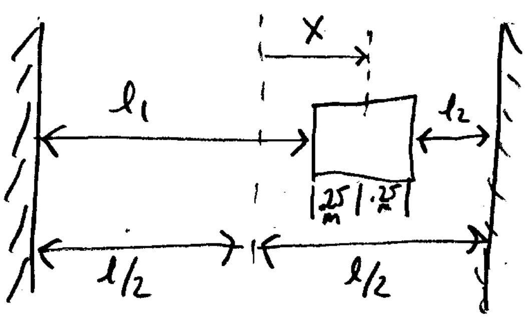
$\nearrow x$ = displacement from middle

FBD



$T_2 = (\ell_2 - \ell_0)\,k$

$T_1 = (\ell_1 - \ell_0)\,k$



$$\Rightarrow \ell_1 = \frac{\ell}{2} + x - 0.25\,\mathrm{m}$$

$$\ell_2 = \frac{\ell}{2} - x - 0.25\,\mathrm{m}$$

LMB $\quad \{\sum \underline{F}_i = m\underline{a}\}\cdot\hat{\imath}$

$$T_2 - T_1 = m\ddot{x}$$

$$(\ell_2 - \ell_0)\,k - (\ell_1 - \ell_0)\,k = m\ddot{x}$$

(cont'd)

$$\left[\underbrace{\left(\frac{\ell}{2} - x - .25m\right)}_{\ell_2} - \ell_0\right]k - \left[\underbrace{\left(\frac{\ell}{2} + x - 0.25m\right)}_{\ell_1} - \ell_0\right]k = m\ddot{x}$$

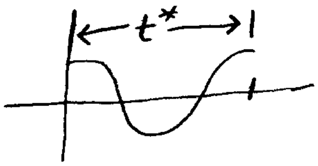
(ℓ, ℓ_0, .25m all drop out!)*

$$\Rightarrow \quad -2kx = m\ddot{x}$$

$$\boxed{\ddot{x} + \left(\frac{2k}{m}\right)x = 0}$$

Classic Harmonic Oscillator eqn.

ODE Soln: $x = A\cos\left(\underbrace{\sqrt{2k/m}\,t}\right) + B\sin\left(\underbrace{\sqrt{\frac{2k}{m}}\,t}\right)$



at one oscillation this whole $(\,) = 2\pi$

$$\sqrt{2k/m}\;t^* = 2\pi \Rightarrow t^* = \frac{2\pi}{\sqrt{2k/m}}$$

$$t^* = \frac{2\pi}{\sqrt{2\cdot(5N/m)/(7\,kg)}}$$

$1N = 1\,kg\cdot m/s^2$

$$t^* = \frac{2\pi}{\sqrt{(10/7)/s^2}}$$

$$\boxed{t^* = \frac{2\pi}{\sqrt{10/7}} \text{ seconds}}$$

(a) &
(b)

*Note: stretching a linear spring increases the force but not the stiffness.

"Solutions"

Your Name: Andy Ruina

Section day and time: ____________

T&AM 203 Prelim II

Tuesday Oct 26, 2004

Draft September 27, 2004

3 problems, 25 points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

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  II.) clear, and
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- □ your answers are $\boxed{\text{boxed}}$ in; and
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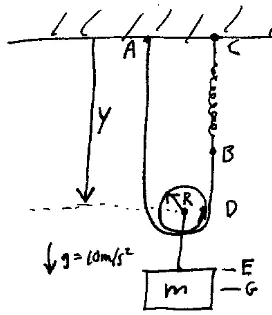
c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/25

1) The system below is released from rest at $y = 1.25$ m. In terms of the quantities given, what is y at $t = 5$ s?

FBD



$L_{AB} = 2$ m = length of string

$R = \cancel{0.25}$ m $(0.5/\pi)$ m

L_s = rest length of spring = 1 m

$m = 3$ kg

$k = 70$ N/m

$y(0) = 1.25$ m

$y(5s) = ?$

Spring Law: $T = K\Delta$ ← spring stretch.

total spring length

Kinematics: $2y + \pi R = L_{AB} + \underbrace{(L_s + \Delta)}_{\text{total spring length}}$

$$\Delta = 2y + \pi R - L_{AB} - L_s$$

LMB $\quad \sum F_y = m\ddot{y} \Rightarrow -2T + mg = \ddot{y}m$

$$T = K\Delta = K(2y + \pi R - L_{AB} - L_s)$$

$$\Rightarrow \quad \ddot{y} + \frac{4K}{m}y = \frac{2K}{m}(L_{AB} + L_s - \pi R) + g$$

$$y_h = A\cos\sqrt{4K/m}\,t + \underbrace{B}_{0 \text{ because } \dot{y}(0)=0}\sin\sqrt{\frac{4K}{m}}\,t$$

$$y_p = \frac{L_{AB} + L_s - \pi R}{2} + \frac{mg}{4K}$$

$$y(t) = A\cos\sqrt{\frac{4K}{m}}\,t + \frac{L_{AB} + L_s - \pi R}{2} + \frac{mg}{4K}$$

$$y(0) = y_0 \Rightarrow \quad y(t) = \left[y_0 - \left(\frac{L_{AB} + L_s - \pi R}{2} + \frac{mg}{4K}\right)\right]\cos\sqrt{4K/m}\,t$$

$$+ \frac{L_{AB} + L_s - \pi R}{2} + \frac{mg}{4K}$$

1) cont'd

$$y(t) = \left[\left[1.25 - \left(\frac{2+1-1/2}{2}\right) - \frac{30}{280}\right] \cos\sqrt{\frac{280}{30}}\, t/s + \left(1.25 + \frac{30}{280}\right)\right] m$$

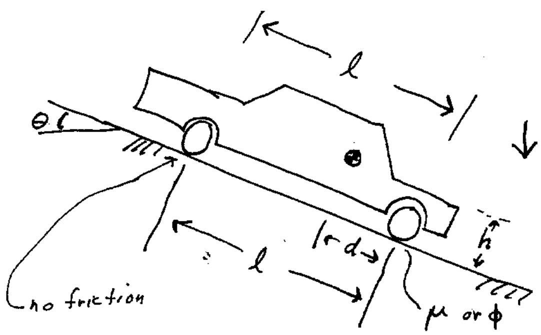
$$= \left[\frac{-3}{28}\cos\left(\sqrt{\frac{280}{3}}\,5\right) + \frac{5}{4} + \frac{3}{28}\right] m$$

$t = 5\,s$

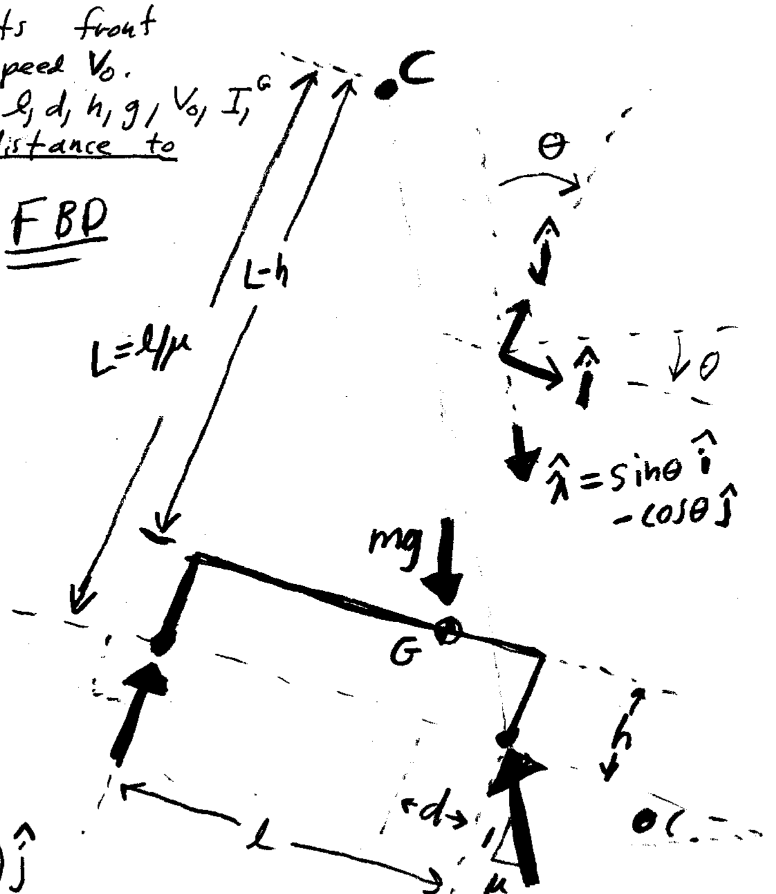
$$\boxed{y(5s) = \left[\frac{3}{28}\left(1 - \cos\left(5\sqrt{\frac{280}{3}}\right)\right) + \frac{5}{4}\right] m}$$

Prob 1

2) The car shown slams on its front brakes when it is going a speed V_0. In terms of some or all of ℓ, d, h, g, V_0, I^G, μ (or ϕ) and θ what is <u>the distance to stop</u>?



FBD



$$\text{AMB}_{/C}: \sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$$

$$\underline{r}_{G/C} \times (mg\hat{\lambda}) = \underline{r}_{G/C} \times (m\underline{a}_G)$$

$$\underline{a}_G = a\hat{i} \, ; \, \underline{r}_{G/C} = (\ell - d)\hat{i} - (L-h)\hat{j}$$

$$\Rightarrow \left[(\ell-d)\hat{i} - (L-h)\hat{j}\right] \times \left[mg(\sin\theta\,\hat{i} - \cos\theta\,\hat{j})\right] = \left[(\ell-d)\hat{i} - (L-h)\hat{j}\right] \times (ma\hat{i})$$

$$\left\{ \cancel{m}g\hat{k}\left[-(\ell-d)\cos\theta + (L-h)\sin\theta\right] = \cancel{m}a(L-h)\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad a = \left[\frac{-(\ell-d)}{(L-h)}\cos\theta + \sin\theta\right] g < 0 \quad \text{(slowing down)}$$

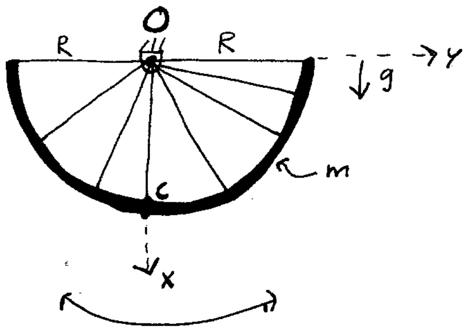
$$a = \frac{dv}{dt} \Rightarrow \quad av = v\frac{dv}{dt} \Rightarrow \quad a\,v\,dt = v\,dv \Rightarrow a\,dx = d(v^2/2) \Rightarrow a\Delta x = \Delta v^2/2$$

$$\left[\Delta(v^2/2) = -V_0^2/2\right]$$

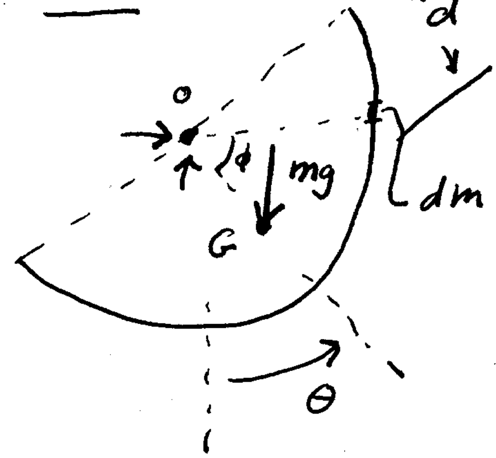
$$\Rightarrow \Delta x = x = \text{stopping distance} = -\cancel{2}V_0^2/\cancel{2}a$$

$$x = \frac{V_0^2}{2\left[\frac{\ell-d}{(L-h)}\cos\theta - \sin\theta\right]g} = \boxed{\frac{V_0^2\left[\ell/\mu - h\right]}{2g\left[(\ell-d)\cos\theta - \left(\frac{\ell}{\mu} - h\right)\sin\theta\right]}}$$

3) A bicycle wheel is cut in half. What is the period of oscillation? Answer in terms of some or all of R, m, g & the release angle θ_0, Neglect the mass & weight of the spokes & hub. Assume small oscillations.



FBD



AMB/O:
$$\sum \underline{M}_{/O} = \dot{\underline{H}}_{/O}$$

$$-mdg\sin\theta\,\hat{k} = \int \underbrace{\underline{r}}_{\text{radial from } O} \times \underbrace{\underline{a}}_{(\text{radial part}) + \overbrace{(\text{circumferential part})}^{R\ddot{\theta}}}\,dm$$

$$\Big\{ \qquad = \ddot{\theta} R^2 m\,\hat{k} \Big\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \quad \ddot{\theta} + \frac{dg}{R^2}\sin\theta = 0$$

small angles $\Rightarrow \quad \ddot{\theta} + \frac{dg}{R^2}\theta = 0 \Rightarrow \theta = A\cos\left(\sqrt{\frac{dg}{R^2}}\,t\right) + B\sin\left(\sqrt{\frac{dg}{R^2}}\,t\right)$

t^* = period: $\sqrt{\frac{dg}{R^2}}\,t^* = 2\pi$

$\boxed{d = 2R/\pi \text{ from next page}}$

$$t^* = 2\pi\sqrt{R^2/dg}$$
$$= 2\pi\sqrt{R\pi/2g}$$

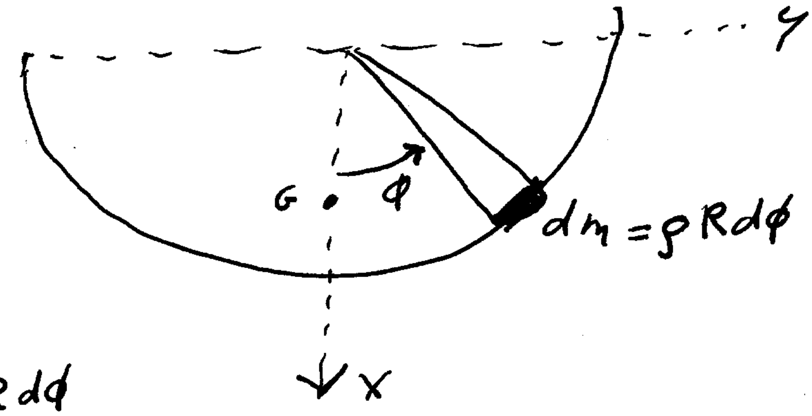
$$\Rightarrow \boxed{t^* = \sqrt{2}\,\pi^{3/2}\sqrt{\frac{R}{g}}}$$ ✓

prob 3

3 cont'd

Find d

$\rho = \dfrac{m}{\pi R}$



$$m X_G = \int x\, dm$$

$$= \int_{-\pi/2}^{\pi/2} (R\cos\phi)\, \rho R\, d\phi$$

$$= R^2\left(\frac{m}{\pi R}\right) \int_{-\pi/2}^{\pi/2} \cos\phi\, d\phi$$

$$= \frac{Rm}{\pi} \sin\phi \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{2Rm}{\pi}$$

$$\Rightarrow \boxed{d = X_G = \frac{2R}{\pi}}$$ ← used on previous page

"SOLUTIONS"

Your Name: Andy Ruina

Section day and time: Tu,Th 9:05-9:55

T&AM 203 Prelim 3

Tuesday Nov 23, 2004

Draft November 23, 2004

3 problems, 25 points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

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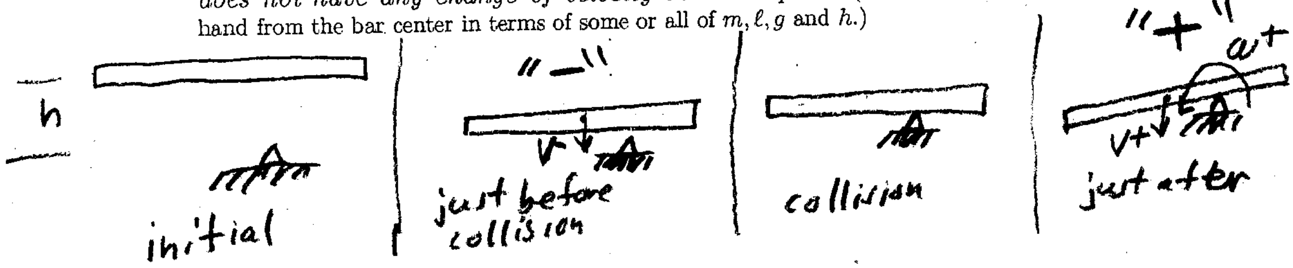
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems **poorly defined**;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

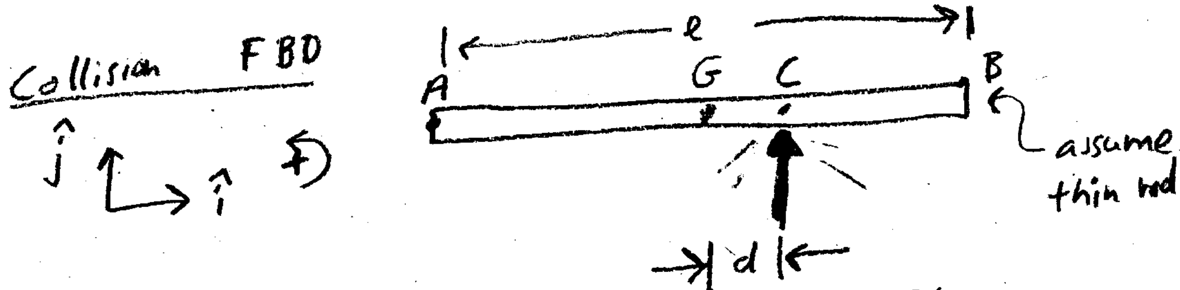
c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/25

1) (25 pt) A uniform horizontal bar with mass m and length ℓ falls a height h without rotation in gravitational field g. The bar is then caught by a stationary hand. The hand does not move, but imparts an impulse but no torque (it freely allows rotation). *Clearly indicate where on the bar the hand should catch the bar so that one end of the bar (indicate which end) does not have any change of velocity at the impact.* (Indicate the distance d of the hand from the bar center in terms of some or all of m, ℓ, g and h.)

$\downarrow g$





AMB /C

$$\underline{H}^-_{/C} = \underline{H}^+_{/C}$$

$$\{ V^- m d \hat{k} = V^+ m d \hat{k} + \underbrace{I^G}_{m\ell^2/12} \omega^+ \hat{k} \}$$

$$V^+ = \omega^+ d$$

$$\{\}\cdot\hat{k} \Rightarrow \quad V^- \cancel{m} d = \omega^+ \cancel{m}\left[d^2 + \tfrac{1}{12}\ell^2\right]$$

$$\Rightarrow \quad \omega^+ = \frac{V^- d}{d^2 + \frac{1}{12}\ell^2} \Rightarrow V_A^+ = \omega^+\left(d + \tfrac{\ell}{2}\right)$$

$$= \frac{V^- d(d + \ell/2)}{d^2 + \ell^2/12}$$

Problem statement: $V_A^+ = V_A^-$

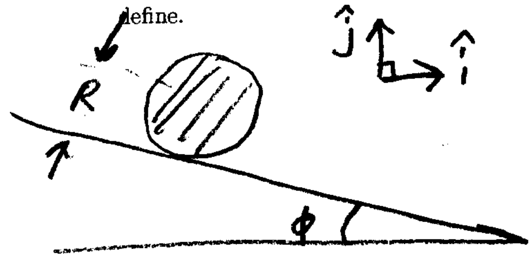
$$\Rightarrow \quad \frac{V^- d(d+\ell/2)}{d^2 + \ell^2/12} = V^-$$

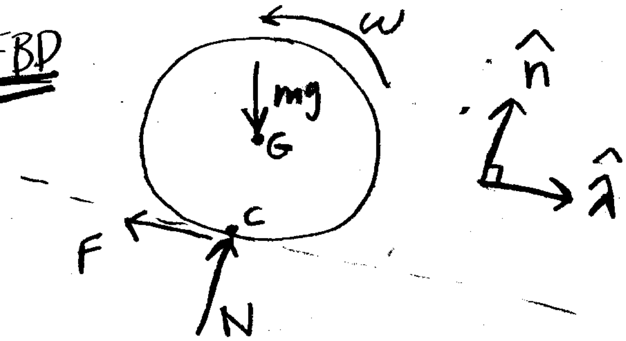
$$\Rightarrow \quad \frac{d\ell}{2} = \frac{\ell^2}{12}$$

$$\Rightarrow \quad \boxed{d = \ell/6}$$

Pt. C is called the center of percussion. Thinking of AB as a baseball bat its where to hit the ball so the hand end at A has no jump in velocity. An approximation of "the sweet spot".

2) **(25 pt)** A uniform disk of mass m and radius R is released from rest at $t = 0$ to roll-without-slip down a slope ϕ (measured relative to the horizontal) as accelerated by gravity g. *At time t what is the acceleration of the point on the disk that is then touching the ground?* Answer in terms of some or all of m, R, g, ϕ, t and any base vectors that you choose that you clearly define.





FBD shown is valid at all times.

Kinematics: $\underline{V}_G = \underbrace{V_G\hat{\lambda} = -\omega R}_{\text{rolling}}\hat{\lambda} \Rightarrow a_G = -\dot{\omega}R$

AMB$_{/C}$: $\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$

$$\underline{r}_{G/C}\times(-mg\hat{j}) = \underline{r}_{G/C}\times m\underline{a}_G + I^G\dot{\omega}\hat{k}$$

$\underline{r}_{G/C} = R\hat{n}$ $\quad$ $\underline{a}_G = -\dot{\omega}R\hat{\lambda}$ $\quad$ $I^G = \frac{1}{2}mR^2$ (see next page)

$\hat{n}\times\hat{j} = \sin\phi\hat{k} \Rightarrow \left\{-Rmg\sin\phi\hat{k} = R^2m\dot{\omega}\hat{k} + \frac{1}{2}mR^2\dot{\omega}\hat{k}\right\}$

$\{\,\}\cdot\hat{k} \Rightarrow -Rg\sin\phi = \left(R^2+\frac{1}{2}R^2\right)\dot{\omega}$

$\Rightarrow \boxed{\dot{\omega} = \frac{-2g\sin\phi}{3R}}$ (right hand side = constant)

$\omega_0 = 0 \Rightarrow \boxed{\omega = \frac{-2g\sin\phi}{3R}t}$

What is accel. of pt. C?

$$\underline{a}_C = \underline{a}_G + \underline{a}_{C/G} = -\dot{\omega}R\hat{\lambda} + \dot{\underline{\omega}}\times\underline{r}_{C/G} + \left(-\omega^2\underline{r}_{C/G}\right)$$

$$\underline{a}_c = -\dot{\omega}R\hat{\lambda} + \dot{\omega}\hat{k}\times(-R\hat{n}) + \omega^2 R\hat{n}$$

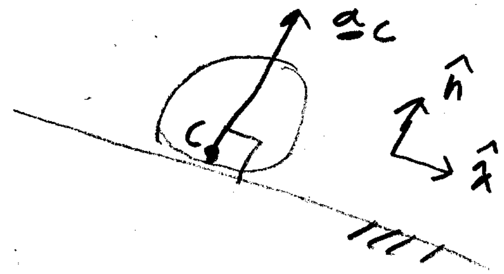
$$\hat{k}\times\hat{n} = -\hat{\lambda}$$

$$= -\dot{\omega}R\hat{\lambda} + \dot{\omega}R\hat{\lambda} + \omega^2 R\hat{n}$$

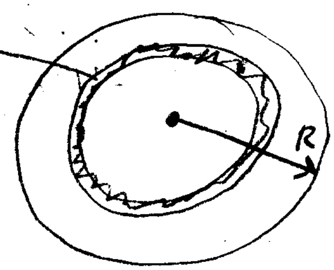
$$= \omega^2 R\hat{n}$$

$$= \left(\frac{-3g\sin\phi}{R}t\right)^2 R\hat{n}$$

$$\boxed{\underline{a}_c = \frac{4g^2\sin^2\phi\, t^2}{9R}\hat{n}}$$



Calculate I^G for uniform disk $\qquad \rho = \frac{m}{A} = \frac{m}{\pi R^2}$



$$I^G = \int r^2\, dm$$

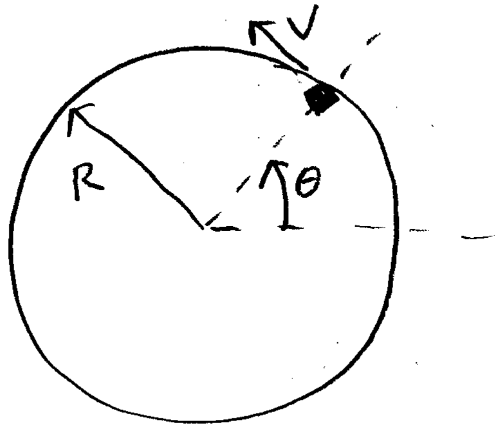
$$= \int_0^R r^2(\rho 2\pi r\, dr)$$

$$= 2\pi\rho\int_0^R r^3\, dr = 2\pi\frac{m}{\pi R^2}\left(\frac{r^4}{4}\right)\Bigg|_0^R$$

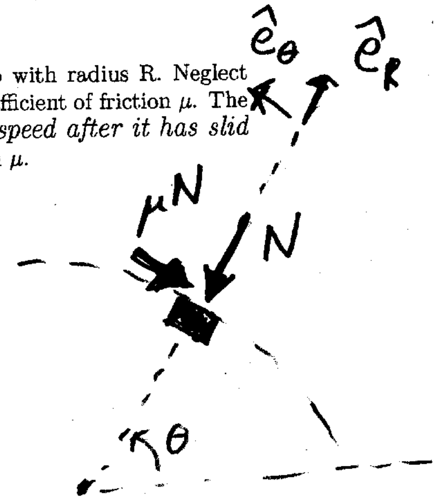
$$= 2\cancel{\pi}\frac{m}{\cancel{\pi}R^2}\frac{R^4}{4}$$

$$\boxed{I^G = mR^2/2}$$

3) (25 pt) A small bead with mass m slides on a rigid stationary circular hoop with radius R. Neglect gravity. The bead slides loosely on the wire (does not pinch it) with coefficient of friction μ. The initial speed of the bead is v_0 (along the circle). *What is the bead speed after it has slid once around the hoop?* Answer in terms of some or all of m, R and μ.



FBD



LMB

$$\underline{F} = m\underline{a} \Rightarrow -N\hat{e}_R - \mu N\hat{e}_\theta = m\left[(\cancel{\ddot{R}}^{\,0} - R\dot{\theta}^2)\hat{e}_R + (R\ddot{\theta} + 2\cancel{\dot{R}\dot{\theta}}^{\,0})\hat{e}_\theta\right]$$

$$\left\{-N\hat{e}_R - \mu N\hat{e}_\theta = -mR\dot{\theta}^2\hat{e}_R + mR\ddot{\theta}\hat{e}_\theta\right\}$$

$$\{\ \}\cdot\hat{e}_R \Rightarrow \quad N = mR\dot{\theta}^2$$

$$\{\ \}\cdot\hat{e}_\theta = \quad -\mu N = mR\ddot{\theta}$$

$$\Rightarrow mR\ddot{\theta} = -\mu mR\dot{\theta}^2$$

$\boxed{\text{Define } \omega = \dot{\theta}}$

$$\Rightarrow \ddot{\theta} = -\mu\dot{\theta}^2 \Rightarrow \boxed{\dot{\omega} = -\mu\omega^2} \ *$$

Now think of ω as $\omega(\theta)$. $\boxed{\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt}}$

Why? Because we are given final θ not final t.

$$\Rightarrow \frac{d\omega}{d\theta}\omega = -\mu\omega^2$$

$$\Rightarrow \frac{d\omega}{d\theta} = -\mu\omega$$

$$\Rightarrow \omega = \omega_0 e^{-\mu\theta}$$

[Note $V = \omega R$]

$$\Rightarrow V = \underbrace{\omega_0 R}_{V_0} e^{-\mu\theta}$$

$$\Rightarrow V = V_0 e^{-\mu\theta}$$

$\underbrace{\theta = 2\pi}$, $\Rightarrow \boxed{V = V_0 e^{-2\pi\mu}}$

↑ one revolution

Alternative soln. to ODE *, (the long way around)

$$\frac{d\omega}{dt} = -\mu\omega^2 \Rightarrow \frac{d\omega}{\omega^2} = -\mu\, dt$$ (seperable 1st order ODE)

$$\Rightarrow -\omega^{-1} - (-\omega_0^{-1}) = -\mu t \Rightarrow \frac{1}{\omega_0} - \frac{1}{\omega} = -\mu t$$

$$\Rightarrow \frac{1}{\omega} = \frac{1}{\omega_0} + \mu t \Rightarrow \boxed{\omega = \frac{1}{\mu t + 1/\omega_0} = \frac{\omega_0}{1+\omega_0\mu t}} \quad (1)$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\omega_0}{1+\omega_0\mu t} \Rightarrow d\theta = \frac{\omega_0\, dt}{1+\omega_0\mu t}$$ (seperable again)

$$\Rightarrow \int_0^\theta d\theta' = \int_0^t \frac{\omega_0\, dt'}{1+\omega_0\mu t'}$$

substitution;
let $u = 1+\omega_0\mu t$
$du = \omega_0\mu\, dt$

$$\theta = \int_1^u \frac{1}{\mu}\frac{du'}{u'} = \frac{1}{\mu}\left[\ln(u) - \ln(1)\right] = \frac{1}{\mu}\ln u$$

$$\theta = \frac{1}{\mu}\ln(1+\omega_0\mu t)$$

$\theta = 2\pi \Rightarrow$ $2\pi\mu = \ln(1+\omega_0\mu t)$

$$\Rightarrow e^{2\pi\mu} = 1+\omega_0\mu t$$

$$t = \frac{e^{2\pi\mu}-1}{\omega_0\mu} \quad (2)$$

Apply (2) to (1) $\Rightarrow$ $$\omega\big|_{\theta=2\pi} = \frac{\omega_0}{1+\omega_0\mu\left[\frac{e^{2\pi\mu}-1}{\omega_0\mu}\right]} = \frac{\omega_0}{e^{2\pi\mu}}$$

$$\Rightarrow \omega = \omega_0 e^{-2\pi\mu} \Rightarrow \omega R = \omega_0 R e^{-2\pi\mu}$$

$$\Rightarrow \boxed{V = V_0 e^{-2\pi\mu}}$$ (again)

at $\theta = 2\pi$

Solutions

1

Your Name: Manish Agarwal

Section time: ______________

T&AM 203 Prelim 1

Tuesday September 26, 2006

Draft September 25, 2006

3 problems, 25^+ points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1: ______/25

Problem 2: ______/25

Problem 3: ______/25

1) (25 pt) Pulleys. In the problems below you are asked "What is the relation" between this and that. This means you should write the simplest possible equation in which this and that are the only unknowns.)

a) (1 point) Please read all the rules and hints at the front of the exam. Write here: "I read the cover page.":

____

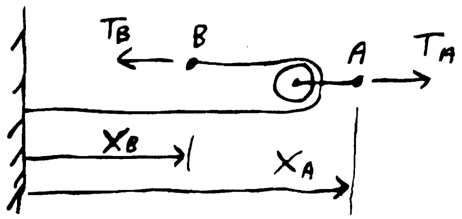
b) (3 points) The ideal pulley system (make the usual assumptions) in (b,c) below shown is part of a larger mechanism. What is the relation between T_A and T_B? Clearly justify your work from first principles.

c) (10 points) For the same pulley system what is the relation between a_A and a_B? Clearly justify your work from first principles.

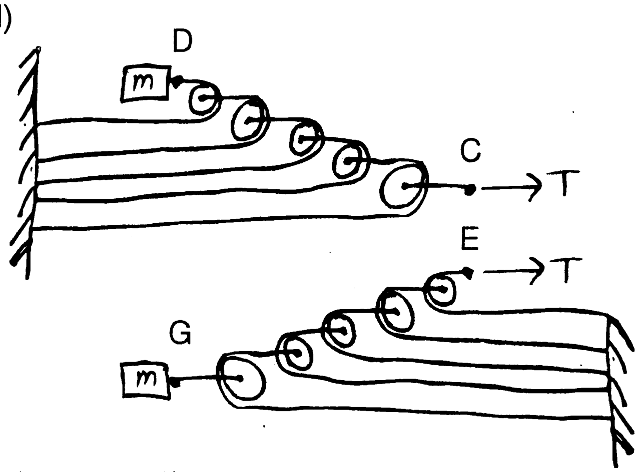
[Part (d) will only be graded if (b) and (c) above are correct.]

d) (11 points) The two pulley systems below (d) are treated as having all ideal components. What is the relation between a_C and a_E? You may use the results from parts (b) and (c) above without re-deriving them again and again. When comparing the systems use $m = m$ and $T = T$.

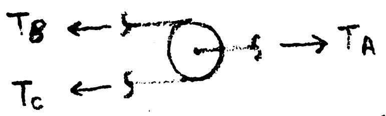
b,c)



d)



b)



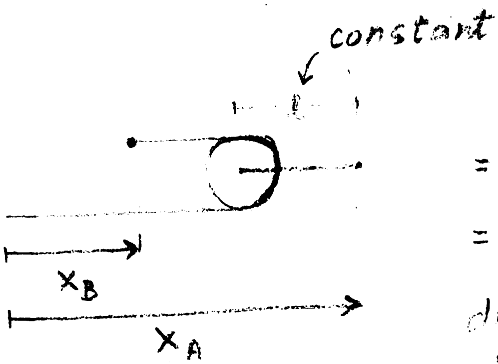
assuming massless pulley

$T_A - T_B - T_c = 0$

also $T_B = T_c$ assuming massless, inextensible string.

$\Rightarrow$ $\boxed{T_A = 2T_B}$

c)



length of the string

$= (X_A - \ell) + (X_A - \ell - X_B)$

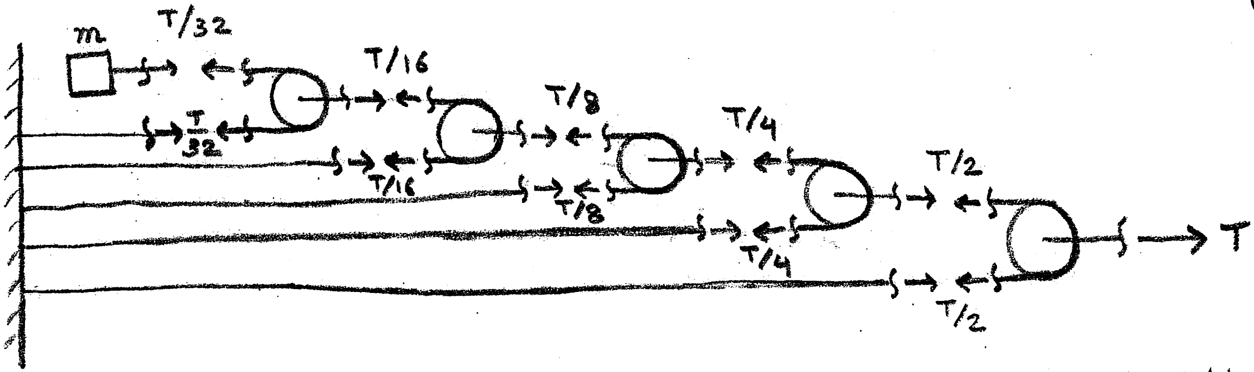
$=$ constant

differentiating wrt time ($\because \dot{\ell} = 0$)

$\dot{X}_A + \dot{X}_A - \dot{X}_B = 0$

differentiating again $2\ddot{X}_A = \ddot{X}_B \Rightarrow \boxed{2a_A = a_B}$

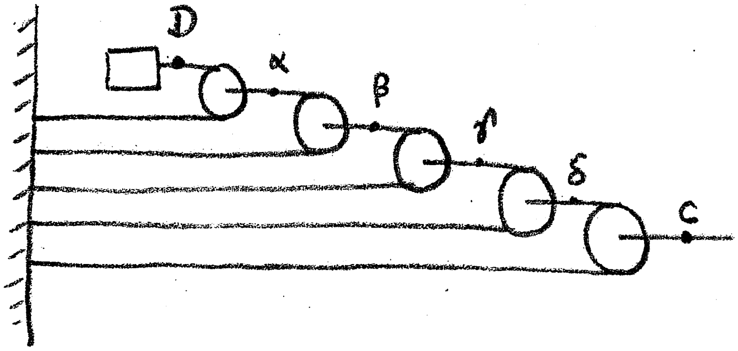
d)



using part b) the above FBDs can be established

N2L on m gives

$$\frac{T}{32} = m\,a_D \quad \Rightarrow \quad a_D = \frac{T}{32m}$$

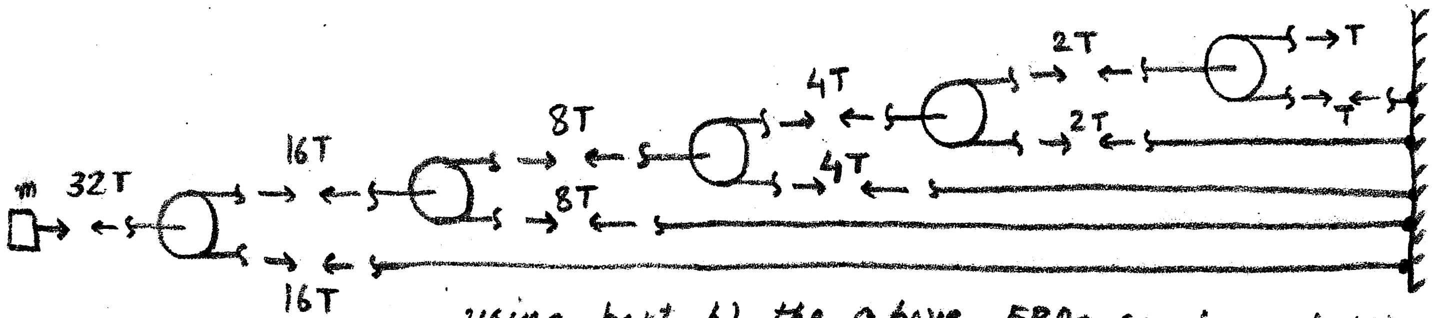


by part c)

$$2a_\alpha = a_D$$
$$2a_\beta = a_\alpha$$
$$2a_\gamma = a_\beta$$
$$2a_\delta = a_\gamma$$
$$2a_c = a_\delta$$

$$\Rightarrow \quad 2a_c = \frac{a_\gamma}{2} = \frac{a_\beta}{4} = \frac{a_\alpha}{8} = \frac{a_D}{16} = \frac{T}{32m} \cdot \frac{1}{16}$$

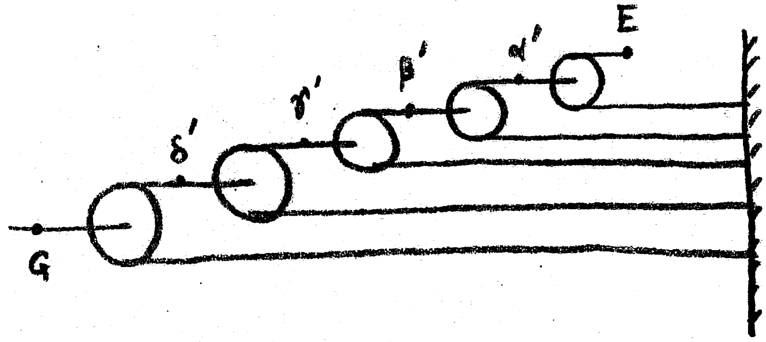
$$\Rightarrow \quad a_c = \frac{T}{m}\left(\frac{1}{32}\right)^2 \qquad \text{— } (1)$$



using part b) the above FBDs can be established

N2L on m gives $32T = m\,a_G \qquad a_G = 32\frac{T}{m}$

by part c)



$$2a_{\alpha'} = a_E$$
$$2a_{\beta'} = a_{\alpha'}$$
$$2a_{\gamma'} = a_{\beta'}$$
$$2a_{\delta'} = a_{\gamma'}$$
$$2a_G = a_{\delta'}$$

$$\Rightarrow \quad a_E = 2a_{\alpha'} = 4a_{\beta'} = 8a_{\gamma'} = 16a_{\delta'} = 32a_G$$

$$\Rightarrow \quad a_E = 32 \cdot \underbrace{32\frac{T}{m}}_{\text{substituting the value of } a_G} = (32)^2 \frac{T}{m} \quad — \text{②}$$

dividing ① by ②

$$\frac{a_c}{a_E} = \left(\frac{1}{32}\right)^4 \qquad \boxed{a_E = (32)^4 a_c}$$

$$a_E \approx 1{,}000{,}000 \; a_c \quad !$$

5

2) (25 pt) **MATLAB etc.** The block of code shown calculates motions from a dynamics problem relevant to this course. It runs without error. (This code is no good outside a test, obviously, because the variable names are not suggestive, intermediate variables aren't used, and there is no commenting.)

a) (10 points) Write a mechanics question (with values, units, basic assumptions etc.) that the output of this code answers. Your question should make no reference to matlab or computers but should be in the language of mechanics.

[Grading of parts (b) and (c) below depend on the answer to (a) above being correct. So be confident before moving on.]

b) (10 points) Assume that the command `plot(t,z(:,2))` is added just below the command `t(end)`. Draw, as accurately as you can, the resulting plot. Label (give numerical values) key points and asymptotes which you find using your own pencil-and-paper analysis. Label the axes (even though the code does not do this).

c) (5 points) Get as far as you can towards finding a numerical value for `t(end)` without using the computer. Ultimately you will be stuck without a calculator. But get to a point where the job of the calculator is clear.

this is what the code prints. So the 'mechanics question' should demand time.

initial conditions

```
function prelim1q2
options=odeset('events',@fSally);
[t,z,tev, zev, i] = ode45(@fgeorge,[0 1000],[0 10],options);
t(end)
end

function zdot = fgeorge(t,z);
zdot = [z(2) -2*z(2)-10]';
end

function [value,done,dir] = fSally(t,z)
value = z(1) ; done = 1; dir= -1;
end
```

Remember g!

This suggests a diffrential equation of the form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_2 - 10$$

requires to detect an event corresponding to $x_1 = 0$ when x_1 crosses from +ive to -ive

a) realizing $x_1 = x$ (position)
$x_2 = v$ (velocity)

we get the diff equ^n as

$$\dot{x} = v$$
$$\dot{v} = -2v - 10$$

or $$\ddot{x} = -2\dot{x} - 10$$

noting that its a one-dimensional motion and comparing it with the standard equ^n in mechanics

$$m\ddot{x} + c\dot{x} + kx = F$$

or $$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}$$

or $$\ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x + \frac{F}{m}$$

we see $\frac{c}{m} = 2$ $\quad k = 0$ $\quad \frac{F}{m} = -10 = -g$!

linear damping

no stiffness

force opposing the motion in all probability should be gravity!

the mechanics question can, hence, be

" A projectile of mass 1 kg is projected vertically up with velocity 10 m/s. The air applies a linear drag proportional to the velocity (drag constant 2 N/(m/s)). Find the time projectile take to hit the ground again."

b) plot (t, z(:,2)) wants a plot of v vs t

now $\quad \dot{v} = -2v - 10$

$$\int_{10}^{v} \frac{dv}{2v+10} = -\int_0^t dt \quad \Rightarrow \quad \frac{1}{2} \ln(2v+10)\Big]_{10}^{v} = -t$$

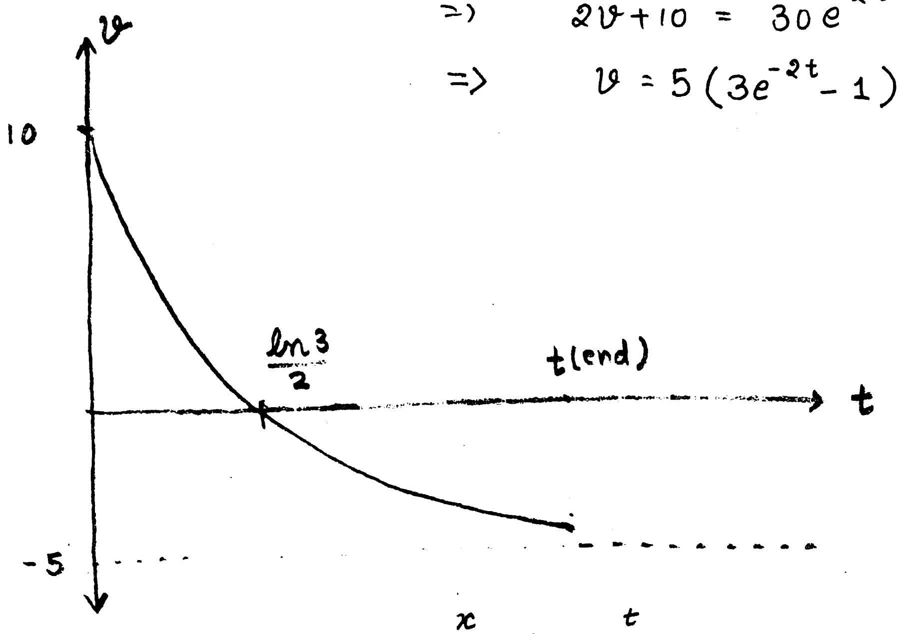
initial condition $\longrightarrow$ 10

$$\Rightarrow \quad \ln\left(\frac{2v+10}{30}\right) = -2t$$

$$\Rightarrow \quad 2v + 10 = 30e^{-2t}$$

$$\Rightarrow \quad v = 5(3e^{-2t} - 1)$$

$$\boxed{\begin{aligned} &V = 0 \\ \Rightarrow\ & e^{-2t} = 1/3 \\ & 2t = \ln 3 \\ & t = \tfrac{1}{2}\ln 3 \end{aligned}}$$



c) $\dfrac{dx}{dt} = 5(3e^{-2t} - 1) \quad \Rightarrow \quad \int_0^x dx = 5\int_0^t (3e^{-2t} - 1)\, dt$

$$\Rightarrow \quad x = 5\left\{\frac{3e^{-2t}}{-2} - t\right\}_0^t \quad \Rightarrow \quad x = 5\left[-\frac{3}{2}e^{-2t} - t + \frac{3}{2}\right]$$

$$x = 5\left[\frac{3}{2}(1 - e^{-2t}) - t\right]$$

now t(end) occurs at $x = 0$

$$\Rightarrow \quad \boxed{\frac{3}{2}(1 - e^{-2\,t(\text{end})}) - t(\text{end}) = 0}$$

is to be solved for t(end)

now use the calculator!

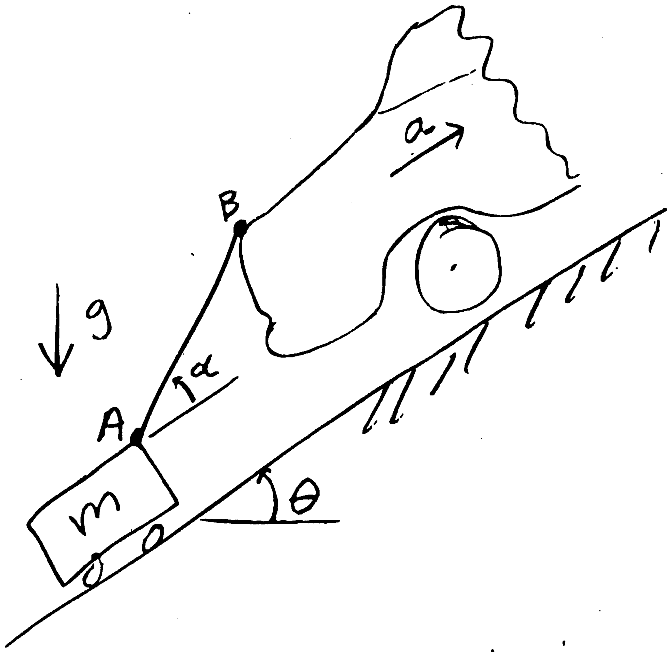
⑦

3) (25 pt) A car drags a cart. A car with <u>known acceleration a</u> accelerates up a hill dragging a cart.

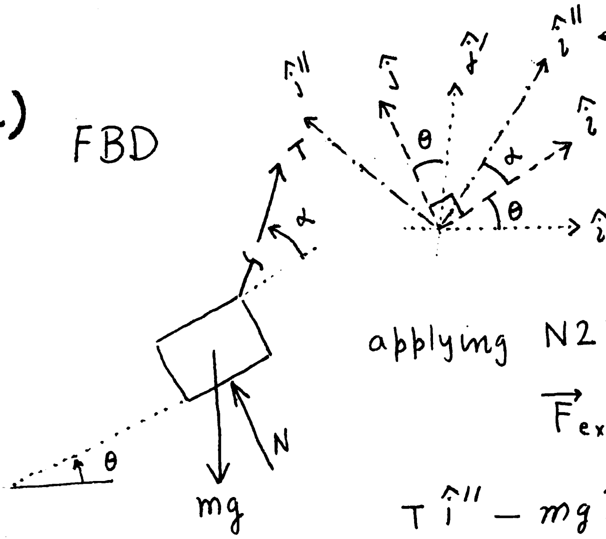
a) (15 points) Assume no friction. Find the force of the ground on the cart. Answer in terms of some or all of m, g, a, θ, α and L_{AB}.

[Your score is the better of the two scores from part (a) and part (b).]

b) (25 points) Assume there is friction between the cart and the ground. Find the tension in the cable AB. Answer in terms of some or all of m, g, a, θ, α, L_{AB} and the friction coefficient μ (or the friction angle ϕ, defined as $\tan\phi = \mu$).



a) FBD



$\hat{i}''$ ← direction ~~of~~ tension

$\hat{i}$ ← direction along the plane (up)

$\hat{i}'$ ← horizontal direction

applying N2L on mass m gives

$$\vec{F}_{ext} = m\vec{a}$$

$$T\hat{i}'' - mg\hat{j}' + N\hat{j} = m(\underbrace{a_t}_{\text{tangential acceleration}}\hat{i} + \underbrace{a_n}_{\text{normal acceleration } = 0}\hat{j})$$

$$\{\}\cdot\hat{i} \Rightarrow T\underbrace{(\hat{i}''\cdot\hat{i})}_{\cos\alpha} - mg\underbrace{(\hat{j}'\cdot\hat{i})}_{\cos(\frac{\pi}{2}-\theta)} + N\underbrace{(\hat{j}\cdot\hat{i})}_{0} = m a_t$$

$$\Rightarrow T\cos\alpha - mg\sin\theta = m a_t = m a \quad -①$$

($a_t = a$ assuming the string remains tight.)

$\{\ \}\cdot\hat{j} \Rightarrow \quad T\,\underbrace{(\hat{i}''\cdot\hat{j})}_{\cos(\frac{\pi}{2}-\alpha)} - mg\,\underbrace{(\hat{j}'\cdot\hat{j})}_{\cos\theta} + N\,\underbrace{(\hat{j}\cdot\hat{j})}_{1} = 0$

$$T\sin\alpha - mg\cos\theta + N = 0 \quad — \quad (2)$$

from (2)

$$N = mg\cos\theta - T\sin\alpha \quad — \quad (3)$$

from (1)

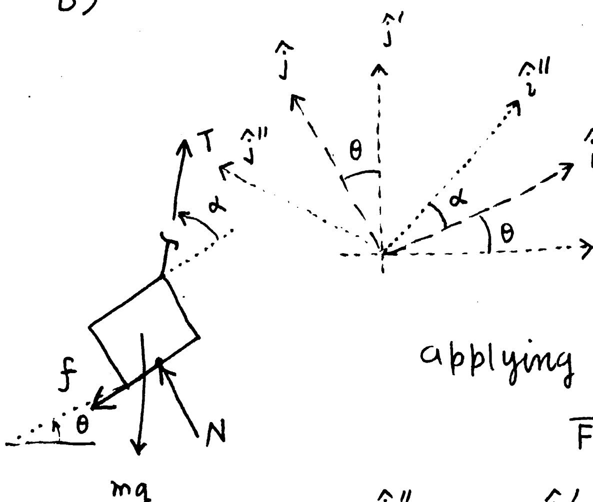
$$T = \frac{mg\sin\theta + ma}{\cos\alpha}$$

Substituting in (3)

$$N = mg\cos\theta - \frac{\sin\alpha}{\cos\alpha}(mg\sin\theta + ma)$$

$$\boxed{N = m\left[g(\cos\theta - \tan\alpha\sin\theta) - a\tan\alpha\right]}$$

b) AN extra frictional force appear in this part !



applying N2L on mass m gives

$$\vec{F}_{ext} = m\vec{a}$$

$T\hat{i}'' - mg\hat{j}' + N\hat{j} - f\hat{i} = ma\hat{i}$ noting $f = \mu N$

$\{\ \}\cdot\hat{i} \quad T\,\underbrace{(\hat{i}''\cdot\hat{i})}_{\cos\alpha} - mg\,\underbrace{(\hat{j}'\cdot\hat{i})}_{\cos(\frac{\pi}{2}-\theta)} + N\,\underbrace{(\hat{j}\cdot\hat{i})}_{0} - \mu N\,\underbrace{(\hat{i}\cdot\hat{i})}_{1} = ma\,\underbrace{(\hat{i}\cdot\hat{i})}_{1}$

$\Rightarrow \quad T\cos\alpha - mg\sin\theta - \mu N = ma \quad — \quad (1)$

$\{\ \}\cdot\hat{j} \quad T\,\underbrace{(\hat{i}''\cdot\hat{j})}_{\cos(\frac{\pi}{2}-\alpha)} - mg\,\underbrace{(\hat{j}'\cdot\hat{j})}_{\cos\theta} + N\,\underbrace{(\hat{j}\cdot\hat{j})}_{1} - \mu N\,\underbrace{(\hat{i}\cdot\hat{j})}_{0} = ma\,\underbrace{(\hat{i}\cdot\hat{j})}_{0}$

$\Rightarrow \quad T\sin\alpha - mg\cos\theta + N = 0 \quad — \quad (2)$

from (2)

$$N = mg\cos\theta - T\sin\alpha$$

⑨ Substituting the value of N in ①

$$T\cos\alpha - mg\sin\theta - \mu(mg\cos\theta - T\sin\alpha) = ma$$

$$\Rightarrow \quad T(\cos\alpha + \mu\sin\alpha) = ma + mg\sin\theta + \mu mg\cos\theta$$

$$\boxed{T = \frac{m(a + g\{\sin\theta + \mu\cos\theta\})}{\cos\alpha + \mu\sin\alpha}}$$

Note :- from ② and answer above

$$N = mg\cos\theta - T\sin\alpha$$

$$= mg\cos\theta - \frac{m\sin\alpha(a + g\{\sin\theta + \mu\cos\theta\})}{\cos\alpha + \mu\sin\alpha}$$

put $\mu = 0$ to get the answer to part a)

$$N(\mu=0) = mg\cos\theta - m\tan\alpha(a + g\sin\theta)$$

$$= m[g(\cos\theta - \tan\alpha\sin\theta) - a\tan\alpha]$$

which is what we found in a) !

Your Name: MANISH AGARWAL

Section time: ______________________

T&AM 203 Prelim 2

Tuesday October 24, 2006

Draft October 24, 2006

3 problems, 25^+ points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	4:	/25
Problem	5:	/25
Problem	6:	/25

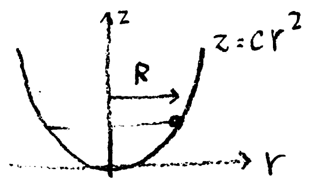
4) (25 pt) **Particle sliding in circles in a parabolic bowl.** As if in a James Bond adventure (in a big slippery Cornell-managed radio-telescope bowl in Puerto Rico), a particle-like human with mass m is sliding with negligible friction around in level circles at speed v. The equation describing the bowl is $z = CR^2 = C(x^2 + y^2)$

a) (20 points) Find v in terms of any or all of R, g, and C.

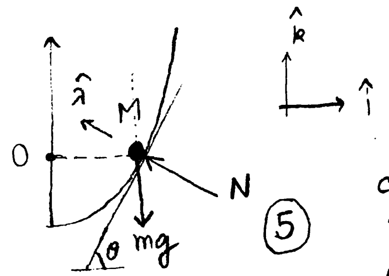
b) (5 points) Now say you are given ω, C and g. Find v and R if you can. Explain any oddities.

Sol

(a)



FBD



5

mass M executes circular motion about center O.

3

note $\quad \tan\theta = \left.\dfrac{dz}{dr}\right|_{r=R} = 2CR \quad$ — ①

also $\quad \hat{\lambda} = \cos\theta\,\hat{k} - \sin\theta\,\hat{i} \quad$ — ②

applying N2L

$$\vec{F}_{ext} = m\vec{a}$$

$$-mg\hat{k} + N\hat{\lambda} = m(\underbrace{a_z}_{0}\hat{k} + \overset{-\frac{v^2}{R}}{\overset{\parallel}{a_r}}\,\hat{i})$$

$\because$ motion in Level circle

$$\Rightarrow \quad -mg\hat{k} + N(\cos\theta\hat{k} - \sin\theta\hat{i}) = -\frac{mv^2}{R}\hat{i} \qquad [\text{using ②}]$$

$\{\ \}\cdot\hat{k} \Rightarrow \quad -mg + N\cos\theta = 0 \Rightarrow N\cos\theta = mg$ — ③ 5

$\{\ \}\cdot\hat{i} \Rightarrow \quad -N\sin\theta = -\dfrac{mv^2}{R} \Rightarrow N\sin\theta = \dfrac{mv^2}{R}$ — ④ 5

dividing ④ by ③ and using ①

$$\tan\theta = 2CR = \frac{v^2}{Rg}$$

$$\Rightarrow \boxed{v = R\sqrt{2Cg}} \quad ②$$

(b) ω, C, g given, v and R to be found

(2) $\omega = \frac{v}{R} = \sqrt{2Cg}$ from answer of part (a)

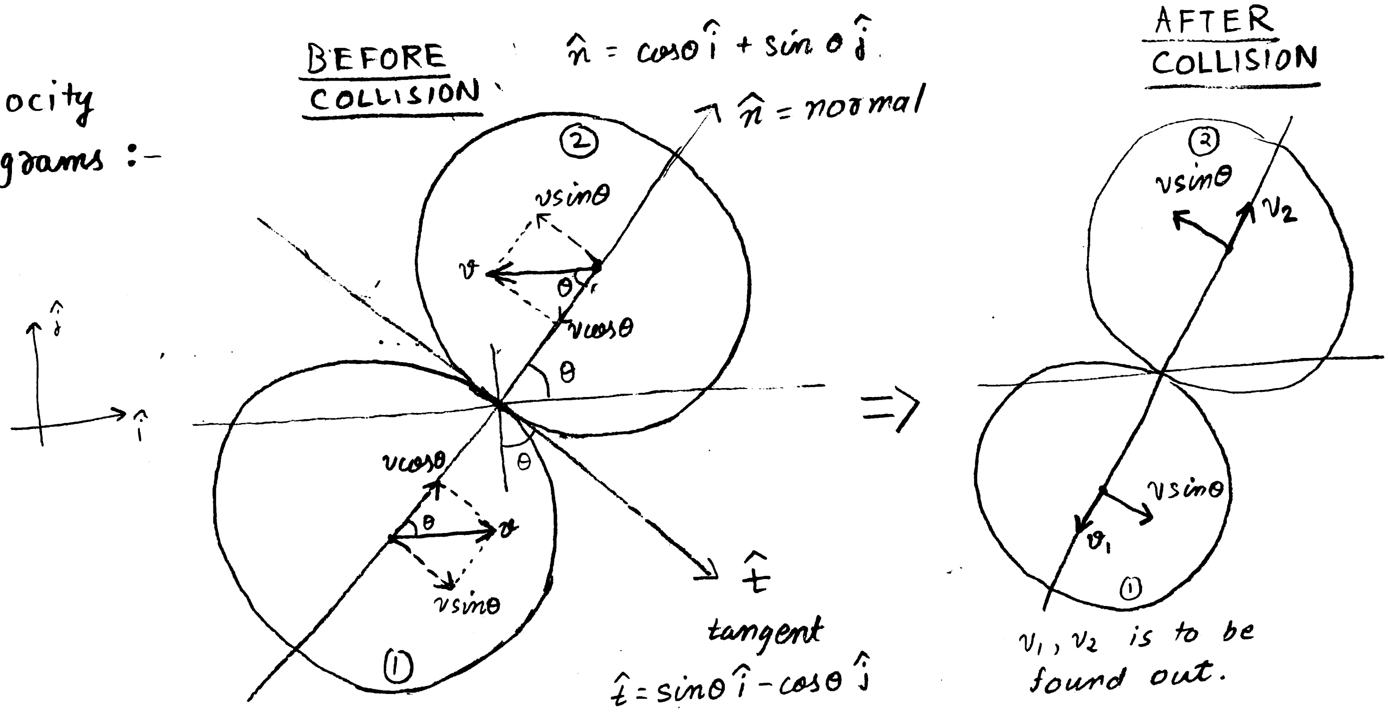
$\Rightarrow$ v and R can have any values untill

(3) $\frac{v}{R} = \sqrt{2Cg} = \omega$

this has to match in the given data.

5) (25 pt) **Collision.** Two equal mass m spherical particles have a frictionless collision with coefficient of restitution e. Before the collision their two velocities are $\underline{\mathbf{v}}_1^- = v\,\hat{\mathbf{i}}$ and $\underline{\mathbf{v}}_2^- = -v\,\hat{\mathbf{i}}$. The normal to their common tangent plane at contact is $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$. In terms of some or all of $v, m, e, \theta, \hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, find the velocity of particle 2 after the collision.

Velocity diagrams :-



since impact is only in $\hat{n}$ direction, only normal component of velocities change.

by conservation of momentum for system in $\hat{n}$ direction

$$m v\cos\theta - m v\cos\theta = m V_2 - m V_1$$

(10) $\Rightarrow$ $\boxed{V_1 = V_2}$ — (1)

by definition of coefficient of restitution

$$e = \frac{\text{velocity of seperation}}{\text{velocity of approach}}$$

$$e = \frac{v_1 + v_2}{V\cos\theta + V\cos\theta} = \frac{v_2}{V\cos\theta} \qquad [\ v_1 = v_2 \text{ from (1) }]$$

(10)

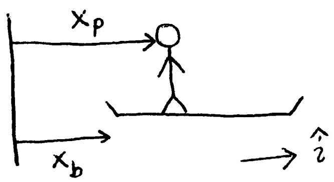
$$\Rightarrow \quad v_2 = eV\cos\theta$$

$\Rightarrow$ velocity of particle (2) after impact $= -v\sin\theta\,\hat{t} + eV\cos\theta\,\hat{n}$

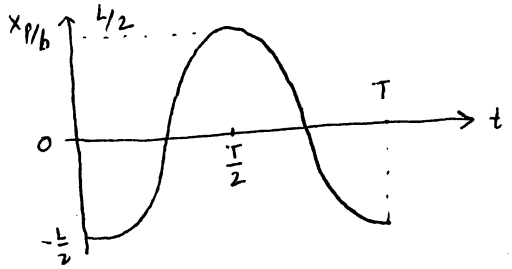
$= -v\sin\theta\,(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) + ev\cos\theta\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

(5) $\boxed{v\left\{(-\sin^2\theta + e\cos^2\theta)\,\hat{i} + \sin\theta\cos\theta\,(e+1)\,\hat{j}\right\}}$

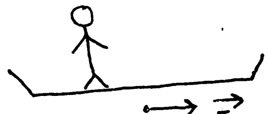
6) (25 pt) A person mass m_p walks up and back, all the way to the bow and to the stern, in a boat mass m_b. The person walks continuously and repeatedly, over and over and over again, moving relative to the boat sinusoidally in time, with period T. The length of the boat is L and the drag force on the boat from the water is $F = cv_b$. After a while the boat just moves back and forth also. How far does the boat go back and forth? (That is, the bow of the boat goes back and forth between two points, what is the distance between those two points?) Answer in terms of some or all of m_p, m_b, L and T.



motion of person wrt boat



FBD



(3)

$$x_{P/b} = -\frac{L}{2}\cos\left(\frac{2\pi}{T}t\right)$$

$$\Rightarrow v_{P/b} = \frac{2\pi}{T}\frac{L}{2}\sin\left(\frac{2\pi}{T}t\right) \qquad \text{let } \frac{2\pi}{T} = \omega$$

Material property

$$\vec{F}_D = -c\vec{V}_b \quad (3)$$

$$\Rightarrow a_{P/b} = \frac{L}{2}\omega^2\cos(\omega t) \quad \text{—} \circledast$$

(5)

LMB $\sum\vec{F} = \dot{\vec{L}}$

$$\vec{F}_D = m_p\vec{a}_p + m_b\vec{a}_b$$

$$-c\vec{V}_b = m_p\vec{a}_p + m_b\vec{a}_b \quad (3)$$

by, <u>KINEMATICS</u>

$$\vec{a}_p = \vec{a}_{P/b} + \vec{a}_b$$

given by $\circledast$

$$\Rightarrow \left\{ -c\vec{V}_b = (m_p + m_b)\vec{a}_b + m_p\vec{a}_{P/b} \right\}$$

$$\{\ \}\cdot\hat{i} \Rightarrow \quad -cV_b = (m_p+m_b)\dot{v}_b + m_p\frac{L}{2}\omega^2\cos\omega t \qquad [a_b = \dot{v}_b]$$

$$\dot{v}_b + \underbrace{\frac{c}{m_p+m_b}}_{C_1} V_b = \underbrace{-\frac{m_p\frac{L}{2}\omega^2}{m_p+m_b}}_{A}\cos\omega t$$

(5)

$$\Rightarrow \boxed{\dot{v}_b + C_1 V_b = A\cos\omega t} \text{ — (1)}$$

$\omega = 2\pi/T$

$C_1 = c/(m_p+m_b)$

$A = -\frac{m_p\frac{L}{2}\omega^2}{m_p+m_b}$

we need to solve for V_b and then x_b

The exponential part of solution $\to 0$ as $t \to \infty$. For the oscillating part guess the solution of the form

$$V_b = B\cos\omega t + D\sin\omega t$$

Plug it in (1) to get

$$-B\omega\sin\omega t + D\omega\cos\omega t + BC_1\cos\omega t + DC_1\sin\omega t = A\cos\omega t$$

equating $\sin\omega t$ and $\cos\omega t$ terms

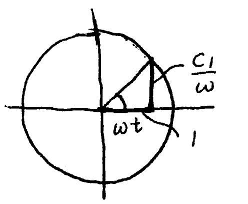
$$-B\omega + DC_1 = 0$$

$$D\omega + BC_1 = A$$

Solving it we get $\quad D = \dfrac{A\omega}{\omega^2 + C_1^2} \quad \& \quad B = \dfrac{AC_1}{\omega^2 + C_1^2}$

$$\Rightarrow \quad V_b = \frac{A}{\omega^2 + C_1^2}\left(C_1\cos\omega t + \omega\sin\omega t\right)$$

$$\Rightarrow \quad X_b = \frac{A}{\omega^2 + C_1^2}\left(\frac{C_1}{\omega}\sin\omega t - \cos\omega t\right) + \text{constant of integration}$$



$$\Rightarrow \quad \text{amplitude} = \frac{A}{\omega^2 + C_1^2}\sqrt{\left(\frac{C_1}{\omega}\right)^2 + 1} = \frac{A}{\omega\sqrt{C_1^2 + \omega^2}}$$

so, distance boat goes back and forth is $2 \times |\text{Amplitude}|$

$$\left|\frac{2A}{\omega\sqrt{C_1^2 + \omega^2}}\right| \qquad (6)$$

which is

$$\boxed{\dfrac{\left(\dfrac{m_p}{m_p + m_b}\right)\left(\dfrac{2\pi}{T}\right)L}{\sqrt{\left(\dfrac{c}{m_p + m_b}\right)^2 + \left(\dfrac{2\pi}{T}\right)^2}}} \quad \text{Ans.}$$

in dimensionless form dividing by $\frac{2\pi}{T}$ througout

$$\boxed{\dfrac{\left(\dfrac{m_p}{m_p + m_b}\right)L}{\sqrt{\left[\dfrac{cT}{2\pi(m_p + m_b)}\right]^2 + 1}}} \quad \text{Ans}$$

Solutions to Homework exam

Your Name: Manish Agarwal
ma327

Section time: ____________

T&AM 203 Homework Exam

Tuesday Dec 12, 2006

Draft December 12, 2006

4 problems, 25^{+} points each, 4 hours.

If you can do all the homework you are guaranteed a grade of C. These 4 problems are based on homework problems (see next page), or parts of homework problems, with slight changes so that memorizing answers won't help. If you can do 3 of them *fully correctly* (good work, correct answer) in 4 hours you are guaranteed a grade of at least C.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, or books allowed. You can bring a one-sided formula sheet, but not any worked out HW etc.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 5/91:	/25
Problem 6/15:	/25
Problem 6/79:	/25
Problem 6/168:	/25

Replace the text from the book (copied with the figures on the next page) with the problem statements below. For all problems all quantities are described with letters and not numbers. Answers should be expressed with letters. Don't use the numbers given with the book problem statement. Ignore the numbers in the figures.

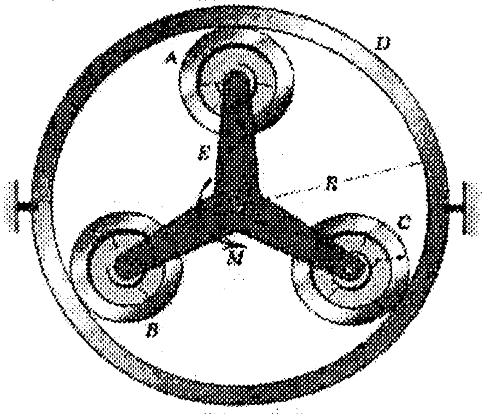
5/91 Given $\dot{a}, L_{CD}, L_{AB}, a$ and b find $\dot{b}$. (Note $\underline{\mathbf{v}}_A = -\dot{a}\,\hat{\mathbf{i}}$.)

6/15 The car has mass m and gravity points down with constant g. The coefficient of friction between wheels and road is μ. For rear wheel drive what is the maximum possible car acceleration? (Answer in terms of some or all of a, b, h, m, g and μ.)

6/79 A uniform disk with radius R and mass m rolls down a slope θ. The friction coefficient μ is large enough so the disk rolls without slipping. Gravity g points down. Find the component of the force that acts on the disk from the ramp that is tangent to the ramp. Answer in terms of some or all of R, m, g, θ and μ.

6/168 The angular velocity and acceleration of the spider are given as ω_s and α_s. Find the acceleration of a point on one of the planet gears that is, at the instant in question, in contact with the ring gear. Answer in terms of some or all of R, r, ω_s, α_s and unit vectors you clearly define.

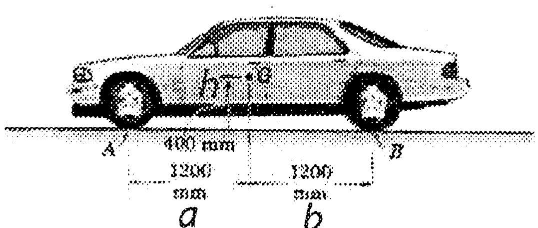
6/168 A planetary gear system is shown, where the gear teeth are omitted from the figure. Each of the three identical planet gears A, B, and C has a mass of 0.8 kg, a radius $r = 50$ mm, and a radius of gyration of 30 mm about its center. The spider E has a mass of 1.2 kg and a radius of gyration about O of 60 mm. The ring gear D has a radius $R = 150$ mm and is fixed. If a torque $M = 5$ N·m is applied to the shaft of the spider at O, determine the initial angular acceleration α of the spider.



Problem 6/168

6/15 The 1650-kg car has its mass center at G. Calculate the normal forces N_A and N_B between the road and the front and rear pairs of wheels under conditions of maximum acceleration. The mass of the wheels is small compared with the total mass of the car. The coefficient of static friction between the road and the rear driving wheels is 0.8.

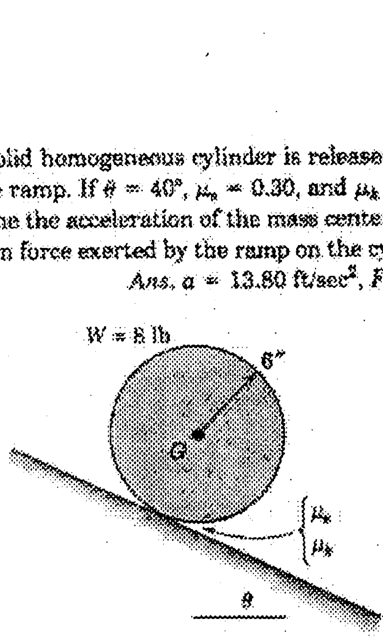
Ans. $N_A = 6.85$ kN, $N_B = 9.34$ kN



Problem 6/15

6/79 The solid homogeneous cylinder is released from rest on the ramp. If $\theta = 40°$, $\mu_s = 0.30$, and $\mu_k = 0.20$, determine the acceleration of the mass center G and the friction force exerted by the ramp on the cylinder.

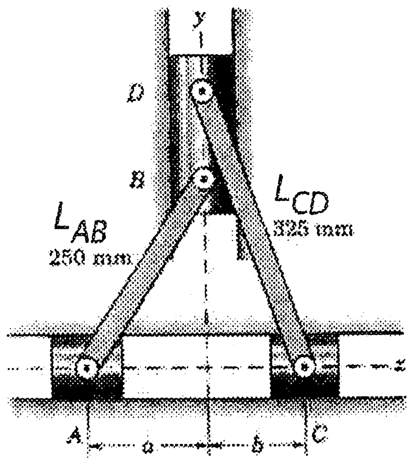
Ans. $a = 13.80$ ft/sec^2, $F = 1.714$ lb



Problem 6/79

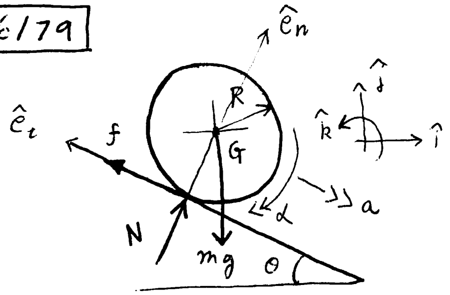
►5/91 At the instant represented, $a = 150$ mm and $b = 125$ mm, and the distance $a + b$ between A and C is decreasing at the rate of 0.2 m/s. Determine the common velocity v of points B and D for this instant.

Ans. $v = 0.0536$ m/s



Problem 5/91

6/179



LMB: $-mg\,\hat{j} + N\,\hat{e}_n + f\,\hat{e}_t = -m\,a\,\hat{e}_t$

$\{\}\cdot\hat{e}_t \Rightarrow -mg\sin\theta + f = -ma$

$-f + mg\sin\theta = ma \quad -(1)$

$\{\}\cdot\hat{e}_n \Rightarrow \quad N = mg\cos\theta \quad -(2)$

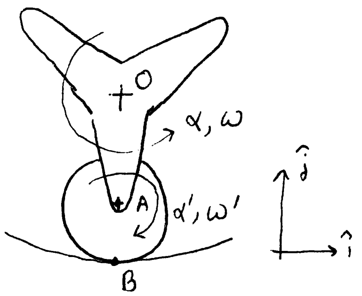
AMB: about G

$-fR = -I\alpha \qquad fR = \frac{1}{2} m R^2 \alpha \quad -(3)$

No slip condition: $\alpha R = a \quad -(4)$

(1), (3) and (4) $\Rightarrow$ $\boxed{f = \dfrac{mg\sin\theta}{3}}$ Ans

6/168



$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$$= \cancel{\vec{a}_O}^{\,0} + \vec{a}_{A/O} + \vec{a}_{B/A}$$

$$= \vec{\alpha} \times \vec{OA} + \vec{\omega} \times (\vec{\omega} \times \vec{OA}) + \vec{\alpha}' \times \vec{AB} + \vec{\omega}' \times (\vec{\omega}' \times \vec{AB})$$

$$= \alpha(R-r)\,\hat{i} + \omega^2(R-r)\,\hat{j} - \alpha' r\,\hat{i} + \omega'^2 r\,\hat{j}$$

$$= [\alpha(R-r) - \alpha' r]\,\hat{i} + [\omega^2(R-r) + \omega'^2 r]\,\hat{j} \qquad 10$$

by no slip condition, tangential acceleration is zero.

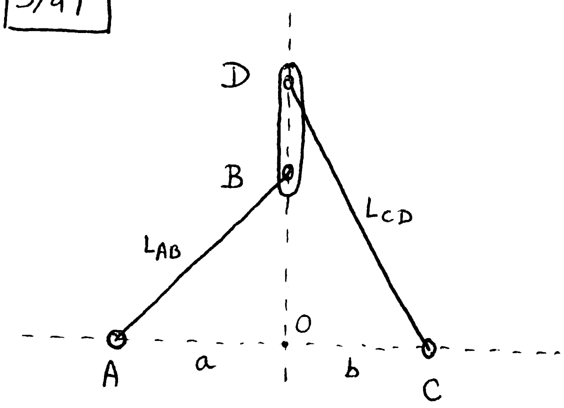
$$\vec{a}_B \cdot \hat{i} = 0 \quad \Rightarrow \quad \alpha(R-r) - \alpha' r = 0$$

$$\Rightarrow \quad \omega(R-r) - \omega' r = 0 \Rightarrow \quad \omega' = \frac{\omega(R-r)}{r} \qquad 10$$

$$\Rightarrow \quad \vec{a}_B = \left\{ \omega^2(R-r) + \left[\frac{\omega(R-r)}{r}\right]^2 \right\} \hat{j} \qquad 5$$

$$= \boxed{\frac{\omega^2 (R-r) R}{r}\,\hat{j}}$$ Ans

5/91



$\mathring{a}$ given

to find $\mathring{b}$

$$OB = \sqrt{L_{AB}^2 - a^2}$$

$$OD = \sqrt{L_{CD}^2 - b^2}$$

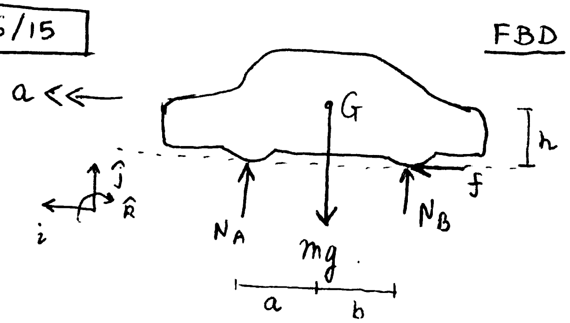
$$DB = OD - OB = \sqrt{L_{CD}^2 - b^2} - \sqrt{L_{AB}^2 - a^2}$$

$\frac{d(DB)}{dt} = 0$ since DB is of fixed length

$$\Rightarrow \quad \frac{1}{\cancel{2}} \frac{-\cancel{2} b \mathring{b}}{\sqrt{L_{CD}^2 - b^2}} - \frac{1}{\cancel{2}} \frac{-\cancel{2} a \mathring{a}}{\sqrt{L_{AB}^2 - a^2}} = 0$$

$$\Rightarrow \quad \boxed{\mathring{b} = \frac{a}{b} \sqrt{\frac{L_{CD}^2 - b^2}{L_{AB}^2 - a^2}} \, \mathring{a}}$$ Ans

6/15



LMB: $f\hat{i} + N_A\hat{j} + N_B\hat{j} - mg\hat{j} = m a_x \hat{i}$

$\{\} \cdot \hat{i}$ $\quad f = ma$ — ①

$\{\} \cdot \hat{j}$ $\quad N_A + N_B = mg$ — ②

AMB: about point G

$$N_A a + f h = N_B b \quad — ③$$

acceleration is max for max f

$\Rightarrow \quad f = \mu N_B$ — ④

②, ③ and ④ $\Rightarrow$ $N_B = \dfrac{mga}{a+b-\mu h}$

$$\Rightarrow \quad a = \frac{f}{m} = \frac{\mu N_B}{m} = \frac{\mu g a}{a+b-\mu h} = \boxed{\frac{\mu g}{1 + \frac{b}{a} - \mu \frac{h}{a}}}$$ Ans

Your Name: Joe Burns

Your TA: Andy Ruina

T&AM 203 FINAL EXAM

Wednesday May 17, 2000

Draft May 9, 2000

4 problems, 100 points, and 150 minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Six pages of formulas from the front and back of the text are provided. The back of the test can be used for tentative scrap work. Ask for extra scrap paper if you need it.

b) Full credit if

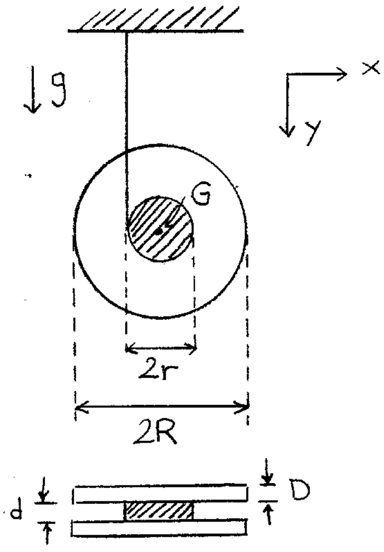
- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25
Problem	3:	/25

TOTAL: enough to pass /100

1)(25 pts) Yo-yo. A yo-yo of mass $2m$ is made of two identical disks (mass m, radius R, thickness D) glued on either side of a massless spindle (radius r, thickness d). A string is wrapped around the spindle and unwinds without friction. The string has total length L and is infinitesimally thin and massless. G is at the yo-yo's center of mass.



a) (3 pts) Does G move in the x-direction as the yo-yo falls and unwinds? Why or why not?

b) (6 pts) Find G's vertical acceleration. Comment on the two cases:
i) $R << r$ and ii) $R >> r$,

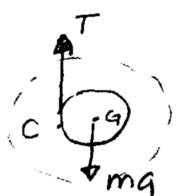
c) (4 pts) Find the tension in the string.

d) (5 pts) Write an expression for the total kinetic energy of the yo-yo when G's speed is v.

e) (5 pts) If $r << L$ and the yo-yo starts from rest, find v when the string is fully unwound.

f) (2 pts) Under what circumstances will the yo-yo rewind completely?

FBD



a.) Since $\Sigma \underline{F}$ is only in the $\hat{j}$ direction, $\boxed{\underline{a}_G \text{ is only in } \hat{j}}$, $\underline{v}_0 = 0$ $\therefore$ no x motion.

b.) Method I

$\Sigma \underline{M}_{/C} = \dot{\underline{H}}_C$

$r\hat{i} \times mg\hat{j} = \dot{\underline{H}}_G + \underline{r} \times m\underline{a}$

$\cancel{m}rg\,\hat{k} = \frac{1}{2}\cancel{m}R^2\dot{\omega}\,\hat{k} + r\hat{i} \times \cancel{m}a\hat{j}$

$rg = \frac{1}{2}R^2\frac{a}{r} + ra = \left(\frac{1}{2}\frac{R^2}{r} + \frac{r^2}{r}\right)a$

$\therefore$ $\boxed{a = \dfrac{g}{1 + \frac{1}{2}\frac{R^2}{r^2}}}$ $\approx g$ if $R \ll r$; $a \ll g$ if $r \ll R$

Method II

Kinematics: $\dot{\omega} r = a$ — (1)

$(LMB)\cdot\hat{j} \Rightarrow -T + mg = ma$ — (2)

$AMB_G \Rightarrow \frac{1}{2}mR^2\dot{\omega} = Tr$ — (3)

3 eqns for the 3 unknowns $\dot{\omega}, a, T$

c.) Plug a back into (2) $T = m(g-a) = mg\left(\dfrac{1 + \frac{1}{2}\frac{R^2}{r^2} - 1}{1 + \frac{1}{2}\frac{R^2}{r^2}}\right) = \boxed{\dfrac{mg}{1 + \frac{2r^2}{R^2}} = T}$

d.) For planar motion

$KE = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$ $\left[\text{or} = \frac{1}{2}I_C\omega^2 \text{ since in pure rotation about } C\right]$

$= \frac{1}{2}mv^2 + \frac{1}{2}\left[\frac{1}{2}mR^2\frac{v^2}{r^2}\right] = \boxed{\frac{mv^2}{2}\left[1 + \frac{1}{2}\frac{R^2}{r^2}\right] = KE}$

e.) Use conservation of energy with PE measured from top

$PE + KE = -mgR + 0 = -\cancel{m}gL + \frac{1}{2}\cancel{m}v^2\left(1 + \frac{1}{2}\frac{R^2}{r^2}\right)$

E@ TOP E@ BOTTOM

We ignore r & R compared to L

$\Rightarrow \boxed{v = \sqrt{\dfrac{2g(L-R)}{1 + \frac{1}{2}\frac{R^2}{r^2}}}}$

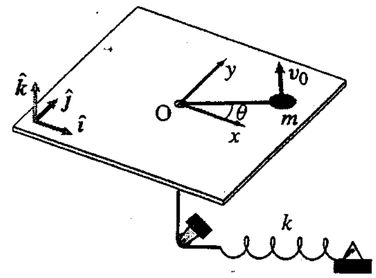
could also get by $v = \sqrt{2aL}$

f.) As long as energy is conserved (which it never will be in a real system), the yo-yo will rewind perfectly.

loss in string, slip & "bounce" at bottom.

2)(25 pts) Particle on a springy leash. A particle with mass m slides on a rigid horizontal frictionless plane. It is held by a string which is in turn connected to a linear elastic spring with constant k. The string length is such that the spring is relaxed when the mass is on top of the hole in the plane. The position of the particle is $\vec{r} = x\hat{\imath} + y\hat{\jmath}$. For each of the statements below, state the circumstances in which the statement is true (assuming the particle stays on the plane). Justify your answer with convincing explanation and/or calculation.

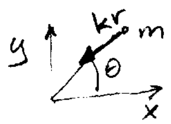
a) (2 pts) The force of the plane on the particle is $mg\hat{k}$.
b) (2 pts) $\ddot{x} + \frac{k}{m}x = 0$
c) (2 pts) $\ddot{y} + \frac{k}{m}y = 0$
d) (3 pts) $\ddot{r} + \frac{k}{m}r = 0$, where $r = |\vec{r}|$
e) (2 pts) $r =$ constant
f) (3 pts) $\dot{\theta} =$ constant
g) (3 pts) $r^2\dot{\theta} =$ constant
h) (2 pts) $m(\dot{x}^2 + \dot{y}^2) + kr^2 =$ constant
i) (3 pts) The trajectory is a straight line segment.
j) (3 pts) The trajectory is a circle.



a.) Since the particle has no vertical ($\hat{k}$) motion, $\sum F_z = 0$

$\Rightarrow N = mg$ Always true

b.) c.) Writing LMB in horizontal plane and using Cartesian coords,



$$\sum \underline{F} = m\underline{a} \Rightarrow -k\underline{r} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) = -k(x\hat{\imath} + y\hat{\jmath})$$

$$\therefore \boxed{\begin{array}{l}\ddot{x} + \frac{k}{m}x = 0 \\ \ddot{y} + \frac{k}{m}x = 0\end{array}}$$

always true as long as a frictionless

d.) If we express $\underline{a}$ in polar coords

$$-kr\hat{e}_r = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right] \quad *$$

$$\hat{e}_r \cdot (LMB) \Rightarrow -kr = m(\ddot{r} - r\dot{\theta}^2) \Rightarrow \boxed{\ddot{r} + \frac{k}{m}r = 0 \text{ if } \dot{\theta} = 0}$$

motion is purely radial

e.) $r =$ constant when $m(-r\dot{\theta}^2) = -kr$ (part of radial $*$)

$$\therefore \dot{\theta} = \sqrt{\frac{k}{m}} \text{ and is const}$$

f.) This is a variant of e.), $\dot{\theta}$ will be constant if it is $= \sqrt{\frac{k}{m}}$ and r is constant

g.) $r^2\dot{\theta}$ is angular momentum per unit mass (i.e., for planar motion $H = m\,r(r\dot{\theta})$; or $r^2\dot{\theta} = \frac{H}{m}$)

From $*$, we see the $\hat{e}_\theta$ component is $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{2\cancel{r}\dot{r}\dot{\theta} + r^{\cancel{2}}\ddot{\theta}}{\cancel{r}} = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

Since there is no $\hat{e}_\theta$ force component, this is always true $\Rightarrow r^2\dot{\theta} =$ const.

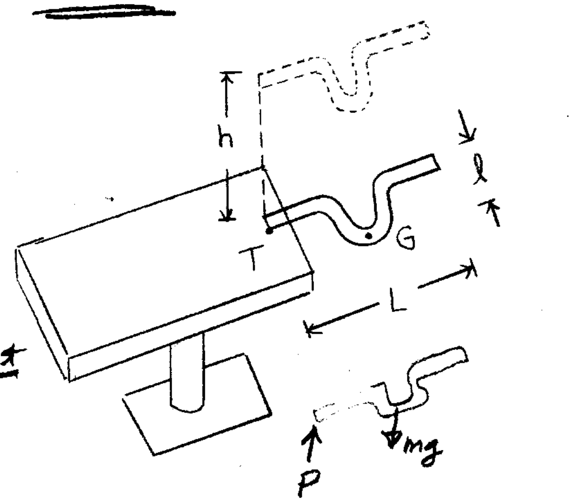
h.) This is twice total energy $2\left(\frac{1}{2}mv^2 + \frac{1}{2}kr^2\right)$. Thus always true for this conservative system.

i.) As part d) showed, get straight line motion if purely radial motion

j.) As parts e) and f) mentioned, circular motion is possible when $\dot{\theta} = \sqrt{\frac{k}{m}}$

3)(25 pts) The problems below (a-d) are independent.

a) (5 pts) A falling wire. A U-shaped wire (of the given dimensions) falls from a height h without rotation and strikes a tabletop at T completely inelastically. Briefly defend your answers to the following questions.



During impact:

i) (1 pt) Is the wire's linear momentum conserved?
ii) (1 pt) Is the wire's angular momentum about G constant?
iii) (1 pt) Is the wire's angular momentum about T constant?
iv) (1 pt) Is the wire's angular velocity conserved?
v) (1 pt) Is the wire's total mechanical energy (kinetic + potential) constant?

i.) LMB $\Rightarrow$ $\underline{F} = \frac{d}{dt} m\underline{v}$. External forces act ∴ linear momentum not const

ii.) AMB $\Rightarrow$ $\underline{M}_{/G} = \dot{\underline{H}}_{/G}$ $\quad$ $\underline{P}$ has moment about G ∴ $\underline{H}_{/G} \neq$ constant

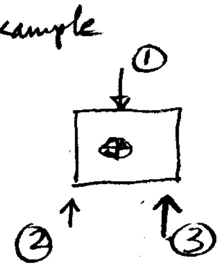
iii) The impulsive force $\underline{P}$ has no moment about T; we can ignore mg in comparison to P and thus $\underline{H}_T$ is approximately constant

iv.) It wasn't rotating beforehand and will afterward (due to $\underline{M}_G$). $\underline{\omega}$ not conserved.

v.) Energy is lost in collisions generally unless they are elastic. This is an inelastic collision ∴ not constant E
(heating, sound)

b) (5 pts) Mars Polar Lander. Last December when the Mars Polar Lander was lost, some blamed its simple thrusters (devices that eject gas and thus are capable of providing an impulse in a single direction). Argue using dynamical principles, the minimum number of thrusters required to have complete three-dimensional control.

The full dynamical description of a body is given by the 3-D position of the CM (LMB) plus three angles to give its orientation (AMB). For each of these, you need to be able to go + and −. We can control the angular orientations with the same thrusters. For example

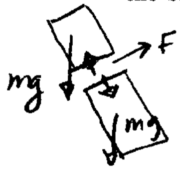


1 for downward motion
2+3 for pure upward
1+3 for + rotation
2+1 for − rotation

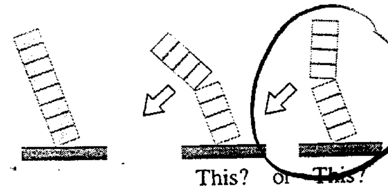
$\boxed{3 \times 3 = 9 \text{ thrusters}}$

c) (5 pts) A falling tower. Frequently parents will build a tower of blocks for their children. Just as frequently, kids knock them down. In falling (even when they start to topple aligned), these towers invariably break in two (or more) pieces at some point along their length. Why does this occur? What condition is satisfied at the point of the break? Will the stack bend towards or away from the floor after the break?

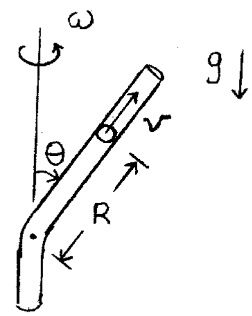
$\dot{\theta}, \ddot{\theta}$



The tower will rotate as a single piece only as long as the forces between the various blocks are compressive and friction sufficient to prevent sliding. Once they become tensile, it will separate since unglued blocks cannot support tension. Top block has largest (largest r) acceleration & least compressive force. Thus the top breaks free first. And then the others follow. The angular motion of the solid stack is given by $\underline{M} = I\dot{\omega}\hat{k}$ where $I \sim h^2$ by $M \sim h$. Thus taller stacks fall more slowly. Hence the break-aways will lag behind & will appear to bend away from the floor.

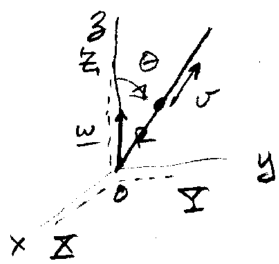


d) (10 pts) A pea shooter. A pea of mass m is being blown out of a tube at constant speed v. The tube itself is at a constant angle θ to the vertical and spins at constant angular velocity $\omega_o \hat{\mathbf{k}}$ (i.e., it sweeps out a cone). At the instant shown, the tube is in the yz-plane and the pea is at a distance R along the tube.

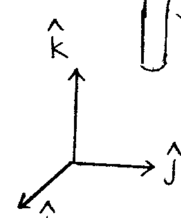


i) (7 pts) What is the pea's acceleration?

ii) (3 pts) What force acts on it?



Choose XYZ (inertial) as given.
Fix moving system xyz such that it spins with the tube and is instantaneously parallel to XYZ (as shown).



$$\underline{a}_{pea} = \underline{a}_{rel} + \underline{a}_o + \underline{\omega} \times (\underline{\omega} \times \underline{R}) + \dot{\underline{\omega}} \times \underline{R} + 2\underline{\omega} \times \underline{v}$$

$\underline{a}_{rel} = 0$ pea moves with constant speed along tube

$\underline{a}_o = 0$ origins are together always

$\dot{\underline{\omega}} = 0$ moving system spins at constant rate

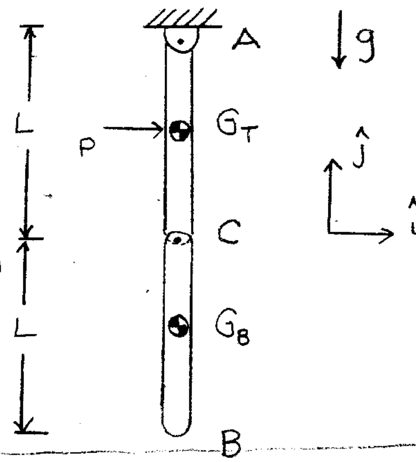
$$\therefore \underline{a}_{pea} = \omega\hat{k} \times (\omega\hat{k} \times R[\sin\theta\,\hat{j} + \cos\theta\,\hat{k}]) + 2\omega\hat{k} \times v(\sin\theta\,\hat{j} + \cos\theta\,\hat{k})$$

$$= \omega\hat{k} \times \omega R\sin\theta(-\hat{i}) - 2\omega v\sin\theta\,\hat{i}$$

$$\boxed{\underline{a}_{pea} = -\omega^2 R\sin\theta\,\hat{j} - 2\omega v\sin\theta\,\hat{i}}$$

LMB ⇒ $\boxed{\underline{F}_{EXT} = m\,\underline{a}_{pea}}$ with $\underline{a}_{pea}$ given above. This force is caused by wall forces (contact) gravity and gas pressure.

4) **(25 pts) Double pendulum.** Two identical homogeneous slender bars (weight W, length L, frictionless hinges) hang vertically in a gravity field. They are initially at rest when a horizontal force $P\hat{\imath}$ is suddenly applied at the center of the top bar.



a) (10 pts) Write out expressions for the accelerations of the centers of mass G_T and G_B in terms of the angular motions of the bars.

b) (10 pts) Write down sufficient equations to solve for the reaction forces at A and C, and for the angular motions.

c) (5 pts) Describe how to use Matlab to solve these equations. You need not solve them.

Instantaneously,

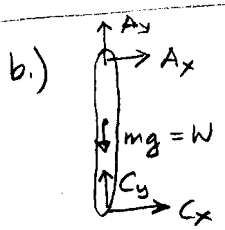
a.) $\underline{a}_{G_T} = \underline{\alpha}_T \times \frac{L}{2}(-\hat{j}) = \alpha_T \frac{L}{2}\hat{\imath}$

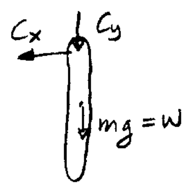
Does not include $\underline{\omega} \times (\underline{\omega} \times \frac{L}{2})$ because $\underline{\omega} = 0$ initially

$\underline{a}_{B_T} = \underline{a}_C + \underline{a}_{B/C} = \alpha_T L\hat{\imath} + \alpha_B \hat{k} \times \frac{L}{2}(-\hat{j})$

$= \left(\alpha_T L + \alpha_B \frac{L}{2}\right)\hat{\imath}$

b.)





We have 6 unknowns : $\underline{A}$ & $\underline{C}$ (2 components each) and 2 αs and 6 eqns (LMB for top and bottom bars - 2 components each) and AMB for each bar.

TOP : LMB $\underline{\Sigma F}_{EXT} = -W\hat{j} + \underline{A} + \underline{C} = m\underline{a}_{G_T}$

$$\boxed{\frac{W}{g}\alpha_{G_T}\frac{L}{2}\hat{\imath} = (A_x + C_x)\hat{\imath} + (A_y + C_y - W)\hat{j}}$$

AMB $\Sigma_i \underline{M}_{G_T} = \underline{\dot{H}}_{G_T} = \frac{1}{12}\frac{W}{g}L^2\dot{\omega}_{G_T}\hat{k}$

$$\boxed{(C_x - A_x)\frac{\cancel{L}}{\cancel{2}} = \frac{WL^{\cancel{2}}}{\cancel{12}_{6}\,g}\alpha_T}$$

Bottom

LMB $\Sigma \underline{F}_{EXT} = -W\hat{j} - \underline{C} = m\underline{a}_{G_B}$

$$\boxed{-C_x\hat{\imath} - (+W + C_y)\hat{j} = \frac{W}{g}L\left(\alpha_T + \frac{\alpha_B}{2}\right)\hat{\imath}}$$

AMB $\Sigma \underline{M}_{G_B} = \underline{\dot{H}}_{G_T} = \frac{W}{12g}L^2\alpha_B\hat{k}$

$$\boxed{C_x\frac{L}{2} = \frac{W}{12g}L^2\alpha_B}$$

(c) The boxed equations are 6 algebraic eqns for the unknowns $A_x, A_y, C_x, C_y, \alpha_T$ & α_B. We write them in matrix form $Ax = b$, where x is a column vector of these, and solve $x = A^{-1}B$. In Matlab we would write

$\underline{\underline{A}} = [6\ ;\ 6\ ;\ 6\ ;\ 6\ ;\ 6\ ;\ 6]$; (entries)

$b = [\ ;\ ;\ ;\ ;\ ;\ ]$;

$x = A\backslash b$

"Solutions" *

Your Name: ANDY RUINA

Your TA: BURNS

T&AM 203 Prelim 1

Tuesday February 29, 2000 7:30 — 9:00$^+$ PM

Draft February 26, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Two pages of formulas, from the front of the text, and a blank page for tentative scrap work are provided at the back. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;

↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
± all signs and directions are well defined with sketches and/or words;

→| reasonable justification, enough to distinguish an informed answer from a guess, is given;
* you clearly state any reasonable assumptions if a problem seems *poorly defined*;

- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);

□ your answers are $\boxed{\text{boxed}}$ in; and

≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

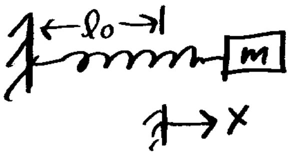
* The quotations are because often my "Solutions" have some remaining errors. (error in 1c corrected here)

I hope!

Problem	1:	30/30
Problem	2:	35/35
Problem	3:	35/35
TOTAL:		100 /100

1)(30 pts) The problems below are independent. MATLAB commands are not allowed except in part (c).

1a) (10 pts) A mass m is connected to a spring k and released from rest with the spring stretched a distance d from its static equilibrium position. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? (Answer in terms of some or all of m, k, and d.) [Neglect gravity and friction.]



FBD: $\leftarrow\square$ kx

LMB: $F = ma$

$\Rightarrow -kx = m\ddot{x}$

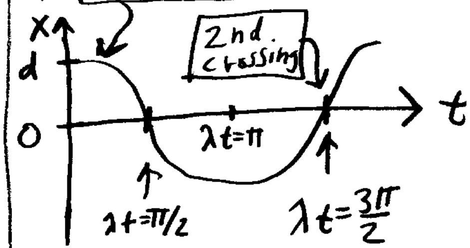
$\Rightarrow \ddot{x} + (k/m)x = 0$

$\Rightarrow x = A\cos(\lambda t) + B\sin(\lambda t)$

$\lambda = \sqrt{k/m}$

$x(0) = d \Rightarrow A = d$, $\dot{x}(0) = 0 \Rightarrow B = 0$

$\Rightarrow x = d\cos(\lambda t)$,



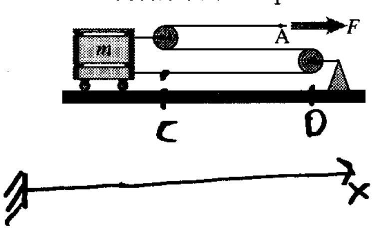
At second crossing

$\lambda t = 3\pi/2$

$\Rightarrow t = 3\pi/2\lambda$

$$\Rightarrow \boxed{t = \frac{3\pi}{2\sqrt{k/m}}}$$

1b) (10 pts) Assuming equal masses and equal forces in the two cases, what is the ratio of the acceleration of point A to that of point B? [assume massless ideal pulleys etc]



FBD: $\square$ with $\rightarrow F$, $\rightarrow F$, $\rightarrow F$

LMB

$3F = m\ddot{x}_c$

$\ddot{x}_c = 3F/m$ (1)

Kinematics

$const = \ell_{CA} + 2\ell_{CD}$

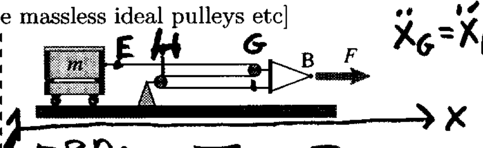
$= (x_A - x_c) + 2(x_D - x_c)$ $\rightarrow 0$

$(\ddot{const}) = (\ddot{x}_A - \ddot{x}_c) + 2(\cancel{\ddot{x}_D} - \ddot{x}_c)$

$0 = \ddot{x}_A - 3\ddot{x}_c$ (2)

$\Rightarrow \ddot{x}_A = 3\ddot{x}_c$

(1)&(2) $\Rightarrow$ $\ddot{x}_A = 9F/m$ (3)



$\ddot{x}_G = \ddot{x}_B$

FBDs: $\square \rightarrow T$

$T \leftarrow$, $T \leftarrow$, $T \leftarrow$, $\rightarrow F$

$\Rightarrow T = F/3$

LMB

$T = m\ddot{x}_E$

$F/3 = m\ddot{x}_E$ (4)

Kinematics

$const = \ell_{GE} + 2\ell_{GH}$

$= (x_G - x_E) + 2(x_G - x_H)$

$(\ddot{const}) = (\ddot{x}_G - \ddot{x}_E) + 2(\ddot{x}_G - \cancel{\ddot{x}_H})$ $\rightarrow 0$

$0 = 3\ddot{x}_G - \ddot{x}_E$

$\Rightarrow \ddot{x}_G = \ddot{x}_B = \ddot{x}_E/3$ (5)

(4)&(5) $\Rightarrow$ $\ddot{x}_B = F/9m$

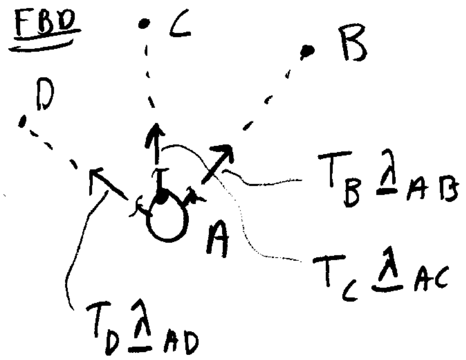
Comparison

$$\frac{\ddot{x}_A}{\ddot{x}_B} = \frac{9F/m}{F/9m}$$

$= \boxed{81}$ (6)

(Point B "feels" 81 times less massive than pt. A)

1c) (10 pts) In three-dimensional space with no gravity a particle with $m = 3\,\mathrm{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\underline{\mathbf{a}} = (1\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{\mathbf{k}})\,\mathrm{m/s^2}$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]



LMB $\quad T_B \underline{\lambda}_{AB} + T_C \underline{\lambda}_{AC} + T_D \underline{\lambda}_{AD} = m\underline{a}$

$$\Rightarrow \underbrace{\overbrace{\begin{bmatrix} \begin{bmatrix}\underline{\lambda}_{AB}\end{bmatrix} & \begin{bmatrix}\underline{\lambda}_{AC}\end{bmatrix} & \begin{bmatrix}\underline{\lambda}_{AD}\end{bmatrix} \end{bmatrix}}^{A}}_{\text{comps of unit vectors in columns}} \begin{bmatrix} T_A \\ T_B \\ T_C \end{bmatrix} = m \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\underbrace{[A][T] = m[a]}_{\text{need to solve for } [T]}$$

```
% a MATLAB script file to find 3 tensions
m    = 3;
a    = [ 1 2 3]';

rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';

uAB = rAB/norm(rAB); % norm gives vector magnitude
% You write the code below (4 to 5 lines).
% Don't copy any of the numbers above.
% Don't do any arithmetic on the side.
```

This code does the job.

```
uAC = rAC/norm(rAC);   % The other two
uAD = rAD/norm(rAD);   % unit vectors
A = [uAB  uAC  uAD];   % assemble A
T = A\(m*a)            % Solve with backslash
```

2)(35 pts) A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\underline{\mathbf{v}}|$, where $\underline{\mathbf{v}}$ is the velocity. Do not neglect gravity.

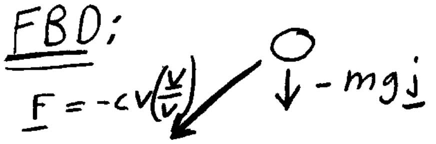
a) (10 ptss) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) (15 pts) Starting with a free body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (10 pts) (challenge, do last, long calculation) Assume the particle is thrown from $\underline{\mathbf{r}} = \underline{\mathbf{0}}$ with $\underline{\mathbf{v}} = v_{x0}\,\hat{\mathbf{i}} + v_{y0}\,\hat{\mathbf{j}}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.)

[Hint: i) find $x(t)$ and $y(t)$ like in the homework, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.). If you use Matlab, start your code by assigning any non-trivial values to all constants.]

FBD:



LMB : $\underline{F} = m\underline{a}$

$$\Rightarrow \{-c\underline{v} - mg\underline{j} = m(\dot{v}_x\underline{i} + \dot{v}_y\underline{j})\}$$

$$\{\}\cdot\underline{i} \Rightarrow \; -cv_x = m\dot{v}_x \Rightarrow m\ddot{x} + c\dot{x} = 0$$

$$\{\}\cdot\underline{j} \Rightarrow \; -cv_y - mg = m\dot{v}_y \Rightarrow m\ddot{y} + c\dot{y} = -mg$$

$$\boxed{\begin{aligned} m\ddot{x} + c\dot{x} &= 0 \\ m\ddot{y} + c\dot{y} &= -mg \end{aligned}} \quad (b)$$

Steady State $\Rightarrow \dot{v}_x = 0,\ \dot{v}_y = 0 \Rightarrow v_x = 0,\ v_y = \frac{-mg}{c}$

$\Rightarrow$ steady state falling speed $= \boxed{mg/c}$ (a)

Trajectory eqs

(1) $\dot{v}_x + \frac{c}{m}v_x = 0$

(3) $\dot{x} = v_x$

(5) $\dot{v}_y + \frac{c}{m}v_y = -g$

(7) $\dot{y} = v_y$

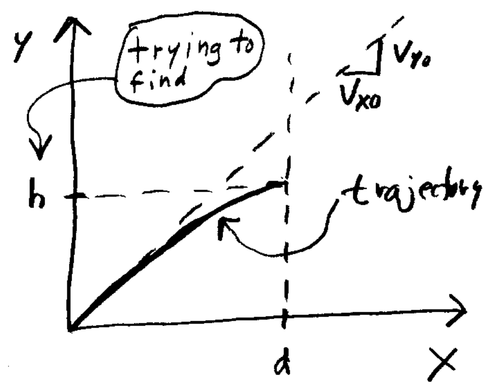
IC's

(2) $v_x(0) = v_{x0}$

(4) $x(0) = 0$

(6) $v_y(0) = v_{y0}$

(8) $y(0) = 0$



Find $x(t)$

(1) $\Rightarrow \quad v_x = c_1 e^{-(c/m)t}$, (2) $\Rightarrow c_1 = v_{x0} \Rightarrow v_x = v_{x0}e^{-(c/m)t}$

(3) $\Rightarrow \quad x(t) = -\frac{mv_{x0}}{c}e^{-(c/m)t} + c_2$, (4) $\Rightarrow c_2 = \frac{mv_{x0}}{c} \Rightarrow \boxed{x = \frac{mv_{x0}}{c}\left(1 - e^{-(c/m)t}\right)}$

(like homework)

Find t from x

$$\frac{cx}{mv_{x0}} = 1 - e^{-(c/m)t} \Rightarrow e^{-(c/m)t} = 1 - \frac{cx}{mv_{x0}} \Rightarrow -(c/m)t = \ln\left(1 - \frac{cx}{mv_{x0}}\right)$$

$$\Rightarrow \boxed{t = -\frac{m}{c}\ln\left(1 - \frac{cx}{mv_{x0}}\right)} \quad (1)$$

{note, this inversion is possible because particle always moves to right.}

Problem 2 continued.

Solve for $y(t)$

$$(5) \Rightarrow \quad v_y = c_3 e^{-(c/m)t} - \frac{mg}{c}, \quad (6) \Rightarrow c_3 = V_{y0} + \frac{mg}{c}$$

$$\Rightarrow \quad V_y(t) = \left(V_{y0} + \frac{mg}{c}\right) e^{-(c/m)t} - \frac{mg}{c}$$

$$(7) \Rightarrow \quad y(t) = -\frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right) e^{-(c/m)t} - \frac{mg}{c} t + c_4$$

$$(8) \Rightarrow \quad 0 = -\frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right) + c_4 \Rightarrow c_4 = \frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right)$$

$$\Rightarrow \quad \boxed{y(t) = \frac{m}{c}\left(V_{y0} + \frac{mg}{c}\right)\left(1 - e^{-(c/m)t}\right) - \frac{mg}{c} t} \quad (2)$$

(like homework)

[Define: char. time, $t_c = m/c$ | V steady state, $V_s = \frac{mg}{c}$
{to simplify algebra} max. x, $x_m = mV_{x0}/c$]

$$(1) \Rightarrow \quad t = -t_c \ln(1 - x/x_m) \qquad (3)$$

$$(2) \Rightarrow \quad y(t) = t_c (V_{y0} + V_s)\left(1 - e^{-t/t_c}\right) - V_s t \qquad (4)$$

Substitute (4) into (3)

$$y = t_c (V_{y0} + V)\left(1 - (1 - x/x_m)\right) + V_s t_c \ln(1 - x/x_m)$$

$$y = t_c (V_{y0} + V_s)\left(\frac{x}{x_m}\right) + V_s t_c \ln(1 - x/x_m)$$

$$\Rightarrow h = t_c (V_{y0} + V_s)\left(\frac{d}{x_m}\right) + V_s t_c \ln(1 - d/x_m)$$

$$\boxed{h = V_s t_c \left[\left(\frac{V_{y0}}{V_s} + 1\right)\left(\frac{d}{x_m}\right) + \ln(1 - d/x_m)\right]} \quad (c)$$

$x_m = mV_{x0}/c$

$V_s = mg/c$

$t_c = m/c$

Checks:

$d \to 0 \Rightarrow$

$\ln(1 - d/x_m) \approx -d/x_m$

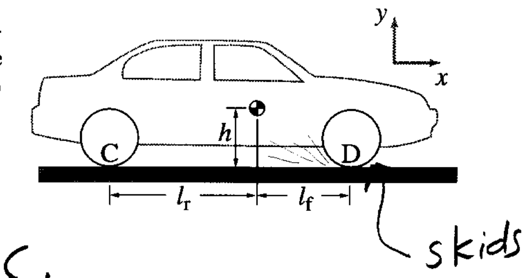
$$\Rightarrow h \sim \cancel{V_s} t_c \left(\frac{V_{y0}}{\cancel{V_s}}\right)\frac{d}{x_m} = \frac{\cancel{m}}{\cancel{c}} V_{y0} \frac{d}{\cancel{m}V_{x0}/\cancel{c}} = \frac{V_{y0}}{V_{x0}} d \quad \text{(which makes sense, see sketch)}$$

$$c \to 0 \Rightarrow \ln(1 - d/x_m) \approx -d/x_m - \frac{1}{2}(d/x_m)^2 \Rightarrow h \sim \frac{V_{y0}}{V_{x0}} d - \frac{1}{2}\left(\frac{m}{c}\right)\left(\frac{mg}{c}\right)\left(\frac{d}{mV_{x0}/c}\right)^2$$

Taylor series

$$\sim \frac{V_{y0}}{V_{x0}} - \frac{1}{2} g \left(\frac{d}{V_{x0}}\right)^2 \quad \text{(parabolic trajectory)}$$

3) 35 pts) Car accelerating. A car (mass $=m$) with a big motor, front-wheel drive, and a stiff suspension accelerates to the right with the front wheels over-powered and skidding (friction coefficient $=\mu$) and back wheels turning freely.



a) (5 pts) Assuming the car starts from rest and has constant acceleration a, how far has it travelled in time t? (Answer in terms of a and t.) [Not a trick, just easy.].

$$v(t) = \int a(t)\,dt = \int a\,dt = at + C_1$$

$$v(0) = 0 \Rightarrow \quad v(t) = at$$

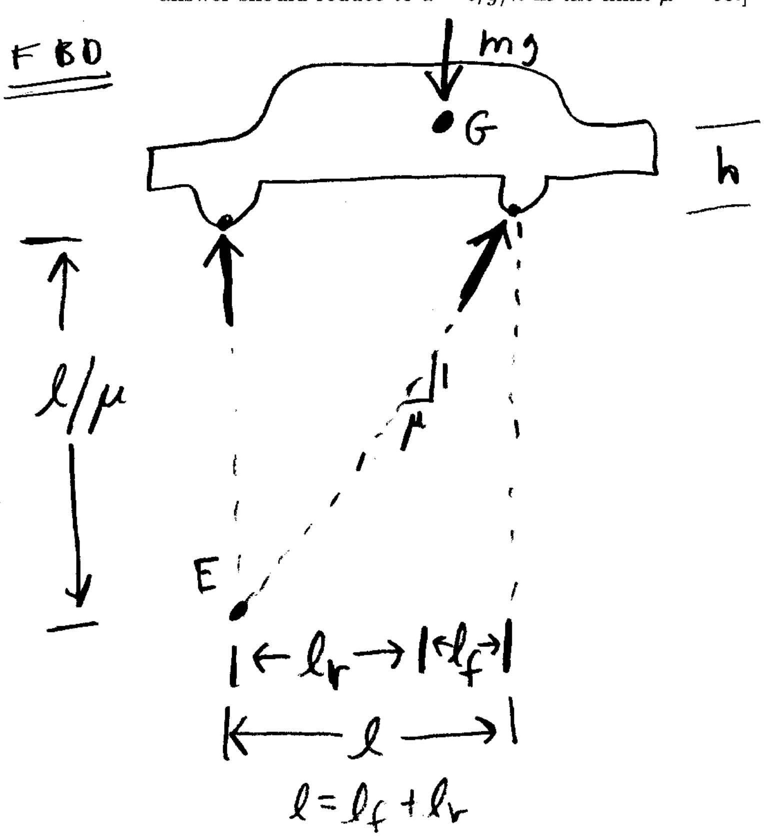
$$x(t) = \int v(t)\,dt = \int at\,dt = \frac{at^2}{2} + C_2$$

$$x(0) = 0 \Rightarrow C_2 = 0$$

$$\Rightarrow \boxed{x(t) = \tfrac{1}{2}at^2} \quad (a)$$

b) (30 pts) Find a in terms of any or all of ℓ_r, ℓ_f, h, m, g and μ. [Hint: all the directions on the cover page apply. Your answer should reduce to $a = \ell_r g/h$ in the limit $\mu \to \infty$.]

FBD



Assume: massless tires, rigid body,

$$\underline{a}_G = \underline{a} = a\,\underline{i} = a_G\,\underline{i}$$

AMB

$$\sum \underline{M}_{/E} = \dot{\underline{H}}_{/E} \quad (1)$$

Evaluate left hand side of (1)

$$\sum \underline{M}_{/E} = \underline{r}_{EG} \times (-mg\,\underline{j})$$

$$= \left(\ell_r\underline{i} + \left(\frac{\ell}{\mu} + h\right)\underline{j}\right) \times (-mg\,\underline{j})$$

$$= -\ell_r mg\,\underline{k} \quad (2)$$

Ev. right hand side of (1)

$$\dot{\underline{H}}_{/E} = \underline{r}_{EG} \times m\underline{a}_G$$

$$= \left(\ell_r\underline{i} + \left(\frac{\ell}{\mu} + h\right)\underline{j}\right) \times (ma\,\underline{i})$$

$$= -ma\left(\frac{\ell}{\mu} + h\right)\underline{k} \quad (3)$$

Plug (2) & (3) back into (1)

$$\Rightarrow \quad \sum \underline{M}_{/E} = \dot{\underline{H}}_{/E}$$

$$\left\{ -l_r m g \underline{k} = -ma\left(\frac{l}{\mu} + h\right)\underline{k} \right\}$$

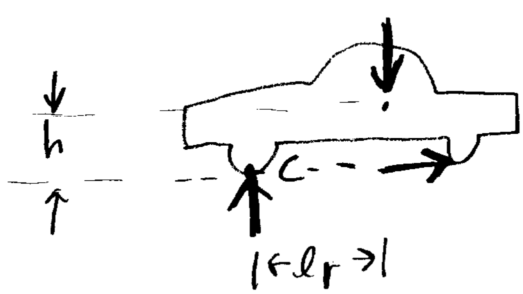
$$\{\ \}\cdot\underline{k} \Rightarrow l_r \cancel{m} g = \cancel{m} a\left(\frac{l}{\mu} + h\right)$$

$$a = g\frac{l_r}{\left(\frac{l}{\mu} + h\right)}$$

$$\boxed{a = g\left[\frac{l_r}{\frac{l_r + l_f}{\mu} + h}\right]} \quad (b)$$

Check: $a \underset{\mu\to\infty}{=} g\dfrac{l_r}{h}$ [No matter how big is μ, a front wheel drive car can't have more accel. than this.]

Why? when $\mu \to 0$ FBD looks like this



If we look at

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{C}$$

$$\Rightarrow mgl_r = ahm$$

$$\Rightarrow a = gl_r/h$$

(gravity "balances" acceleration.)

"SOLUTIONS"

Your Name: STAFF

Your TA: "

T&AM 203 Prelim 2

Tuesday March 28, 2000 7:30 — 9:00$^+$ PM

Draft March 28, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Six pages of formulas from the front and back of the text are provided. The back of the test can be used for tentative scrap work. Ask for extra scrap paper if you need it.

b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- * you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot\theta_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

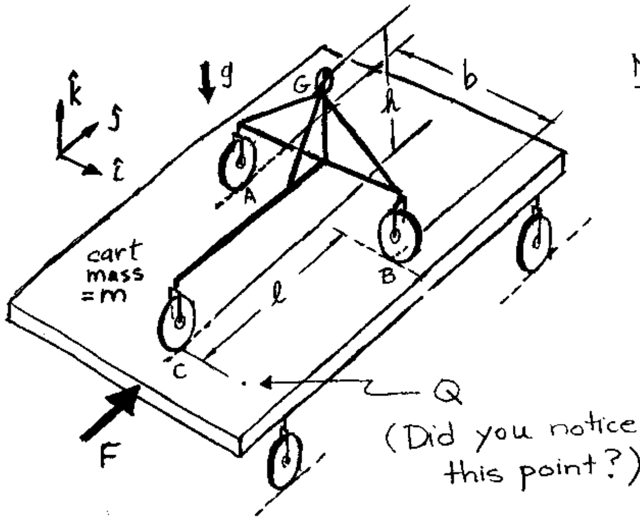
Hopefully,

Problem	1:	30 /30
Problem	2:	35 /35
Problem	3:	35 /35
TOTAL:		100 /100

1)(30 pts) **3-wheeled robot.** A 3-wheeled robot with mass m is being transported on a level flatbed trailer also with mass m. The trailer is being pushed with a force $F\,\hat{\mathbf{j}}$. The ideal massless trailer wheels roll without slip. The ideal massless robot wheels also roll without slip. The robot steering mechanism has turned the wheels so that wheels at A and C are free to roll in the $\hat{\mathbf{j}}$ direction and the wheel at B is free to roll in the $\hat{\mathbf{i}}$ direction. The center of mass of the robot at G is h above the trailer bed and symmetrically above the axle connecting wheels A and B. The wheels A and B are a distance b apart. The length of the robot is ℓ.

Find the force vector $\underline{\mathbf{F}}_A$ of the trailer on the robot at A in terms of some or all of m, g, ℓ, F, b, h, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.

[Hints: Use a free body diagram of the cart with robot to find their acceleration. With reference to a free body diagram of the robot, use angular momentum balance about axis BC to find F_{Az}.]



Note: From the announcement,

$$\boxed{\underline{a}_G = \underline{a}_{CART} = \underline{a}}$$

The robot does not move with respect to the cart.

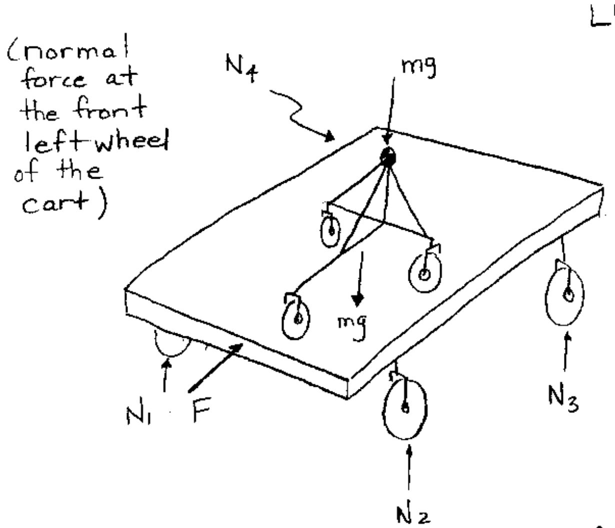
FBD (cart w/robot)

$m_{ROBOT} + m_{CART} = 2m$

LMB : $\Sigma \underline{F} = m_{TOTAL}\, \underline{a}$

LMB $\cdot \hat{j} \Rightarrow F = 2m a_y$

$\therefore \boxed{a_y = \frac{F}{2m}}$



LMB $\cdot \hat{\imath} \Rightarrow 0 = 2m a_x$

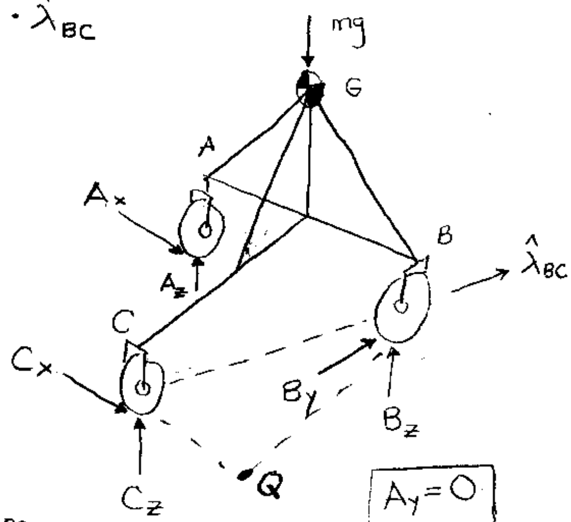
$\therefore \boxed{a_x = 0}$

$a_z = 0$ by the assumption that the cart doesn't leave the ground

$\therefore \boxed{\underline{a} = \frac{F}{2m}\hat{j}}$

(Continue work for problem 1 here)

FBD (robot)



$\boxed{A_y = 0}$

↑ wheel rolls freely in $\hat{j}$-dir.

$\underline{AMB}_{/\text{axis } BC}$: $\{ \Sigma \underline{M}_{/c} = \dot{\underline{H}}_{/c} \} \cdot \hat{\lambda}_{BC}$

where $\hat{\lambda}_{BC} = \dfrac{\underline{r}_{BC}}{|\underline{r}_{BC}|}$

$\Rightarrow \hat{\lambda}_{BC} = \dfrac{\frac{b}{2}\hat{i} + l\hat{j}}{\sqrt{(\frac{b}{2})^2 + l^2}}$

The only forces creating moments about axis BC are A_z, mg :

$\{\Sigma \underline{M}_{/c}\} \cdot \hat{\lambda}_{BC} = \{\underline{r}_{A/c} \times A_z \hat{k} + \underline{r}_{G/c} \times -mg\hat{k}\} \cdot \hat{\lambda}_{BC}$

$= \{(-\frac{b}{2}\hat{i} + l\hat{j}) \times A_z\hat{k} + (l\hat{j} + h\hat{k}) \times -mg\hat{k}\} \cdot \hat{\lambda}_{BC}$

$= \{+A_z \frac{b}{2}\hat{j} + A_z l \hat{i} - mgl\hat{i}\} \cdot \dfrac{\frac{b}{2}\hat{i} + l\hat{j}}{\sqrt{(\frac{b}{2})^2 + l^2}}$

$= \left(\frac{b}{2}l(A_z - mg) + \frac{bl}{2}A_z\right)\dfrac{1}{\sqrt{(\frac{b}{2})^2 + l^2}}$

$\{\dot{\underline{H}}_{/c}\} \cdot \hat{\lambda}_{BC} = \{\underline{r}_{G/c} \times m\underline{a}\} \cdot \hat{\lambda}_{BC}$

$= \{(l\hat{j} + h\hat{k}) \times m(\frac{F}{2m}\hat{j})\} \cdot \hat{\lambda}_{BC}$

$= \{-F\frac{h}{2}\hat{i}\} \cdot \dfrac{\frac{b}{2}\hat{i} + l\hat{j}}{\sqrt{(\frac{b}{2})^2 + l^2}}$

$= -\dfrac{Fbh}{4} \cdot \dfrac{1}{\sqrt{(\frac{b}{2})^2 + l^2}}$

$\Rightarrow \{\Sigma \underline{M}_{/c} = \dot{\underline{H}}_{/c}\} \cdot \hat{\lambda}_{BC}$

$\frac{bl}{2}(A_z - mg) + \frac{bl}{2}A_z = -\frac{Fbh}{4}$

$blA_z = \frac{mgbl}{2} - \frac{Fhb}{4}$

$\therefore \boxed{A_z = \frac{mg}{2} - \frac{Fh}{4l}}$

• Now get A_x by taking $AMB/_Q \cdot \hat{k}$

$\underline{AMB}_{/Q}$: $\{\Sigma \underline{M}_{/Q} = \dot{\underline{H}}_{/Q}\} \cdot \hat{k}$

The only force creating a moment about Q in the $\hat{k}$-direction is A_x !

$\Rightarrow \Sigma \underline{M}_{/Q} \cdot \hat{k} = \{\underline{r}_{A/Q} \times A_x\hat{i}\} \cdot \hat{k}$

$= \{(-b\hat{i} + l\hat{j}) \times A_x\hat{i}\} \cdot \hat{k}$

$= \{-A_x l\hat{k}\} \cdot \hat{k} = -A_x l$

$\dot{\underline{H}}_{/Q} \cdot \hat{k} = \{\underline{r}_{G/Q} \times m\underline{a}\} \cdot \hat{k}$

$= \{(-\frac{b}{2}\hat{i} + l\hat{j} + h\hat{k}) \times m(\frac{F}{2m}\hat{j})\} \cdot \hat{k}$

$= \{\frac{F}{2}(-\frac{b}{2}\hat{k} - h\hat{i})\} \cdot \hat{k}$

$= -\frac{Fb}{4}$

$\Rightarrow \{\Sigma \underline{M}_{/Q} = \dot{\underline{H}}_{/Q}\} \cdot \hat{k}$

$-A_x l = \frac{-Fb}{4} \quad \Rightarrow \boxed{A_x = \frac{Fb}{4l}}$

2)(35 pts) **Slippery money.** A round uniform flat horizontal platform with radius R and mass m is mounted on frictionless bearings with a vertical axis at 0. At the moment of interest it is rotating counter clockwise (looking down) with angular velocity $\underline{\omega} = \omega\hat{\mathbf{k}}$. A force in the xy plane with magnitude F is applied at the perimeter at an angle of 30° from the radial direction. The force is applied at a location that is ϕ from the fixed positive x axis. At the moment of interest a small coin sits on a radial line that is an angle θ from the fixed positive x axis (with $\boxed{\text{mass much much smaller than } m}$). Gravity presses it down, the platform holds it up, and friction (coefficient$=\mu$) keeps it from sliding.

Find the biggest value of d for which the coin does not slide in terms of some or all of $F, m, g, R, \omega, \theta, \phi$, and μ.

Let m_1 = coin mass

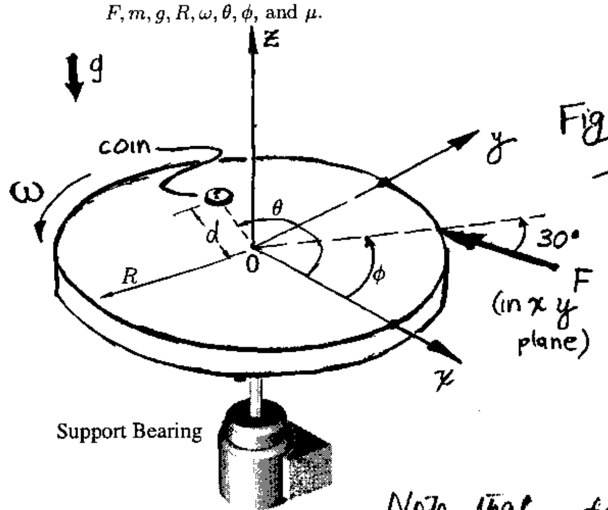
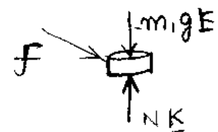


Fig I

FBD of the coin



LMB in z dir:

$N = m_1 g$ — (1)

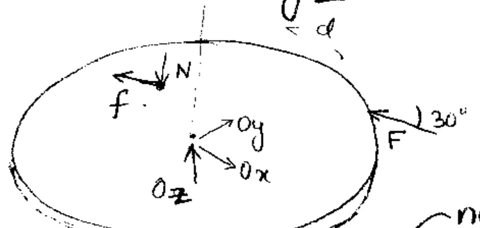
f is the frictional force acting on the coin (the direction is yet unknown, but its in XY plane)

Note that frictional force will act in a direction which provides the acceleration

So let us first calculate the acceleration of the coin

Fig II

FBD of the disc



Let us do AMB about 0

$$\sum \vec{M}_0 = \dot{\vec{H}}_0.$$

$$\sum \vec{M}_0 = F\sin 30° R\,\hat{k} + (\vec{r}\times\vec{f}) + \overbrace{\underbrace{dN(-\hat{j})}_{\substack{\| \\ dm_1 g\ \text{(very small)}}}}^{\text{neglect}}$$

$\vec{r}$ is the vector from 0 to the coin

Now the maximum magnitude of $\vec{r}\times\vec{f} = \mu m_1 g d$

and since $m_1 << m$ & $\mu < 1$

Therefore we can neglect the contribution of f.

So $\sum \vec{M}_0 = RF \sin 30° \hat{k} = \frac{FR}{2}\hat{k}$

Since 0 is a fixed point (from table IV)

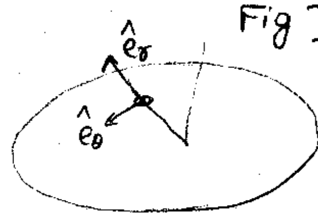
$\vec{H}_0 = I_{0_{zz}} \dot{\vec{\omega}} = \frac{mR^2}{2} \dot{\omega} \hat{k}$

$\therefore \frac{F\cancel{R}}{\cancel{2}} = \frac{mR^{\cancel{2}}}{\cancel{2}} \dot{\omega} \Rightarrow \boxed{\dot{\omega} = \frac{F}{mR}}$ — ② $\dot{\vec{\omega}} = \frac{F}{mR}\hat{k}$

So the acceleration of the coin is $\vec{a}_{coin} = \dot{\vec{\omega}} \times \vec{r}_{coin} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{coin})$

Let $\hat{e}_r$, $\hat{e}_\theta$ be as shown

Fig III



then $\vec{r}_{coin} = d\hat{e}_r$

$\vec{a}_{coin} = \dot{\omega}\hat{k} \times d\hat{e}_r - \omega^2 d\hat{e}_r$

$= d\dot{\omega}\hat{e}_\theta - d\omega^2 \hat{e}_r$

Now from the LMB of the coin's FBD (Fig I)

$\vec{f} = m_1\vec{a}$

We want the limiting case when the coin is about to slip off so in that case f should be at its maximum & its magnitude $= \mu N = \mu m_1 g$

$\therefore |\vec{f}| = |m_1\vec{a}|$

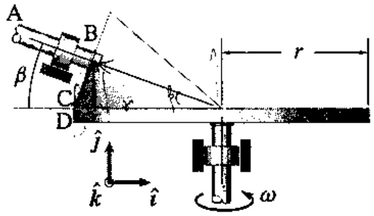
$\Rightarrow \mu \cancel{m_1} g = \cancel{m_1}|\vec{a}| = \cancel{m_1}\sqrt{d^2\dot{\omega}^2 + d^2\omega^4}$

$\Rightarrow d_{max} = \frac{\mu g}{\sqrt{\dot{\omega}^2 + \omega^4}}$ — ③

Substituting ② in ③ $\boxed{d_{max} = \frac{\mu g}{\sqrt{(F/mR)^2 + \omega^4}}}$

3) 35 pts) **Cone on Disk.** A disk rotates with constant rate ω about an fixed axis in the $\hat{\mathbf{j}}$ direction. A right cone held in a fixed bearing at B rolls at constant rate so that the point on the corner of the edge of the cone has the same velocity as the point it touches on the disk, $\underline{v}_C = \underline{v}_D$. Axis AB is in the xy plane.

Find the velocity and acceleration of point C on the cone in terms of some or all of $\omega, r, \beta, \hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$.



$\underline{v} = \underline{\omega} \times \underline{r}$

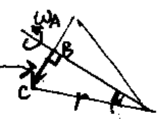
$\therefore \underline{v}_D = \omega \hat{j} \times (-r)\hat{i}$

$= r\omega \hat{k}$

$\underline{v}_C = \omega_A(-\cos\beta \hat{i} + \sin\beta \hat{j}) \times r\sin\beta(-\sin\beta \hat{i} - \cos\beta \hat{j})$

$= r\omega_A \sin\beta(\cos^2\beta \hat{k} + \sin^2\beta \hat{k})$

$= r\omega_A \sin\beta \hat{k}$



$\underline{r}_{C/B} = \underline{r}_{BC} =$

$= r\sin\beta(-\sin\beta \underline{i} - \cos\beta \underline{j})$

Since $\underline{v}_C = \underline{v}_D$

$\therefore \{\ \} \cdot \hat{k}: \quad r\omega = r\omega_A \sin\beta$

$\therefore \omega_A = \frac{\omega}{\sin\beta} = \text{const.}$

$\underline{a} = \cancel{\dot{\underline{\omega}}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$

$= \underline{\omega} \times (\underline{\omega} \times \underline{r})$

$\therefore \underline{a}_C = \underline{\omega}_A \times (\underline{\omega}_A \times \underline{r}_{BC})$

$= \omega_A(-\cos\beta \hat{i} + \sin\beta \hat{j}) \times [\omega_A(-\cos\beta \hat{i} + \sin\beta \hat{j}) \times r\sin\beta(-\sin\beta \hat{i} - \cos\beta \hat{j})]$

$= \omega_A(-\cos\beta \hat{i} + \sin\beta \hat{j}) \times r\omega_A \sin\beta \hat{k}$

$= r\omega_A^2 \sin\beta(\cos\beta \hat{j} + \sin\beta \hat{i})$

$= r \cdot \frac{\omega^2}{\sin^2\beta} \sin\beta(\cos\beta \hat{j} + \sin\beta \hat{i})$

$= \frac{r\omega^2}{\sin\beta}(\sin\beta \hat{i} + \cos\beta \hat{j})$

$\therefore \boxed{\begin{array}{l} \underline{v}_C = r\omega \hat{k} \\ \underline{a}_C = \frac{r\omega^2}{\sin\beta}(\sin\beta \hat{i} + \cos\beta \hat{j}) \end{array}}$

" SOLUTIONS " Your Name: STAFF

Your TA: "

T&AM 203 Prelim 3

Tuesday April 25, 2000 7:30 — 9:00$^+$ PM

Draft April 25, 2000

3 problems, 100 points, and 90$^+$ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Six pages of formulas from the front and back of the text are provided. The back of the test can be used for tentative scrap work. Ask for extra scrap paper if you need it.

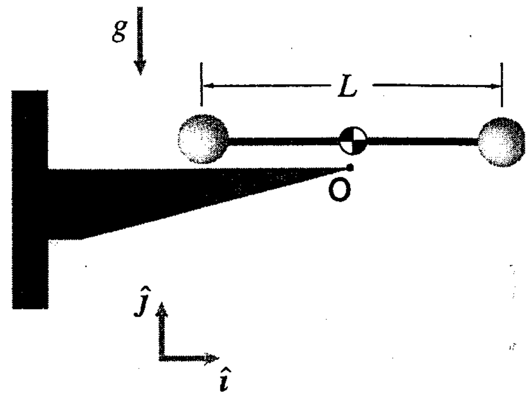
b) Full credit if

- →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/35
Problem 2:	/30
Problem 3:	/35
TOTAL:	/100

1) 35 pts) Bouncing baton. Two equal point masses are connected by a massless rigid rod of length L. While horizontal, the baton falls without rotation until it reaches the speed v and the left ball strikes the rigid surface of a table. At this instant the center of the rod is just over the right edge of the table. The collision is elastic (conserves energy).



a) Immediately after impact, what are the velocity of the rod's center and the angular velocity of the rod? Answer in terms of some or all of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, v, L, g,$ and m.

b) Assuming no other interaction with the table, accurately describe —using equations if appropriate— the subsequent motion and rotation of the baton. Answer in terms of some or all of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, v, L, g,$ and m.

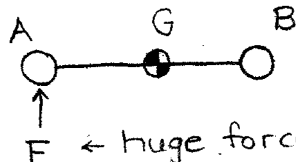
c) What is the minimum value of v for which the left mass will miss the table in its subsequent motion. Assume no subsequent collision of the massless rod with the surface. Answer in terms of some or all of $v, L, g,$ and m.

a) Can use AMB$_{/A}$:

$$\underbrace{\sum \underline{M}_{/A}}_{\underline{0}} = \dot{\underline{H}}_{/A}$$

$\Rightarrow$ $\underline{H}_{/A}$ conserved

FBD during collision



— $E_P = 0$

$\leftarrow$ huge force compared to mg $\Rightarrow$ neglect mg

① just before collision

② " after "

$$\underline{H}_{/A}^{①} = \sum \underline{r}_{i/A} \times m_i \underline{v}_i$$

$$= \cancel{\underline{r}_{A/A}} \times m\underline{v}_A + \underline{r}_{B/A} \times m\underline{v}_B$$

$$= L\hat{i} \times m(-v\hat{j}) = -mLv\hat{k}$$

$$\underline{H}_{/A}^{②} = \sum \underline{r}_{i/A} \times m_i \underline{v}_i$$

$$= \underline{0} + \underline{r}_{B/A} \times m\underline{v}_B$$

$$= L\hat{i} \times m(-v_B\hat{j})$$

$$= -mLv_B\hat{k}$$

$$\{\underline{H}_{/A}^{①} = \underline{H}_{/A}^{②}\} \cdot \hat{k}$$

$$\Rightarrow v_B = v$$

$$\Rightarrow \boxed{\underline{v}_B = -v\hat{j}}$$

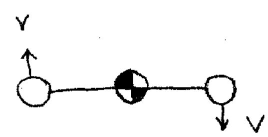
Energy conservation $\Rightarrow$ $E_{K_1} + \cancel{E_{P_1}} = E_{K_2} + \cancel{E_{P_2}}$ (with $E_{P_1} = 0$, $E_{P_2} = 0$)

$$\Rightarrow \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_B^2 + \tfrac{1}{2}mv_A^2$$

$$mv^2 = \tfrac{1}{2}m(-v)^2 + \tfrac{1}{2}mv_A^2$$

$$\Rightarrow v_A = v$$

$$\Rightarrow \boxed{\underline{v}_A = v\hat{j}}$$



(Continue work for problem 1 here)

Use relative motion to get $\underline{v}_{CM}$, $\underline{\omega}$:

$$\underline{v}_{CM} = \underline{v}_A + \underline{v}_{CM/A} = v\hat{j} + -\omega\hat{k} \times \tfrac{L}{2}\hat{i} = (v - \tfrac{1}{2}\omega L)\hat{j} \quad (1)$$

$$\underline{v}_{CM} = \underline{v}_B + \underline{v}_{CM/B} = -v\hat{j} + -\omega\hat{k} \times -\tfrac{L}{2}\hat{i} = (-v - \tfrac{1}{2}\omega L)\hat{j} \quad (2)$$

$\{(1) = (2)\}\cdot\hat{j} \quad \Rightarrow \quad v - \frac{1}{2}\omega L = -v - \frac{1}{2}\omega L$

$\Rightarrow \quad \omega = \frac{2v}{L}, \quad \boxed{\underline{\omega} = -\frac{2v}{L}\hat{k}}$

$\Rightarrow \quad \boxed{\underline{v}_{CM} = 0}$

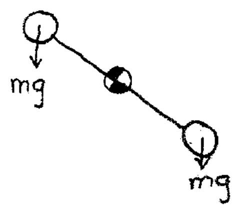
b) LMB: $\Sigma \underline{F} = (m+m)\underline{a}_{CM}$

$-2mg\hat{j} = 2m\underline{a}_{CM}$

$\Rightarrow \underline{a}_{CM} = -g\hat{j}$

$\Rightarrow$ The center of mass falls straight down with acceleration of magnitude g.

FBD after collision

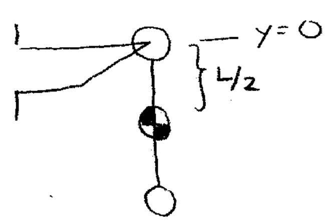


AMB/CM : $\underbrace{\Sigma \underline{M}_{/CM}}_{\underline{0}} = \dot{\underline{H}}_{/CM}$ gravity forces cancel each other

$\Rightarrow \underline{H}_{/CM}$ conserved $\Rightarrow$ $\underline{\omega}$ constant

$\Rightarrow$ The baton rotates at rate $\underline{\omega} = -\frac{2v}{L}\hat{k}$.

c) For the minimum v, the baton should rotate by 90° when the center of mass has fallen a distance $L/2$.



$\omega = \frac{2v}{L}$ = constant

$\Rightarrow \quad \omega = \frac{\Delta\theta}{\Delta t} \leftarrow$ find this

$$\left.\begin{aligned} \underline{a}_{CM} &= -g\hat{j} \\ \Rightarrow \underline{v}_{CM} &= -gt\hat{j} \\ \underline{r}_{CM} &= -\tfrac{1}{2}gt^2\hat{j} \end{aligned}\right\}$$

When $\underline{r}_{CM} = -\frac{L}{2}\hat{j}$

$\frac{L}{2} = \frac{1}{2}g(\Delta t)^2$

$\Rightarrow \Delta t = \sqrt{L/g}$

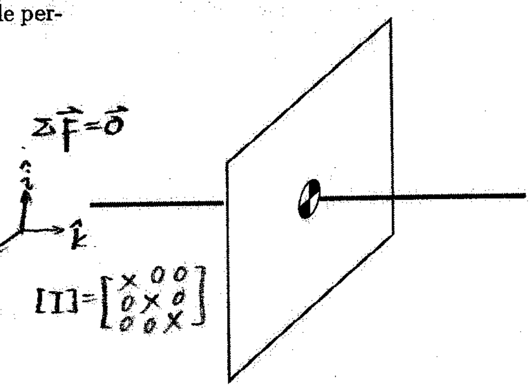
$\therefore \quad \omega = \frac{2v}{L} = \frac{\pi/2}{\sqrt{L/g}}$

$\therefore \quad \boxed{v = \frac{\pi}{4}\sqrt{gL}}$

2)(30 pts) Static and Dynamic Balance A series of bodies, each of uniform density and each with total mass m, rotate at a constant angular speed ω about a fixed horizontal axis. Ignore gravity. For each body state whether the body is (i) *statically* balanced and whether it is (ii) *dynamically* balanced. Give clear arguments using words or equations to support your claims.

(iii, iv) For each body you must *either* (a) add one point mass m or (b) add two point masses each of mass $m/2$ (your choice) that *maintain* static and dynamic balance if they are balanced, or that *make* the bodies statically and dynamically balanced. Justify your placement with words and/or equations. The masses need not be added to the bodies, but could be attached off the bodies by structures with negligible mass. [Hint: none of the placements are unique. You may draw a side view if that helps clarify your placement.]

a) A rectangular plate (height h, length ℓ) mounted with the axle perpendicular to the plate and through its center.



i) Statically balanced? (yes/no) Why?

yes. center of the mass is on the axis. $\Sigma\vec F=\vec 0$

ii) Dynamically balanced? (yes/no) Why?

yes. Look at $\vec H$, since $\vec\omega=\text{const}$

$\therefore \dot{\vec H}=\vec\omega\times\vec H$, $\vec H=[I]\cdot\vec\omega$ $\quad \vec\omega=\omega\hat k$ $\quad [I]=\begin{bmatrix}x&0&0\\0&x&0\\0&0&x\end{bmatrix}$

Since $\vec\omega \parallel [I]\cdot\vec\omega$ $\quad \therefore \dot{\vec H}=\vec 0=\Sigma\vec M$

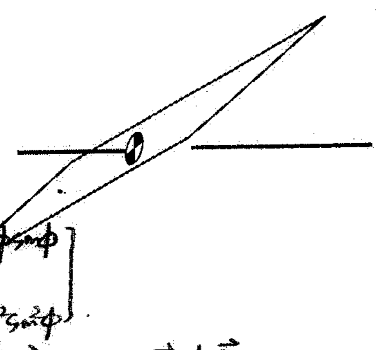
iii) Are you adding one mass or two?

One

iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.

Center of the rectangle , it won't affect the previous answer since for this point mass, $[I]=[0]$

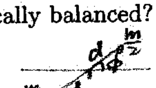
b) The same plate as in (a) above but mounted at an angle $\phi\neq\pi/2$ from the shaft.



i) Statically balanced? (yes/no) Why?

Yes. mass center is on the axis. $\Sigma\vec F=\vec 0$

ii) Dynamically balanced? (yes/no) Why?

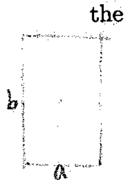
No,  $\vec\omega=\omega\hat k$. $[I]=\begin{bmatrix}md^2\cos^2\phi&0&md^2\cos\phi\sin\phi\\0&md^2&0\\md^2\cos\phi\sin\phi&0&md^2\sin^2\phi\end{bmatrix}$

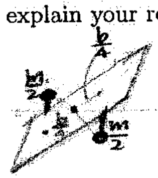
$[I]\cdot\vec\omega=md^2\sin\phi(\cos\phi\hat i+\sin\phi\hat k)\not\parallel\vec\omega$. $\therefore \Sigma\vec M\neq\vec 0$

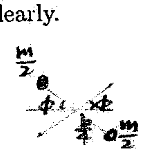
iii) Are you adding one mass or two?

Two.

iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



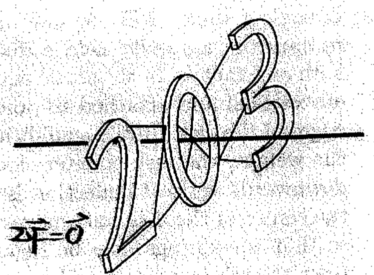




it will make the whole system symmetric. then $I_{xy}=I_{yz}=I_{xz}=0$!

So $[I]\cdot\vec\omega\parallel\vec\omega \Rightarrow \Sigma\vec M=\vec 0$

c) The numerals '203' cut out of a plate and connecteed by massless rods. Each letter has mass $m/3$ and the three center-of-mass points of the individual letters are colinear and equally spaced. The shaft goes through the center of the '0' and is perpendicular to the plane of the letters.



i) Statically balanced? (yes/no) Why?

yes. center of mass is on the axis. $\Sigma \vec{F} = \vec{0}$

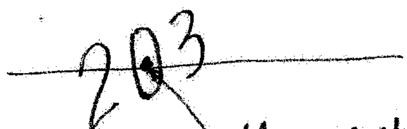
ii) Dynamically balanced? (yes/no) Why?

yes, since $\hat{k}$ is the eigenvector of the [I], and it's perpendicular to the other two eigenvectors which lie in the x-y plane. $\therefore [I]\cdot\vec{\omega} \parallel \vec{\omega} \Rightarrow \dot{\vec{H}} = \vec{\omega} \times ([I]\cdot\vec{\omega}) = \vec{0} = \Sigma\vec{M}$

iii) Are you adding one mass or two?

One.

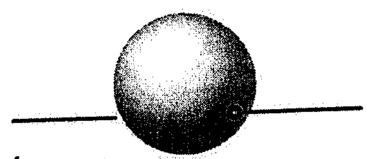
iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



add m at the center of 'O'.

it won't affect the [I].

d) A sphere with radius R where the shaft passes a distance $d < R$ from the center.



i) Statically balanced? (yes/no) Why?

No, center of mass isn't on the axis.
$\therefore$ need extra force to provide centripetal acceleration.

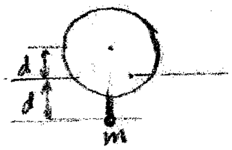
ii) Dynamically balanced? (yes/no) Why?

No, center of the mass is not on the axis.
$I_{xz} \neq 0$. $\therefore [I]\cdot\vec{\omega} \nparallel \vec{\omega}$. $\Sigma\vec{M} \neq \vec{0}$

iii) Are you adding one mass or two?

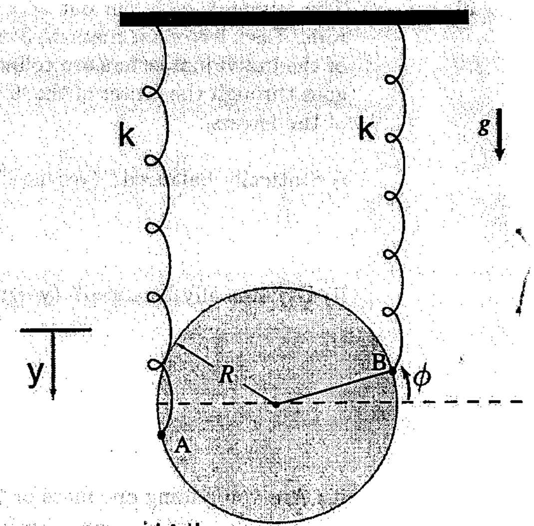
One.

iv) Mark the location (or locations) as accurately as you can on the figure and explain your reasoning clearly.



After adding this point mass, $I_{xz} = 0$.
and $I_{yz} = I_{xy} = 0$
$\therefore [I]\cdot\vec{\omega} \parallel \vec{\omega} \Rightarrow \Sigma\vec{M} = \vec{0}$

3)(35 pts) **Hanging disk, 2-D.** A uniform thin disk of radius R and mass m hangs in a gravity field g from a pair of massless springs each with constant k. In the static equilibrium configuration the springs are vertical and attached to points A and B on the right and left edges of the disk. In the equilibrium configuration the springs carry the weight, the disk counter-clockwise rotation is $\phi = 0$, and the downwards vertical deflection is $y = 0$. Assume throughout that the center of the disk only moves up and down, and that ϕ is small so that the springs may be regarded as vertical when calculating thier stretch ($\sin\phi \approx \phi$ and $\cos\phi \approx 1$).



a) Find $\ddot{\phi}$ and $\ddot{y}$ in terms of some or all of $\phi, \dot{\phi}, y, \dot{y}, k, m, R$, and g.

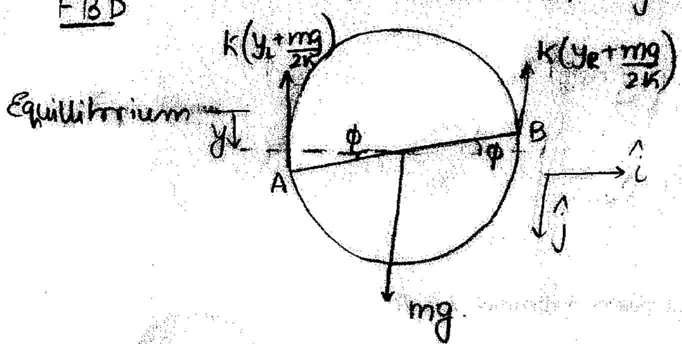
b) Find the natural frequencies of vibration in terms of some or all of k, m, R, and g.

Ans :- Note that 'y' is measured from the equillibrium (and represents the displacement of the CM).

(a)

So at equillibrium springs are stretched by $l_0 = \frac{mg}{2k}$

FBD



$y_R = y - R\phi$

$y_L = y + R\phi$

LMB in '$\hat{j}$' direction

$$m\ddot{y} = -k\left(y - R\phi + \frac{mg}{2k}\right) - k\left(y + R\phi + \frac{mg}{2k}\right) + mg$$

$$\Rightarrow m\ddot{y} = -ky + \cancel{kR\phi} \cancel{-mg} - ky - \cancel{kR\phi} + \cancel{mg}$$

$$\Rightarrow \boxed{\ddot{y} + \frac{2k}{m} y = 0} \quad (1) \quad \Rightarrow \ddot{y} = -\frac{2k}{m} y$$

AMB about CM :

$$k\left(y_R + \frac{mg}{2k}\right) \cdot R - k\left(y_L + \frac{mg}{2k}\right) \cdot R = I\ddot{\phi}$$

(Continue work for problem 3 here)

$$\Rightarrow \; k(y-R\phi)R - k(y+R\phi)R = I\ddot{\phi}$$

$$\Rightarrow \quad \ddot{\phi} + \frac{2kR^2}{I}\phi = 0$$

$$I \text{ for a disk} = \frac{mR^2}{2}.$$

$$\Rightarrow \quad \ddot{\phi} + \frac{2k\cancel{R^2}.2}{m\cancel{R^2}}\phi = 0$$

$$\Rightarrow \quad \boxed{\ddot{\phi} + \frac{4k}{m}\phi = 0} \quad (2)$$

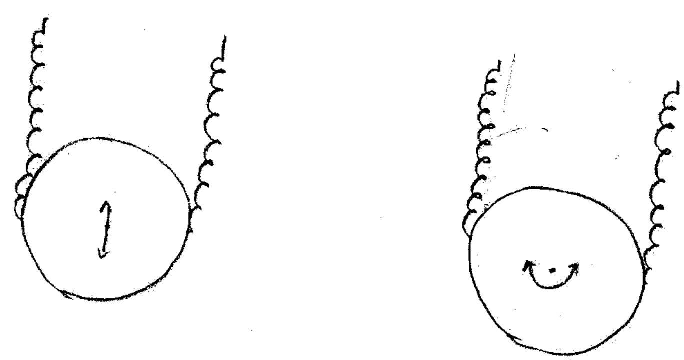
$$\ddot{\phi} = -\frac{4k}{m}\phi.$$

b) Since the system has 2 dof it has two natural frequencies.

which from equations (1) & (2) are

$$\boxed{\omega_1 = \sqrt{\frac{2k}{m}}, \quad \omega_2 = \sqrt{\frac{4k}{m}}}$$

The modes of vibration look like



"Solutions"

Your Name: ANDY RUINA

Section day and time: ______________

T&AM 203 Final exam

Tuesday May 14, 2002

Draft May 14, 2002

5 problems, 100 points, and 150 minutes (no extra).

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problem(s).

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

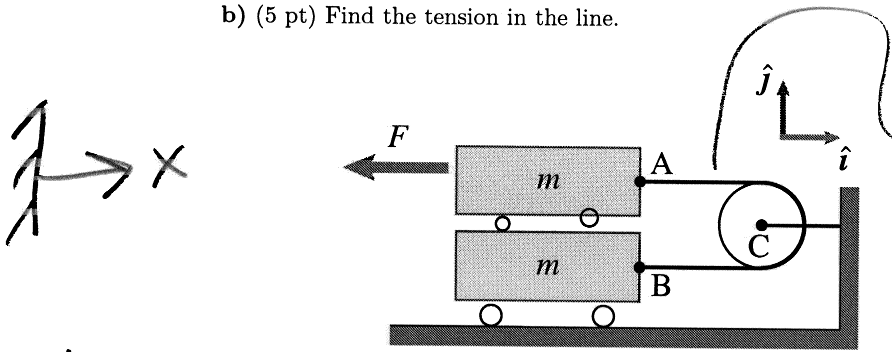
Problem 1:	/15
Problem 2:	/20
Problem 3:	/25
Problem 4:	/20
Problem 5:	/20
TOTAL:	**/100**

1) (15 pt) Two equal masses are stacked an tied together by the pully as shown. All bearings are frictionless. All rotating parts have negligible mass. The line is inextensible.

0) (10 pt) for basic setup diagrams, assumptions, and equations needed to answer the questions below.

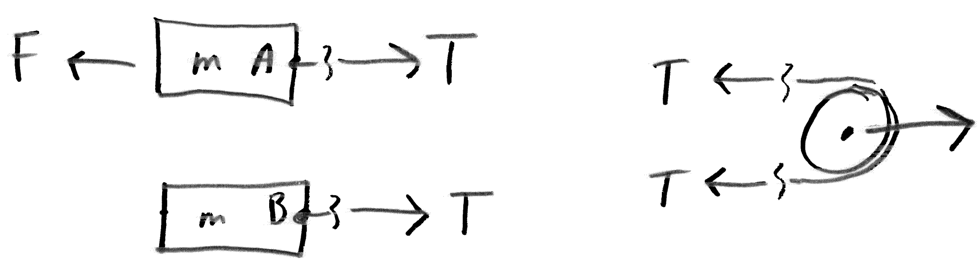
a) (5 pt) Find the acceleration of point A.

b) (5 pt) Find the tension in the line.



length $= \ell$

FBDs:



$\{\text{LMB for A}\}\cdot\hat{i} \Rightarrow \quad -F + T = m\ddot{x}_A$ (1)

$\{\text{LMB for B}\}\cdot\hat{i} \Rightarrow \quad T = m\ddot{x}_B$ (2)

Kimemetics:

$\ell = \text{const.} \Rightarrow \quad \ddot{x}_A = -\ddot{x}_B$ (3)

(1) - (2) $\Rightarrow \quad -F = m\,\ddot{x}_A - \ddot{x}_B$

(3) $\Rightarrow \quad = 2m\ddot{x}_A$

$\Rightarrow \boxed{\underline{a}_A = -\frac{F}{2m}\hat{i}}$ (a)

(1) + (2) $\Rightarrow \quad -F + 2T = m(\ddot{x}_A + \ddot{x}_B)$

(3) $\Rightarrow \quad = 0$

$\Rightarrow \boxed{T = F/2}$ (b)

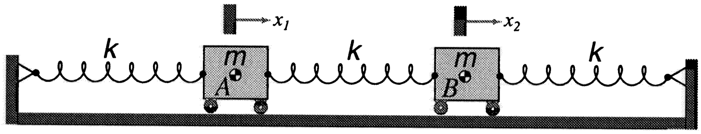
2) (20 pt) Two identical masses ($m = 2\,\text{kg}$) move in a straight line without friction. Three identical springs ($k = 7\,\text{N/kg}$) hold them in place (one between the left mass and a wall, one between the two masses, and one between the right mass and the wall). When the horizontal displacements x_1 and x_2 of the masses are zero all three springs are relaxed.

The system is released from rest at $t = 0$ with $x_1(0) = 0.3\,\text{m}$ and $x_2(0) = -.3\,\text{m}$.

a) (15 points) Write Matlab code where the final output will be the position of mass one at $t = 10\,\text{s}$. Your code should be general enough to handle arbitrary initial conditions. [Do not just use Matlab to evaluate the solution from (b) below.]

b) (5 points) Write a formula for the answer above. That is, evaluate an analytic solution of the resulting differential equations at $t = 10\,\text{s}$. [Hint: Using ideas from the lab makes this problem much easier than blindly grinding through the methods of Math 293, 294].

$\hat{j}$ $\hat{i}$



$X_1 = X_A$

$X_2 = X_B$

FBDs

$T_1 \leftarrow \square \rightarrow T_2 \qquad T_2 \leftarrow \square \rightarrow T_3$

$$T_1 = kX_1, \quad T_2 = k(X_2 - X_1), \quad T_3 = -kX_3$$

$\{LMB \text{ for } A\}\cdot\hat{i} \Rightarrow \quad -T_1 + T_2 = m\ddot{X}_1$

$$-kX_1 + k(X_2 - X_1) = m\ddot{X}_1$$

$$-2kX_1 + kX_2 = m\ddot{X}_1 \qquad (1)$$

$\{LMB \text{ for } B\}\cdot\hat{i} \Rightarrow \quad T_3 - T_2 = m\ddot{X}_B$

$$-kX_2 - (k(X_2 - X_1)) = m\ddot{X}_2$$

$$kX_1 - 2kX_2 = m\ddot{X}_2 \qquad (2)$$

(1) & (2) $\Rightarrow$

$$m\begin{bmatrix}\ddot{X}_1 \\ \ddot{X}_2\end{bmatrix} = k\begin{bmatrix}-2 & 1 \\ 1 & -2\end{bmatrix}\begin{bmatrix}X_1 \\ X_2\end{bmatrix}$$

Given ICs:

$$\left.\begin{aligned} X_1(0) &= .3 \\ X_2(0) &= -.3 \end{aligned}\right\} \text{(using consistent mks units)}$$

$$\left.\begin{aligned} V_1(0) &= 0 \\ V_2(0) &= 0 \end{aligned}\right\} \text{defining } V_1 = \dot{X}_1,\ V_2 = \dot{X}_2$$

(2 cont'd)

```
X10=.3; X20=-.3; V10=0; V20=0;
Z0=[X10 X20 V10 V20];
tspan=[0 10];
[t z]= ode23('twoblocks', tspan, Z0);
answer = z(end,1)          % note, no semicolon
```

driver file

```
function Zdot = twoblocks(t,z)
K=7; m=2;
pos=[z(1) z(2)]';
vel=[z(3) z(4)]';
posdot= vel;
veldot= K*[-2 1;
            1 -2]*pos/m;
Zdot=[posdot' veldot'];
```

twoblocks.m

By inspection a normal mode of this system has blocks moving equally and oppositely. Appling $x_2=-x_1$ to ① $\Rightarrow$

$$m\ddot{x}_1 = -3k\,x_1 \Rightarrow x_1 = A\cos\sqrt{\tfrac{3k}{m}}\,t + B\sin\sqrt{\tfrac{3k}{m}}\,t$$

Init. cond. $\Rightarrow$ $A=.3$, $B=0$

So, in consistent units, at $t=10$

$$\boxed{x_1 = A\cos\sqrt{\frac{3k}{m}}\,t = .3\cos\left(10\sqrt{\frac{21}{2}}\right)}$$ (b)

3) (25 pt) A suitcase on level ground with wheels in front and skids in back is pulled by its handle with a forward force F. The handle is directly above the front wheels. In all cases you are given these quantities and can use them in your answer:

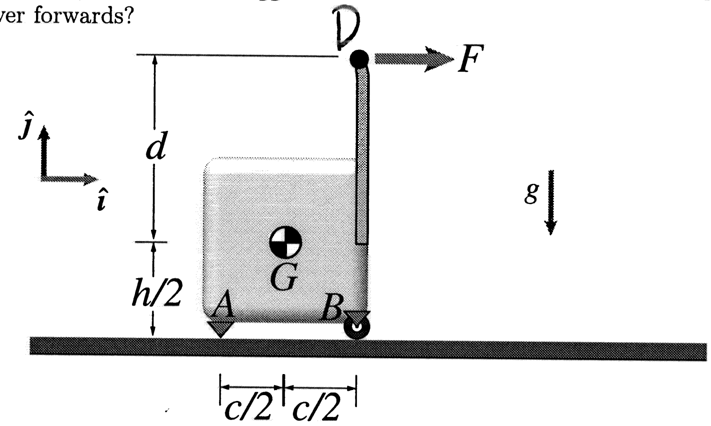
$m =$ total mass of suitcase;
$I^G =$ the polar moment of inertia about an axis through the center of mass G and in the ussual zz direction (perpendicular to the plane on which a side view of the suitcase is drawn);
$h/2 =$ the height of G above the ground;
$c/2 =$ the distance G is forward of the rear ground contact at A;
$c/2 =$ the distance G is behind the front ground contact at B;
$d =$ the height of the handle at D above the center of mass of the suitcase;
$\mu =$ the coefficient of friction between the rear skids and ground;
$g =$ the gravity constant.

0) (10 points) for the basic diagrams, stated assumptions, and mechanics equations needed to solve the parts below.

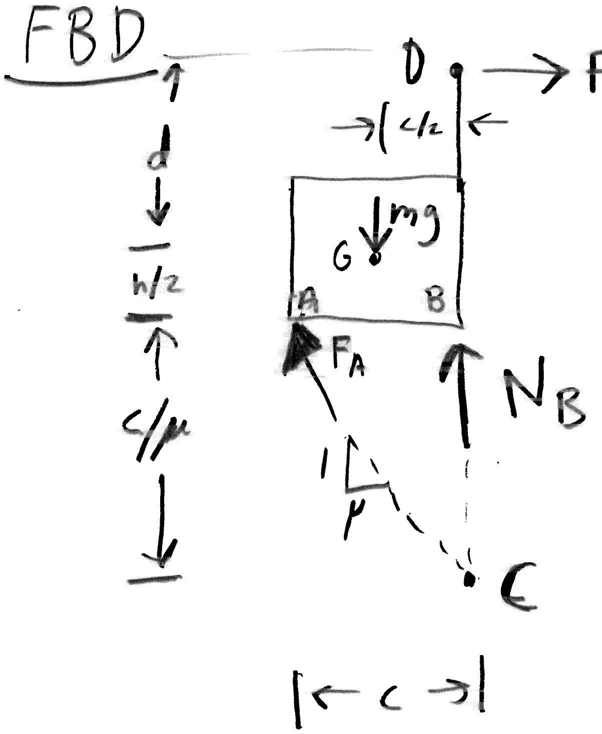
a) (5 points) Given F and that F is big enough for non-zero forward acceleration, and that F is not so big as to cause the suitcase to tip, what is the acceleration of the suitcase?

b) (5 points) What is the smallest F that can make the suitcase move at all?

c) (5 points) What is the biggest acceleration that can be achieved without the suitcase tipping over forwards?



Note: in all cases $F = \mu N$ (slip is, or is about to, occur)



AMB/C

$$\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$$

$$\left\{ \frac{mgc}{2}\hat{k} - F\left(\frac{c}{\mu} + \frac{h}{2} + d\right)\hat{k} = \underline{r}_{G/C} \times m\underline{a}_G + I^G\dot{\omega}\hat{k} \right\}$$

$\underline{a}_G = a\hat{i}$, $\dot{\omega} = 0$

$$\underline{r}_{G/C} = \left[\left(\frac{c}{\mu} + \frac{h}{2}\right)\hat{j} - \frac{c}{2}\hat{i}\right]$$

(a)

$$\{\}\cdot\hat{k} \Rightarrow \frac{mgc}{2} - F\left(\frac{c}{\mu} + \frac{h}{2} + d\right) = -ma\left(\frac{c}{\mu} + \frac{h}{2}\right)$$

$$\Rightarrow \boxed{a = \left[-\frac{mgc}{2} + F\left(\frac{c}{\mu} + \frac{h}{2} + d\right)\right] \Big/ m\left(\frac{c}{\mu} + \frac{h}{2}\right)}$$

$$\underline{a} = a\hat{i}$$

Sanity check: when $d = h = 0$ | 3, Cont'd |

$$\Rightarrow \quad a = \frac{\left(-mgc/2 + F\frac{c}{\mu}\right)}{m\frac{c}{\mu}}$$

$$a = F/m - \mu mg/2$$ (as expected, good)

(b) One approach would be to do prob. again as a statics prob. Another is to set $\underline{a} = \underline{0}$ in soln. above.

$$\Rightarrow \quad \frac{mgc}{2} = F_{min}\left(\frac{c}{\mu} + \frac{h}{2} + d\right)$$

$$\boxed{F_{min} = \frac{mgc}{2\left(\frac{c}{\mu} + \frac{h}{2} + d\right)}} \quad (b)$$

Sanity check: when $d = h = 0$ $\Rightarrow F_{min} = \mu mg/2$ (as expected)

(c) In this case $F_A = 0$ (just barely)

AMB/D $\Rightarrow$ $\sum \underline{M}_{/D} = \dot{\underline{H}}_{/D}$

$$\left\{ \frac{mgc}{2}\hat{k} = \underbrace{\underline{r}_{G/D}}_{-d\hat{j} - \frac{c}{2}\hat{i}} \times m \underbrace{\underline{a}_G}_{a\hat{i}} \right\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \quad \frac{mgc}{2} = dma$$

$$a = \frac{gc}{2d}$$

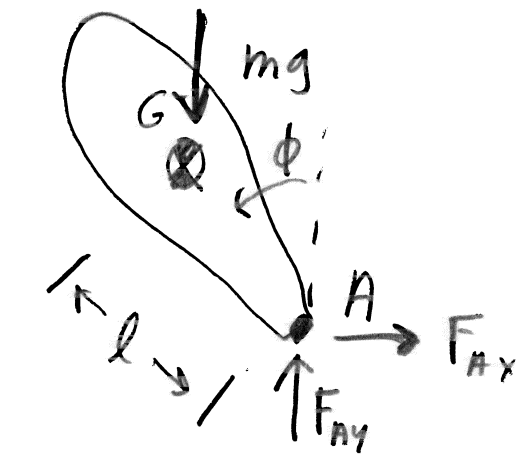
$$\boxed{\underline{a} = a\hat{i} = \frac{gc}{2d}\hat{i}} \quad (c)$$

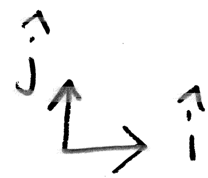
4) (20 pt) An inverted pendulum is supported at one end A by a hinge that moves up and down and, at the instant of interest, has an upwards acceleration a. The pendulum mass is m and its moment of inertia about the center of mass G is I^G. G is a distance ℓ from the end at A. At the instant in question the pendulum is tipped counter-clockwise from the vertical an angle ϕ and is tipping at the rate $\dot{\phi}$. Gravity g is pointing down.

Find $\ddot{\phi}$ in terms of some or all of a, ℓ, m, I^G, g, ϕ, and $\dot{\phi}$.

[Hint: you can check to see if your answer reduces to something you know well when $a = 0$ and $I^G = 0$. Another check is to set $a = -g$.]

FBD





given $\underline{a}_A = a\hat{j}$

$\underline{\omega} = \dot{\phi}\hat{k}$

AMB/A $\Rightarrow$ $\sum \underline{M}_{/A} = \dot{\underline{H}}_{/A}$

$$\underline{r}_{G/A} \times (-mg\hat{j}) = \underline{r}_{G/A} \times (m\underline{a}_G) + I^G\ddot{\phi}\hat{k}$$

$\underline{r}_{G/A} = \ell(\cos\phi\hat{j} - \sin\phi\hat{i})$

$$\underline{a}_G = \underline{a}_A + \underline{a}_{G/A}$$
$$= a\hat{j} + \dot{\underline{\omega}} \times \underline{r}_{G/A} - \omega^2 \underline{r}_{G/A}$$

$\dot{\underline{\omega}} = \ddot{\phi}\hat{k}$

will drop out in the end ($\underline{r}_{G/A} \times \underline{r}_{G/A} = \underline{0}$)

$$\Rightarrow \Big\{ mg\ell\sin\phi\hat{k} = \ell(\cos\phi\hat{j} - \sin\phi\hat{i}) \times m\left[a\hat{j} + \ddot{\phi}\ell(-\cos\phi\hat{i} - \sin\phi\hat{j}) - \omega^2\underline{r}_{G/A}\right] + I^G\ddot{\phi}\hat{k}$$

$[\cos^2 + \sin^2 = 1] \Rightarrow$

$$= -m\ell a\sin\phi\hat{k} + m\ell^2\ddot{\phi}\hat{k} + I^G\ddot{\phi}\hat{k} \Big\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow m(g+a)\ell\sin\phi = (I^G + m\ell^2)\ddot{\phi}$$

$$\Rightarrow \boxed{\ddot{\phi} = \frac{(g+a)m\ell\sin\phi}{I^G + m\ell^2}}$$

checks: 1) $a = 0$, $I^G = 0$

$\Rightarrow \ddot{\phi} = \frac{g}{\ell}\sin\phi$ (simple inverted pendulum)

2) $a = -g$

$\Rightarrow \ddot{\phi} = 0$ (pendulum in falling elevator. Like outer space

5) (20 pt) A "centripital gun" consists of a rod hinged at one end at A and a frictionless collar that slides on the rod at the moving position C. The gun is powerred by the applied torque T. Neglect gravity. At the instant of interest you are given

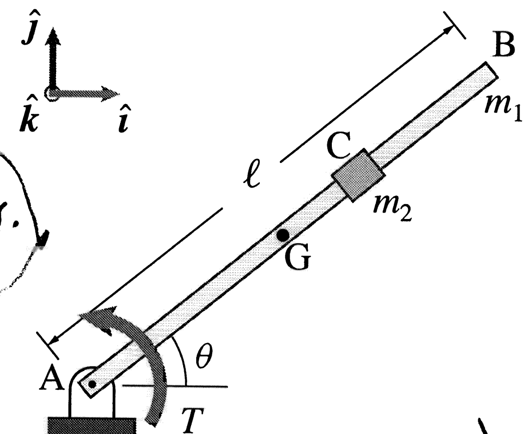
$T =$ the applied torque (counterclockwise is positive);
$m_1 =$ mass of the rod;
$m_2 =$ mass of the colar;
$I^G = m_1\ell^2/12 =$the polar moment of inertia of the rod about an axis through the center of mass G and in the ussual zz direction (perpendicular to the plane on which a side view of the suitcase is drawn);
$\ell/2 =$ the distance from A to G (from end of rod to COM);
$R = R(t) =$ the distance from A to C (the radius of the collar);
$\dot R = \dot R(t) =$ the rate of change of distance from O with time;
$\theta = \theta(t) =$ the counterclockwise angle of the rod relative to a fixed $+x$ axis.
$\dot\theta = \dot\theta(t) = \frac{d}{dt}\theta$;

Find $\ddot R$ in terms of some or all of $T, R, \dot R, \theta, \dot\theta, m_1, m_2, \ell$, and I^G .

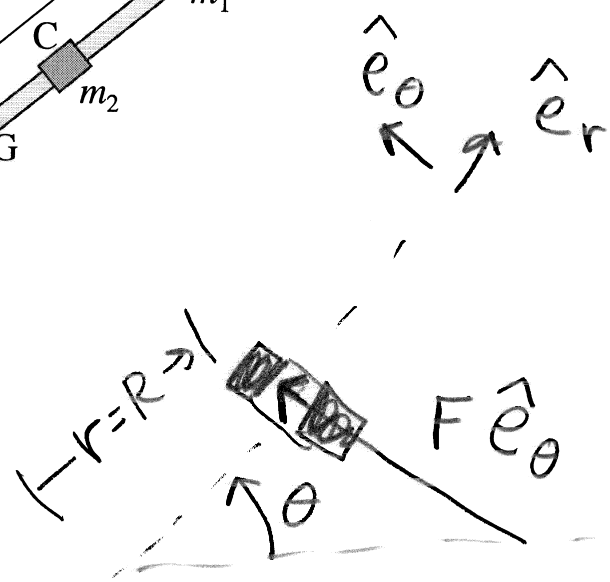
[Simplify your answer until it looks simple.]

Can also draw FBD of rod & look at its dynamics, but no need.

There are longer solutions. Here's a quick one.



FBD of collar!



LMB for collar

$$\sum \underline{F} = m\underline{a}$$

$$\left\{ F\hat{e}_\theta = m\left[(\ddot r - r\dot\theta^2)\hat{e}_r + (2\dot r\dot\theta + r\ddot\theta)\hat{e}_\theta\right]\right\}$$

$$\{\ \}\cdot\hat{e}_r \Rightarrow \quad \ddot r - r\dot\theta^2 = 0$$

$$\boxed{\ddot r = r\dot\theta^2}$$

That's it!

Note:
T has no effect! but to make $\dot\theta$ bigger for later on.

"SOLUTIONS"

Your Name: Andy Ruina

Section day and time: Wed. 10:10

[w/ comments on ODE solutions on pages 9-13]

T&AM 203 Prelim 1

Tuesday Feb 26, 2002

Draft February 26, 2002

3 problems, 100 points, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

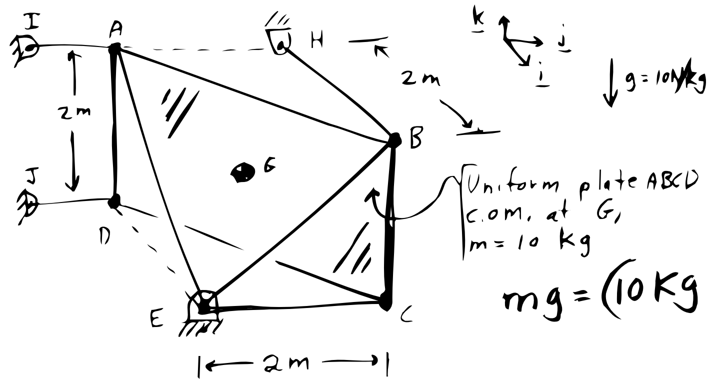
b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/50
TOTAL:	**/100**

1) **(25 pt)** Statics. The sign is held up by 6 bars. Find the tension in bar EB.



$mg = (10\,kg)(10\,N/kg) = 100\,N$

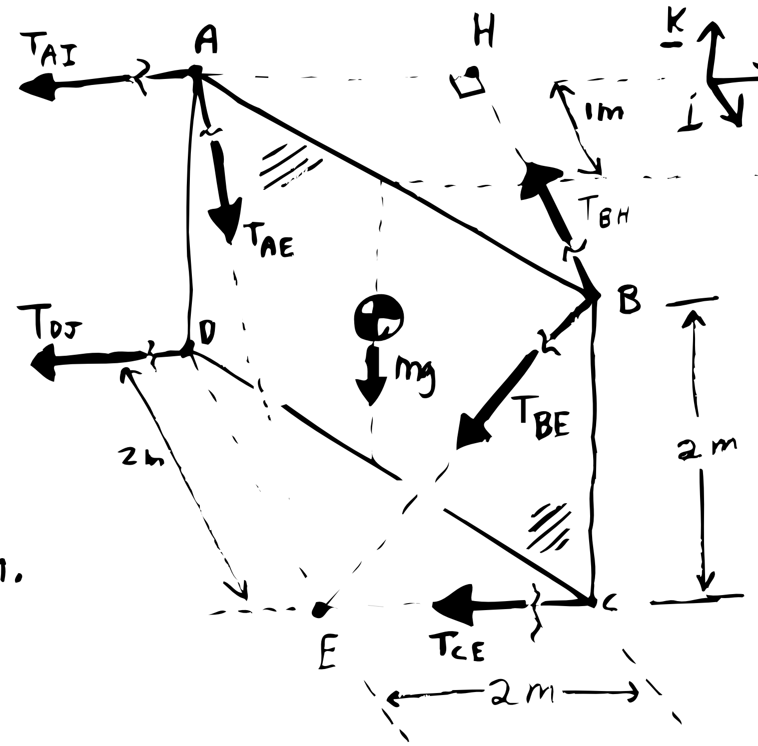
Consider axis AH:

* T_{AI}, T_{DJ}, T_{CE} are ∥ to axis.

* T_{BH} & T_{AE} intersect axis,

⇒ Only T_{BE} and mg contribute to moment about axis AH.

But mg is known.



$$\sum M_{\substack{axis\\AH}} = \left(\sum \underline{M}_{/H}\right)\cdot \underline{j} = 0$$

$$\Rightarrow \quad \underbrace{100\,Nm}_{\text{moment of mg about axis AH}} + \Big(\underline{r}_{B/H} \times \overbrace{T_{BE}\left(\frac{-\underline{j}-\underline{k}}{\sqrt{2}}\right)}^{\underline{T}_{BE}}\Big)\cdot \underline{j} = 0$$

$$\underline{r}_{B/H} = 2\,m\,\underline{i}$$

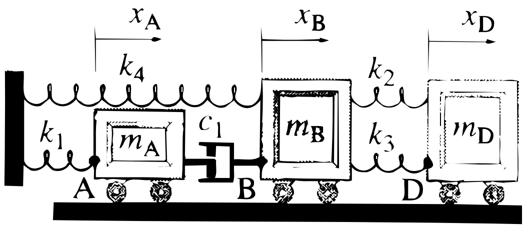
$$\Rightarrow \quad 100\,Nm + \left[2m\,T_{BE}(-\underline{k} + \underline{j})/\sqrt{2}\right]\cdot \underline{j} = 0$$

$$100\,N\cancel{m} + \sqrt{2}\,\cancel{m}\,T_{BE} = 0$$

$$\boxed{T_{BE} = \frac{-100}{\sqrt{2}}\,N \approx -70.7\,N}$$

(BE is in compression)

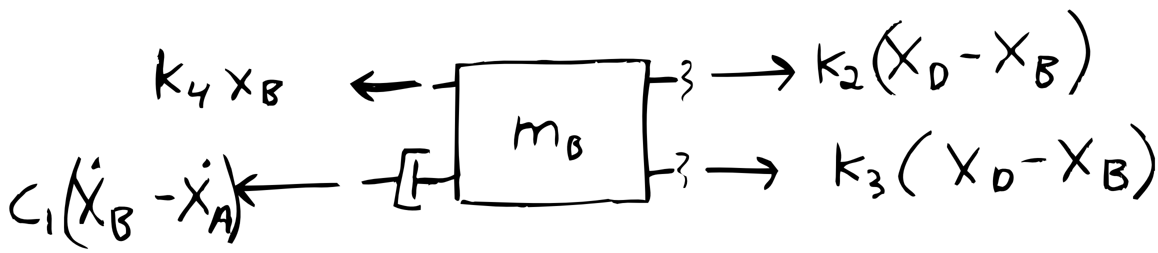
2) (25 pt.) In terms of some or all of $x_A, x_B, x_D, \dot{x}_A, \dot{x}_B, \dot{x}_D, k_1, k_2, k_3, k_4, m_A, m_B, m_D$ and c_1 find $\ddot{x}_B$. Assume the springs are relaxed when $x_A = x_B = x_D = 0$



FBD

$\rightarrow \underline{i}$

When m_B is at position x_B



LMB

$$\left\{ \sum \underline{F}_i = m_B \underline{a}_B \right\} \cdot \underline{i}$$

$$\Rightarrow -c_1(\dot{x}_B - \dot{x}_A) - k_4 x_B + k_2(x_D - x_B) + k_3(x_D - x_B) = m_B \ddot{x}_B$$

$$\boxed{\ddot{x}_B = \frac{1}{m_B}\left[-(k_2 + k_3 + k_4)x_B + (k_2 + k_3)x_D + c_1\dot{x}_A - c_1\dot{x}_B\right]}$$

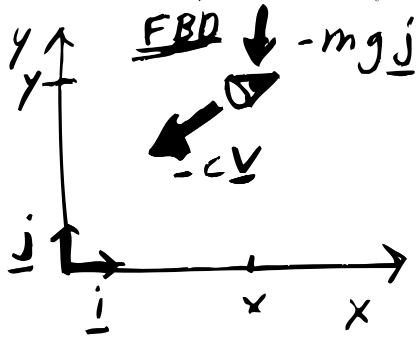
3) **(50 pt)** Trajectory. A 0.02 kg projectile (a badminton birdie, say) is launched from the origin at a 60° upwards angle at a speed of 50 m/s. The projectile stays near the earth so gravity $g = 10\text{m/s}^2$ is well approximated as constant (and all lines towards the center of the earth are effectively parallel). The air drag opposes motion and is proportional to speed with proportionality constant of $c = 0.1$ N/(m/s).

a) (20 pt) Write Matlab code to plot the trajectory, with the same vertical and horizontal scale, for 10 seconds [hints: FBD → LMB → first order ODEs → numerical solution → plotting].

b) (20 pt) Find, analytically, the position $\underline{\mathbf{r}}(t)$. [hints: same as above but use calculus instead of Matlab. The calculation has several steps (4 calculus problems, in one way of counting).]

c) (10 pt) More difficult. As accurately and neatly as you can, plot the trajectory. Label the units on the axis. The plot should go from when the projectile is launched until it hits the ground again. Key quantities to show are the peak height and the distance the projectile goes (which can be calculated very accurately). You can use the solution above or anything else you know or think. This will be graded on its correctness, not its agreement (or not) with the solution (b) above. But you should briefly rationalize your plot.

2D



given:
$$\begin{cases} m = .02\text{ Kg} \\ c = .1\text{ N/(m/s)} \\ g = 10\text{ N/Kg} \\ \underline{V}_0 = V_0\underline{\lambda}_0 \\ \quad = 50\text{ m/s}(\cos 60°\underline{i} + \sin 60°\underline{j}) \end{cases}$$

LMB $\quad \sum \underline{F}_i = m\underline{a}$

$$\{-c\underline{v} - mg\underline{j} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j})\}$$
$$\llcorner\ \underline{v} = \dot{x}\underline{i} + \dot{y}\underline{j}$$

$$\{\}\cdot\underline{i} \Rightarrow -c\dot{x} = m\ddot{x} \Rightarrow \ddot{x} = -\frac{c}{m}\dot{x} \quad (1)$$
$$\{\}\cdot\underline{j} \Rightarrow -c\dot{y} - mg = m\ddot{y} \Rightarrow \ddot{y} = -\frac{c}{m}\dot{y} - g \quad (2)$$

Define $v_x = \dot{x}$, $v_y = \dot{y}$

(1), (2) $\Rightarrow$
$$\boxed{\begin{aligned} \dot{x} &= V_x & (3)\\ \dot{y} &= V_y & (4)\\ \dot{V}_x &= -(c/m)V_x & (5)\\ \dot{V}_y &= -(c/m)V_y - g & (6) \end{aligned}}$$

4 coupled 1st order ODEs

I.C.'s
$$\boxed{\begin{aligned} X_0 &= 0\\ Y_0 &= 0\\ V_{x0} &= 50\cos 60°\text{ m/s}\\ V_{y0} &= 50\sin 60°\text{ m/s} \end{aligned}}$$

Matlab solution (a)

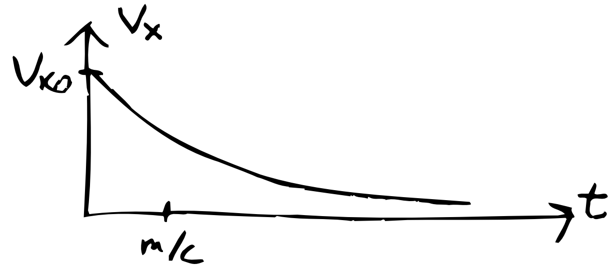
```
% Ruina trajectory soln., assume consistent units
x0=0; y0=0;
Vx0 = 50*cos(60*pi/180); Vy0= 50*sin(60*pi/180);
z0 = [x0 y0 Vx0 Vy0];
tspan = linspace [0, 10, 101];
[t z]= ode23('myrhs', tspan, z0);
x = z(:,1); y= z(:,2);
plot (x,y); xlabel('x'); ylabel('y'); title('trajectory');
axis ('equal')
```

in file myrhs.m

```
function zdot = myrhs(t,z)
x = z(1); y = z(2); Vx = z(3); Vy = z(4);
c=.1; m=.02; g=10;
ẋ = Vx;
ẏ = Vy;
V̇x = -(c/m)*Vx;
V̇y = -(c/m)*Vy -g;
zdot=[ẋ ẏ V̇x V̇y]';
```

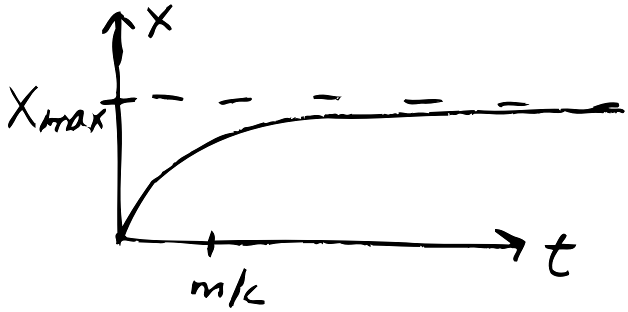
Analytic Soln. (prob 3 cont'd)

(5) $\Rightarrow \quad V_x = V_{xo}\, e^{-(c/m)\, t}$ [See comments on pgs. 9-13 about ODE solutions.]



(3) $\Rightarrow \quad x = x_o + \int_0^t V_x(t')\,dt' = 0 + \int_0^t V_{xo}\, e^{-(c/m)\,t'}\,dt'$

$$= -\frac{m V_{xo}}{c}\, e^{-(c/m)\,t'}\Big|_0^t = \underbrace{\frac{m V_{xo}}{c}\left(1 - e^{-(c/m)\,t}\right)}_{x(t)}$$



$x_{max} \overset{(7)}{=} \frac{m V_{xo}}{c} = \frac{.02\,(kg)\; 25\,(m/s)}{.1\,N/(m/s)}$

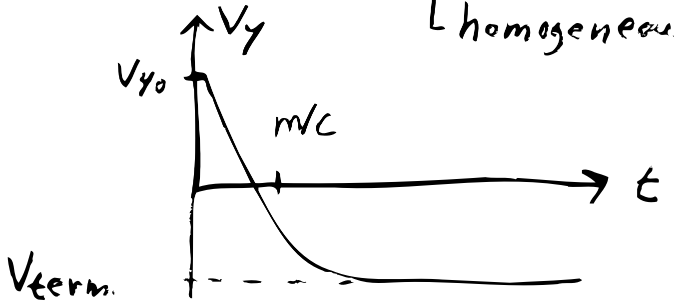
$= \underbrace{5\,m}$

[x expontially approaches 5 m w/ time

(6) $\Rightarrow \quad V_y = \underbrace{\left(V_{yo} + \frac{gm}{c}\right) e^{-\frac{c}{m} t}} \; \underline{- gm/c}$

↑ partic. soln. from inspection (terminal vel.)

homogeneous soln., constant picked to sat. I.C.



V_{term} = terminal vel.

$= -gm/c$ (drag balances weight)

$= -10 \cdot .02/.1\ (m/s)$

$= 2\ m/s$

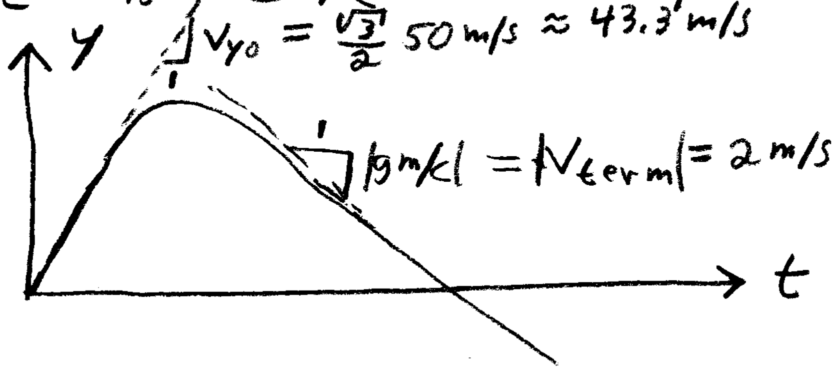
(4) $\Rightarrow \quad y = y_o + \int_0^t V_y(t')\,dt'$

$$= 0 + \int_0^t \left[\left(V_{yo} + \frac{gm}{c}\right) e^{-(c/m)\,t'} - gm/c\right] dt'$$

(prob (3) cont'd)

$$y = \left[-\frac{m}{c}\left(V_{y_0} + \frac{gm}{c}\right) e^{-(c/m)t'} - \frac{gm}{c} t' \right]_0^{t'}$$

$$= \frac{m}{c}\left(V_{y_0} + \frac{gm}{c}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c} t$$



$$\underline{r}(t) = x\,\underline{i} + y\,\underline{j}$$

$$= \frac{mV_{x0}}{c}\left(1 - e^{-(c/m)t}\right)\underline{i}$$
$$+ \left[\left(\frac{mV_{y0}}{c} + \frac{gm^2}{c^2}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c} t\right]\underline{j}$$

$c/m = (.1/.02)\,s^{-1} = 5/s$, $\quad \frac{gm^2}{c^2} = 10 \cdot \frac{(.02)^2}{(.1)^2}\,m = .4\,m$

$\frac{mV_{x0}}{c} = 5\,m$ (see 7)

$\frac{mV_{y0}}{c} = \frac{\sqrt{3}}{2} \cdot 5\,m \approx 8.66\,m$

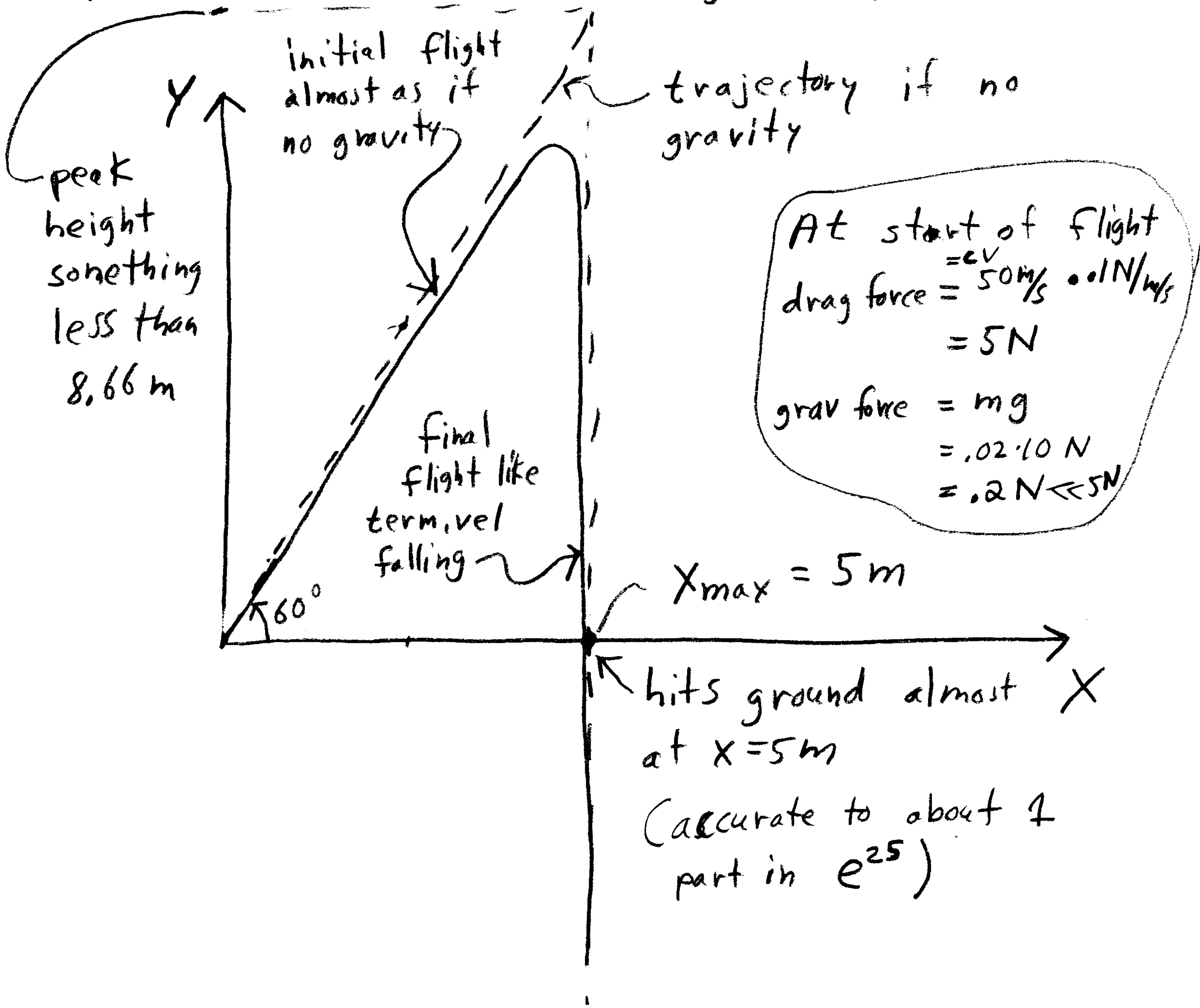
$\frac{gm}{c} = 2\,m/s$

(b)

$$\boxed{\underline{r}(t) = 5m\left(1 - e^{-5t/s}\right)\underline{i} + \left[\left(\frac{5\sqrt{3}}{2} - .4\right)\left(1 - e^{-5t/s}\right)m - 2(m/s)\,t\right]\underline{j}}$$

(prob. 3, cont'd)

c) Plot. All we need from anal. soln. is $\underbrace{\text{eqn. 7}}^{\text{i.e eqn for } x(t)}$.
(No need for soln. of inhomog. soln.)



Some comments on ODE solns.

Solve $\dot{V}_x = -(c/m)V_x$

<u>method 1</u>: guess $V_x = e^{rt}$

plug in $r\cancel{e^{rt}} = -(c/m)\cancel{e^{rt}}$

$$r = -c/m$$

$$\Rightarrow \quad V_x = c_1 e^{-(c/m)t}$$

$\uparrow$ arb. const., pick to sat. I.C.

<u>method 2</u>: (Edwards & Penny 1.4)

$$\frac{dV_x}{dt} = -(c/m)V_x$$

$$\Rightarrow \quad \frac{dV_x}{V_x} = -(c/m)dx$$

$$\Rightarrow \quad \int \frac{dV_x}{V_x} = -\int c/m \, dt$$

$$\Rightarrow \quad \ln V_x = -\frac{c}{m}t + c_1'$$

$$\Rightarrow \quad V_1 = e^{-(c/m)t + c_1'}$$

$$V_x = c_1 e^{-(c/m)t} \qquad (c_1 = e^{c_1'})$$

(again)

ODEs (cont'd)

method 3: Integrating factor
(Edwards & Penny 1.5)

$$\frac{dV_x}{dt} + \frac{c}{m} V_x = 0$$

$e^{c/m\, t}$ is the integrating factor.

$$\Rightarrow \quad e^{\frac{c}{m}t} \frac{dV_x}{dt} + e^{(c/m)t} \frac{c}{m} V_x = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{(c/m)t} V_x\right) = 0$$

$$\Rightarrow \quad e^{(c/m)t} V_x = C_1$$

$$\Rightarrow \quad V_x = C_1 e^{-(c/m)t}$$

(again)

Solve $\quad \ddot{x} + c/m\, \dot{x} = 0$

method 1: guess $x = e^{rt}$

$$\Rightarrow \quad r^2 \cancel{e^{rt}} + r\frac{c}{m} \cancel{e^{rt}} = 0$$

$$\Rightarrow \quad r\left(r + \frac{c}{m}\right) = 0 \quad \Rightarrow \quad r = 0, -c/m$$

$$\Rightarrow \quad x(t) = C_1 e^{-\frac{c}{m}t} + C_2 e^{-0t}$$

$$= C_1 e^{-(c/m)t} + C_2$$

find using ICs.

This is $x(t)$ as found.

Solve $\dot{V}_y + (c/m)\, V_y = -g$

method 1: a) find homog. soln.

$$\dot{V}_y + (c/m) V_y = 0.$$

This is identical to prev. problem which we solved to get

$$\underbrace{V_{y_h}}_{\text{homog. soln.}} = c_1 e^{-(c/m)t}$$

b) Find any "particular" soln. of

$$\dot{V}_y + (c/m) Y = -g.$$

As for spring-mass problem. Easiest guess is a constant. In this case you can get this physically by thinking of falling at terminal velocity.

guess $V_{y_p} = c_2$

$$\cancelto{0}{\dot{c}_2} + (c/m)\, c_2 = g$$

$$c_2 = gm/c$$

Solution is

$$V_y = V_{yh} + V_{yp}$$

$$= c_1 e^{-(c/m)t} + gm/c$$

c_1 ← pick to match I.C.

method 2;
(E 8P 1.4)
(seperable eqn.)

$$\frac{dV_y}{dt} + \frac{c}{m} V_y = -g$$

$$\Rightarrow \frac{dV_y}{dt} = -\frac{c}{m} V_y - g$$

$$\Rightarrow \frac{dV_y}{V_y + \frac{gm}{c}} = -\frac{c}{m}\, dt$$

$$\Rightarrow \int \frac{dV_y}{V_y - \frac{gm}{c}} = -\int \frac{c}{m}\, dt$$

$$\Rightarrow \ln\left(V_y + \frac{gm}{c}\right) = -\frac{c}{m} t + c_1''$$

$$\Rightarrow V_y + \frac{gm}{c} = e^{-\frac{c}{m}t + c_1'}$$

$$\Rightarrow V_y = c_1 e^{-(c/m)t} - gm/c$$

$c_1 = e^{c_1'}$, pick to sat. I.C.

(again)

method 3;
(E 8 P 1.5)
linear ODE using integrating factor

$$\frac{dV_y}{dt} + \frac{c}{m} V_y = -g$$

define $g(t) = e^{(c/m)t}$

integrating factor

mult. through by g

$$\frac{dV_y}{dt} e^{(c/m)t} + \frac{c}{m} V_y e^{(c/m)t} = -g e^{(c/m)t}$$

$$\frac{d}{dt}\left(v_y e^{(c/m)t}\right) = -g e^{(c/m)t}$$

$$\Rightarrow \quad d\left[v_y e^{(c/m)t}\right] = -g e^{(c/m)t} dt$$

$$\Rightarrow \quad \int d\left(v_y e^{(c/m)t}\right) = -\int g e^{c/m\, t} dt$$

$$\Rightarrow \quad v_y e^{(c/m)t} = -\frac{gm}{c} e^{(c/m)t} + c_1$$

$$\Rightarrow \quad v_y = \frac{-gm}{c} + c_1 e^{-(c/m)t}$$

pick to sat. I.C.

(again)

"SOLUTIONS"

Your Name: Andy Ruina

Section day and time: Wed, 11:15

T&AM 203 Prelim 2

Tuesday April 16, 2002

Draft April 16, 2002

3 problems, 100 points, and 90+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	30/30
Problem 2:	40/40
Problem 3:	30/30
TOTAL:	100/100

1) (30 pt) A front-wheel drive car has an engine with arbitrarily large power. It has a stiff suspension and light well-lubricated front wheels. It rides on level ground. Given:

$m =$ total car mass;
$I^G =$ the polar moment of inertia about an axis through the center of mass G and in the ussual zz direction (perpendicular to the plane on which a side view of the car is drawn);
$h =$ the height of G above the ground;
$d =$ the distance G is forward of the rear ground contact at A;
$e =$ the distance G is behind the front ground contact at B;
$\mu =$ the coefficient of friction between the wheels and ground;
$g =$ the gravity constant. ~~In terms of some or all of m, I^G, g, h, d, e, and μ find the maximum possible forward acceleration on level ground.~~

a) (20 points) Assuming the car does not tipe over, find the maximum possible forward acceleration a ($\underline{\mathbf{a}}_G = a\hat{\mathbf{i}}$). Answer in terms of some or all of m, I^G, g, h, d, e, and μ.

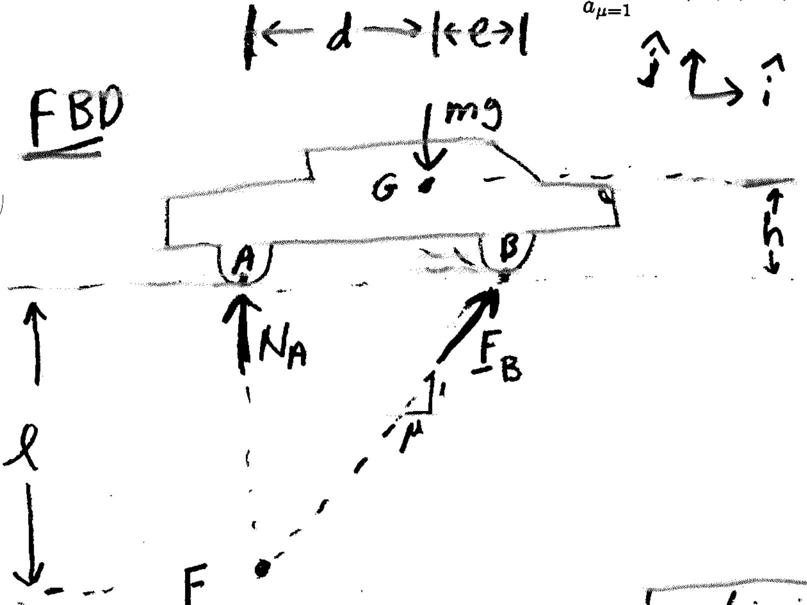
b) (5 points) (Good reasoning may or may not depend on part (a)) Assuming $d = e = h$, what is the largest value of μ that is possible without violating reasonable assumptions (assume that rubber with arbitrarily large μ could be made reasonably)? [Clearly state the reasonable assumptions that you are checking]. The answer is one of these

$$\mu_{max} = 0, .5, 1, \sqrt{2}, \sqrt{3}, 2, e, 2\sqrt{2}, 3, \pi, 4, 8, \text{ or } \infty.$$

c) (5 points) (Good reasoning may or may not depend on part (a)) Assuming $d = e = h$ and that the car does not tip over, by what ratio does the peak acceleration increase if μ is increased from $\mu = 1$ to $\mu = \infty$ (infinite coefficient of friction). The answer is one of these

$$\frac{a_{\mu=\infty}}{a_{\mu=1}} = 0, .5, 1, \sqrt{2}, \sqrt{3}, 2, e, 2\sqrt{2}, 3, \pi, 4, 8, \text{ or } \infty.$$

FBD



Geometry:

$$\frac{d+e}{l} = \frac{\mu}{1}$$

$$\Rightarrow l = \frac{d+e}{\mu}$$

$$\sum \underline{M}_{/E} = \dot{\underline{H}}_{/E}$$

no tipping

$$-dmg\hat{k} = \underline{r}_{G/E} \times m\underline{a}_G + I^G \cancel{\alpha}^{\,0} \hat{k}$$

$$= ((l+h)\hat{j} + d\hat{i}) \times m a \hat{i}$$

$$= -\left(\frac{d+e}{\mu} + h\right) a \hat{k} m$$

$\{\}\cdot\hat{k} \Rightarrow$

$$d m g = \left(\frac{d+e}{\mu} + h\right) a m$$

$$\boxed{a = \frac{dg}{\left(h + \frac{d+e}{\mu}\right)} \qquad \underline{a}_G = \frac{dg\hat{i}}{\left(h + \frac{d+e}{\mu}\right)}}$$

(a)

(prob 1 cont'd)

(b) $\sum \underline{M}_{/B} = \dot{\underline{H}}_{/B}$

$$\left\{ \left(-N_A(d+e) + mge\right)\hat{k} = -mah\hat{k} \right\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow N_A = \frac{m(ah+ge)}{d+e} > 0$$

so rear wheels have no lift off problem. Front wheels can't have a lift off problem because the drive ~~force~~ goes to zero as lifting proceeds.

$\Rightarrow \boxed{\mu = \infty \text{ is o.k.}}$ (b)

c) Could redo problem using assumptions, or just plug in soln. to (a): $(d=e=h)$

$$\boxed{\frac{a_{\mu=\infty}}{a_{\mu=1}} = \frac{\frac{dg}{(d+\frac{2d}{\infty})}}{\frac{dg}{d+\frac{2d}{1}}} = \frac{\frac{g}{1}}{\frac{g}{3}} = 3}\ !$$

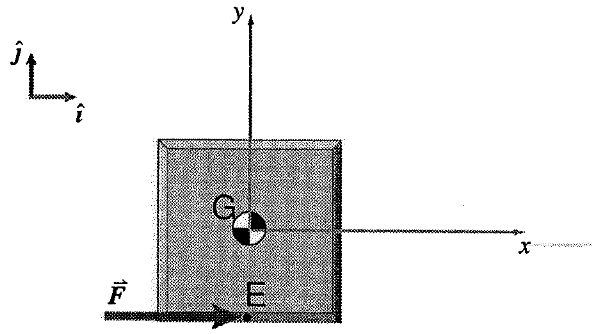
(Front wheel drive car (w/ $d=e=h$) can have accel. of $g/3$ if $\mu=1$ and g if $\mu=\infty$.)

2) (40 pt) A uniform 1kg plate that is one meter on a side is initially at rest in the position shown. A constant force $\underline{\mathbf{F}} = 1\,\mathrm{N}\,\hat{\mathbf{i}}$ is applied at $t = 0$ and maintained henceforth. [If you need to calculate any quantity that you don't know, but can't do the calculation to find it, assume that the value is given.]

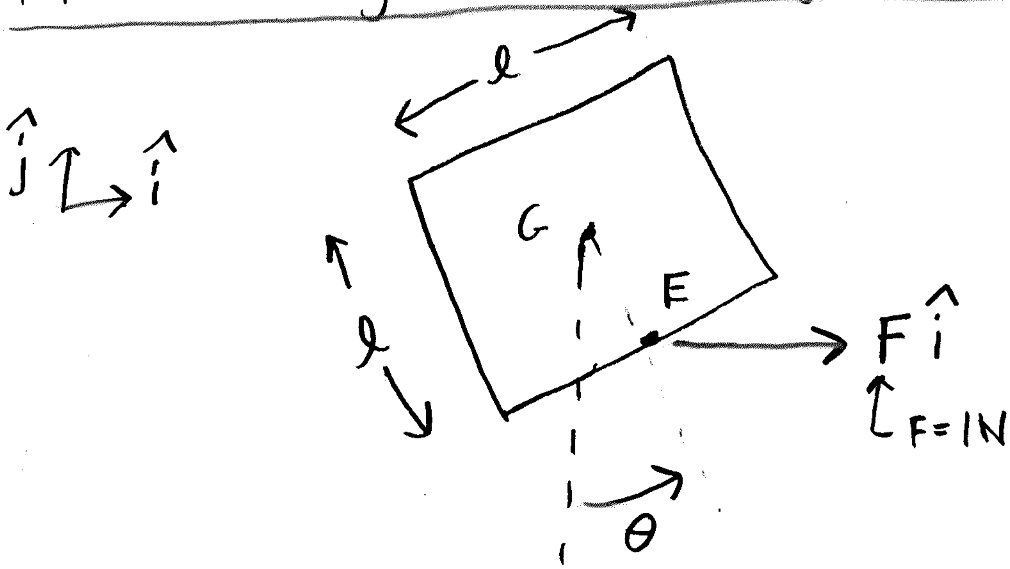
a) Find the position of G as a function of time (the answer should have numbers and units).

b) Find a differential equation, and initial conditions, that when solved would give θ as a function of time. θ is the counterclockwise rotation of the plate from the configuration shown.

c) Write MATLAB commands that would generate a drawing of the outline of the plate at $t = 1\,\mathrm{s}$. You can use hand calculations or Matlab for as many of the intermediate commands as you like. Add enough hand work so your MATLAB reasoning is clear.



FBD in general configuration



$\Rightarrow$

$\underline{r}_G = \frac{F}{m} t^2/2\ \hat{i}$

$= \frac{1N}{1Kg} \frac{t^2}{2} \hat{i}$

$$\boxed{\underline{r}_G = \frac{t^2}{2} \hat{i}\ \frac{m}{sec^2}}$$

(a)

or

$\underline{r}_G = \frac{t^2}{2} \hat{i}$

if all quantities in meters, kg, sec.

LMB: $\sum \underline{F}_i = m \underline{a}_G$

$\{F\hat{i} = m a_{Gx} \hat{i} + m a_{Gy} \hat{j}\}$

$\{\ \}\cdot\hat{i} \Rightarrow a_{Gx} = F/m \Rightarrow V_{Gx} = \frac{F}{m} t \Rightarrow X_G = \frac{F}{m} t^2/2$

$\{\ \}\cdot\hat{j} \Rightarrow a_{Gy} = 0 \Rightarrow Y_G = 0$

b) $\sum \underline{M}_{/G} = \dot{\underline{H}}_{/G}$ (Prob. 2 cont'd)

$$\underline{r}_{E/G} \times F\hat{i} = I^G \alpha \hat{k}$$

$$\frac{\ell}{2}\left(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}\right) \times F\hat{i} = I^G \ddot{\theta}\hat{k}$$

$$\Rightarrow \frac{\ell F}{2}\cos\theta = I^G\ddot{\theta}$$

$$\boxed{\ddot{\theta} = \frac{\ell F}{2 I_G}\cos\theta}$$

$\ell = 1\,m$, $F = 1\,N$, $m = 1\,kg$

$$I^G = \int_{-\ell/2}^{\ell/2}\int_{-\ell/2}^{\ell/2} (x^2+y^2)\frac{m}{\ell^2}\,dx\,dy$$

$$I^G = \left(x^3\ell\Big|_{-\ell/2}^{\ell/2} + y^3\ell\Big|_{-\ell/2}^{\ell/2}\right)\frac{m}{\ell^2}$$

$$I^G = \left(\frac{\ell^4}{12} + \frac{\ell^4}{12}\right)\frac{m}{\ell^2} = \frac{m\ell^2}{6}$$

$$\Rightarrow \ddot{\theta} = \left(\frac{1\cdot 1}{2\cdot\frac{1}{6}}\cos\theta\right)s^{-2} \Rightarrow \boxed{\ddot{\theta} = 3\cos\theta\ s^{-2}}\ (b)$$

c)

all in consistent mKS units

```
pic = [1  -1  -1   1  1;
       1   1  -1  -1  1]/2;
θ0 = 0;  ω0 = 0;
[t z] = ode23('gosh', [0 1], [θ0 ω0])
θ1 = z(end,1);   % final rotation
R = [cos θ1   -sin θ1;
     sin θ1    cos θ1 ];
disp = [1/2  0]';   % displ. from part (a)
homo = [ R  disp;
        [0 0 1] ];   % homog. transf.
newpic = homo * [pic; ones(1,5)];
plot (newpic(1,:), newpic(2,:));
axis ('equal')
```

$[0\ 1]$: tspan; $[\theta_0\ \omega_0]$: z_0

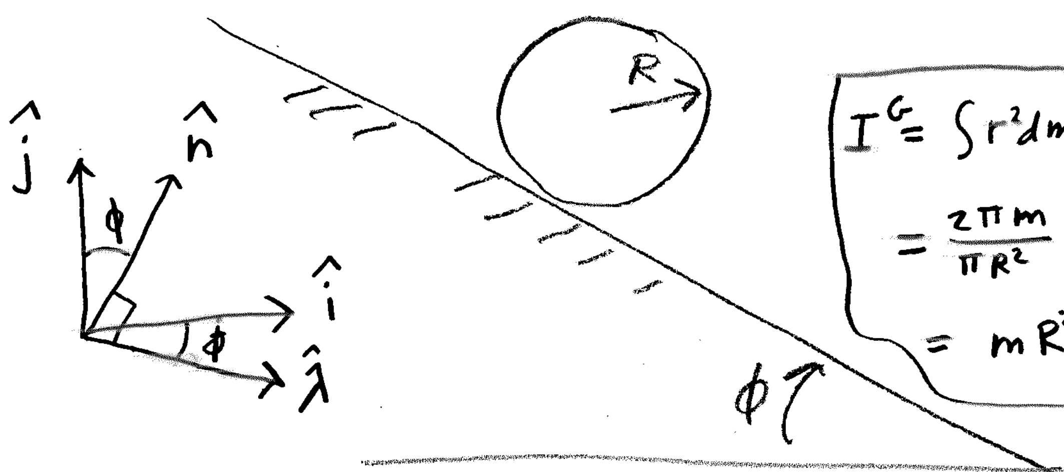
ODE from (b)

gosh.m:

```
function zdot = gosh(t,z)
θ = z(1);  ω = z(2);
θ̇ = ω;  ω̇ = 3 cos θ;
zdot = [θ̇  ω̇]';
```

3) (30 pt) A uniform disk with mass m and radius R is released from rest to roll down a slope that is tipped ϕ from the horizontal. The local gravity constant is g.

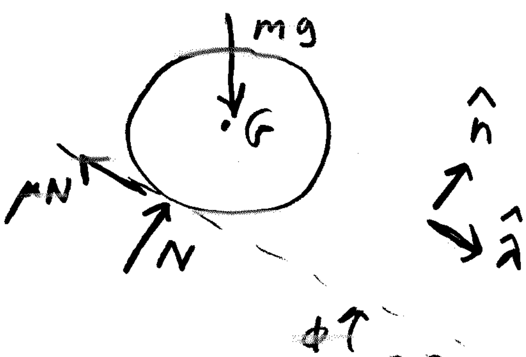
> **a)** (10 pt) Assume that slope is high, or friction coefficient small, so the disk slides down the slope. What is the acceleration of the center of the disk?
>
> **b)** (10 pt) Assume the disk rolls, what is the angular acceleration of the disk?
>
> **c)** (10 pt) If $\mu = .5$, what is the biggest angle ϕ for which the disk will roll and not slide.



$$I^G = \int r^2 dm = \int_0^R \int_0^{2\pi} r^2 \frac{m}{\pi R^2} r\,dr\,d\theta$$
$$= \frac{2\pi m}{\pi R^2} \int_0^R r^3 dr$$
$$= mR^2/2$$

a) FBD

sliding



Kinematics

$\underline{a}_G = a_G \hat{\lambda}$

LMB

$$\sum \underline{F}_i = m \underline{a}_G$$

$$\{ N\hat{n} - \mu N \hat{\lambda} - mg\hat{j} = m a_G \hat{\lambda} \}$$

$$\{\}\cdot\hat{n} \Rightarrow N - mg \underbrace{\hat{j}\cdot\hat{n}}_{\cos\phi} = 0$$

$$\Rightarrow N = mg\cos\phi$$

$\{\}\cdot\hat{\lambda} \Rightarrow$

$$-\mu N - mg \underbrace{\hat{j}\cdot\hat{\lambda}}_{-\sin\phi} = m a_G$$

$\Rightarrow$

$$-\mu \cancel{m} g\cos\phi + \cancel{m} g\sin\phi = \cancel{m} a_G$$

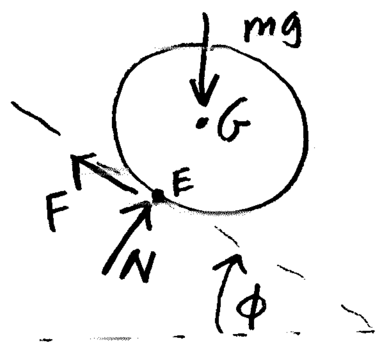
$$a_G = g(\sin\phi - \mu\cos\phi)$$

$$\Rightarrow \boxed{\underline{a}_G = g(\sin\phi - \mu\cos\phi)\hat{\lambda}}$$

(a)

(prob 3 cont'd)

b) FBD rolling



$$\boxed{\begin{array}{ll} \underline{\text{Kinematics}} & \underline{a}_G = -R\alpha\hat{\lambda} \\ R\alpha = -a_G & \underline{\alpha} = \alpha\hat{k} \end{array}}$$

$$\underline{\text{AMB}}_{/E} \Rightarrow \underline{r}_{G/E} \times (-mg\hat{j}) = \underline{r}_{G/E} \times m\underline{a}_G + I^G\alpha\hat{k}$$

$$\Rightarrow R\hat{n} \times (-mg\hat{j}) = R\hat{n} \times (m(-R\alpha\hat{\lambda})) + I^G\alpha\hat{k}$$

$$\Rightarrow \{-Rmg\sin\phi\,\hat{k} = R^2 m\alpha\hat{k} + I^G\alpha\hat{k}\}$$

$$\{\ \}\cdot\hat{k} \qquad \Rightarrow \alpha = \frac{-Rmg\sin\phi}{\underbrace{I^G}_{mR^2/2} + mR^2} = \frac{-2Rg\sin\phi}{3R^2}$$

$$\Rightarrow \boxed{\text{ang. accel.} = \frac{-2g\sin\phi}{3R}\hat{k}} \ \text{(b)}$$

---

c) LMB

$$\Sigma \underline{F}_i = m\underline{a}_G$$

$$\left\{N\hat{n} - F\hat{\lambda} - mg\hat{j} = m\underbrace{a_G}_{\substack{= -\alpha R \\ = 2g\sin\phi/3}}\hat{\lambda}\right\}$$

$$\{\ \}\cdot\hat{n} \Rightarrow N - mg\cos\phi = 0$$
$$N = mg\cos\phi \quad \text{(as for sliding)}$$

$$\{\ \}\cdot\hat{\lambda} \Rightarrow -F + mg\sin\phi = 2mg\sin\phi/3$$
$$\Rightarrow F = mg\sin\phi(1 - 2/3)$$
$$= mg\sin\phi/3$$

critical slope when $\mu = \dfrac{F}{N} = \dfrac{mg\sin\phi/3}{mg\cos\phi}$

$$\Rightarrow \tan\phi = 3\underbrace{\mu}_{0.5 \text{ given}} \Rightarrow \tan\phi = 3/2 \Rightarrow \boxed{\phi = \tan^{-1}(3/2)}$$

"SOLUTIONS"

[w/ comments on ODE solutions on pages 9-13]

Your Name: Andy Ruina

Section day and time: Wed. 10:10

T&AM 203 Prelim 1

Tuesday Feb 26, 2002

Draft February 26, 2002

3 problems, 100 points, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

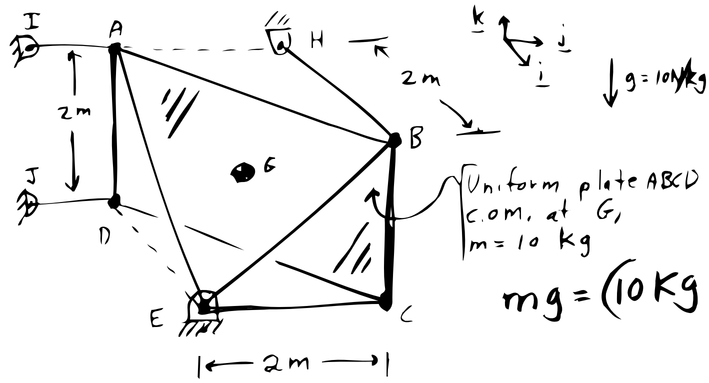
b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems **poorly defined**;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1:	/25
Problem 2:	/25
Problem 3:	/50
TOTAL:	**/100**

1) (25 pt) Statics. The sign is held up by 6 bars. Find the tension in bar EB.



$mg = (10\,kg)(10\,N/kg) = 100\,N$

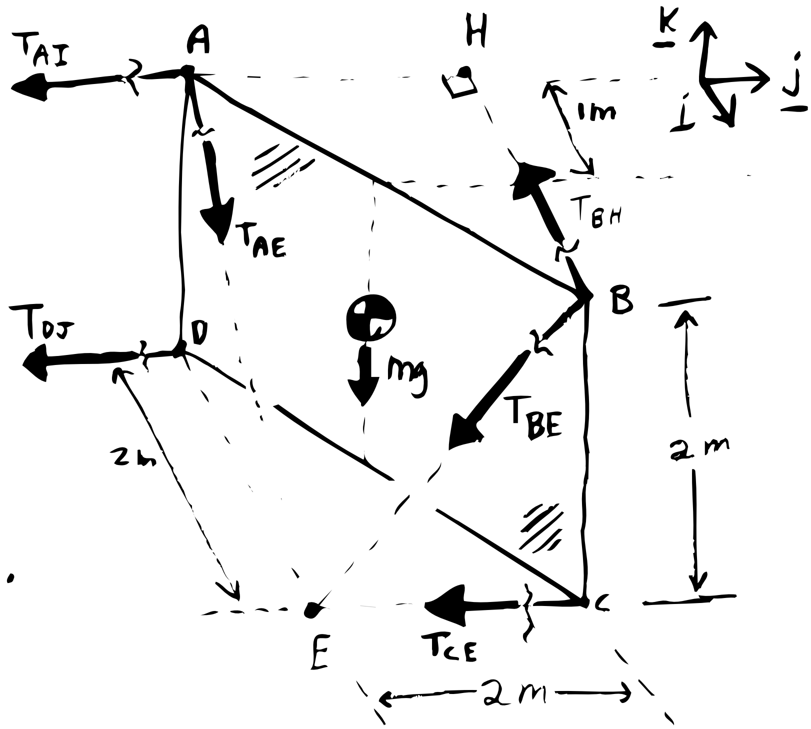
Consider axis AH:

* T_{AI}, T_{DJ}, T_{CE} are // to axis.

* T_{BH} & T_{AE} intersect axis,

$\Rightarrow$ Only T_{BE} and mg contribute to moment about axis AH.

But mg is known.



$$\sum M_{\substack{axis\\AH}} = \left(\sum \underline{M}_{/H}\right)\cdot \underline{j} = 0$$

$$\Rightarrow \underbrace{100\,Nm}_{\text{moment of mg about axis AH}} + \left(\underline{r}_{B/H} \times \overbrace{T_{BE}\left(\frac{-\underline{j}-\underline{k}}{\sqrt{2}}\right)}^{\underline{T}_{BE}}\right)\cdot \underline{j} = 0$$

$$\underline{r}_{B/H} = 2\,m\,\underline{i}$$

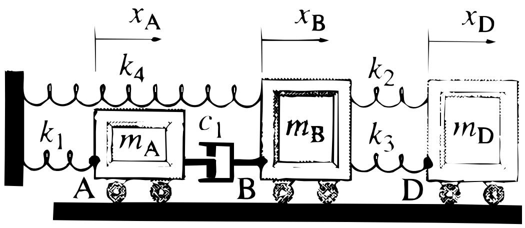
$$\Rightarrow 100\,Nm + \left[2m\,T_{BE}(-\underline{k} + \underline{j})/\sqrt{2}\right]\cdot \underline{j} = 0$$

$$100\,N\cancel{m} + \sqrt{2}\,\cancel{m}\,T_{BE} = 0$$

$$\boxed{T_{BE} = \frac{-100}{\sqrt{2}}\,N \approx -70.7\,N}$$

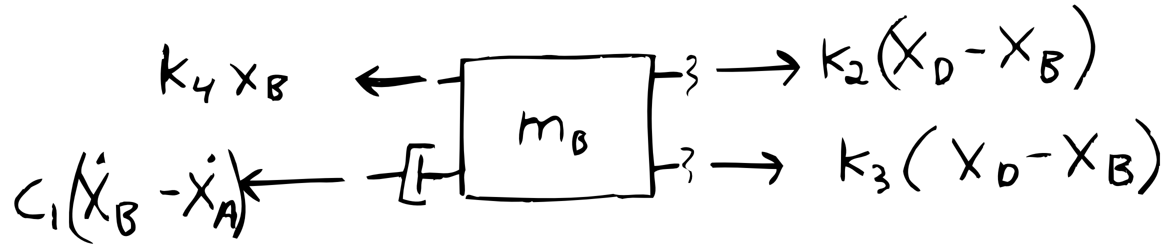
(BE is in compression)

2) (25 pt.) In terms of some or all of $x_A, x_B, x_D, \dot{x}_A, \dot{x}_B, \dot{x}_D, k_1, k_2, k_3, k_4, m_A, m_B, m_D$ and c_1 find $\ddot{x}_B$. Assume the springs are relaxed when $x_A = x_B = x_D = 0$



FBD $\rightarrow \underline{i}$

When m_B is at position x_B



LMB $\left\{ \sum \underline{F}_i = m_B \underline{a}_B \right\} \cdot \underline{i}$

$$\Rightarrow -c_1(\dot{X}_B - \dot{X}_A) - K_4 X_B + K_2(X_D - X_B) + K_3(X_D - X_B) = m_B \ddot{X}_B$$

$$\boxed{\ddot{X}_B = \frac{1}{m_B}\left[-(K_2 + K_3 + K_4) X_B + (K_2 + K_3) X_D + c_1 \dot{X}_A - c_1 \dot{X}_B\right]}$$

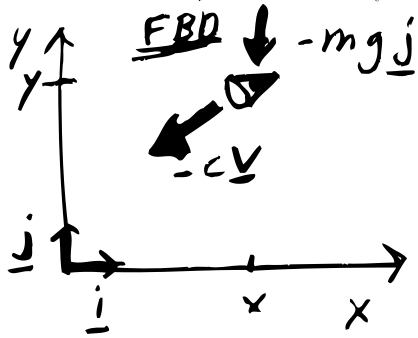
3) **(50 pt)** Trajectory. A 0.02 kg projectile (a badminton birdie, say) is launched from the origin at a 60° upwards angle at a speed of 50 m/s. The projectile stays near the earth so gravity $g = 10\text{m/s}^2$ is well approximated as constant (and all lines towards the center of the earth are effectively parallel). The air drag opposes motion and is proportional to speed with proportionality constant of $c = 0.1$ N/(m/s).

a) (20 pt) Write Matlab code to plot the trajectory, with the same vertical and horizontal scale, for 10 seconds [hints: FBD → LMB → first order ODEs → numerical solution → plotting].

b) (20 pt) Find, analytically, the position $\underline{\mathbf{r}}(t)$. [hints: same as above but use calculus instead of Matlab. The calculation has several steps (4 calculus problems, in one way of counting).]

c) (10 pt) More difficult. As accurately and neatly as you can, plot the trajectory. Label the units on the axis. The plot should go from when the projectile is launched until it hits the ground again. Key quantities to show are the peak height and the distance the projectile goes (which can be calculated very accurately). You can use the solution above or anything else you know or think. This will be graded on its correctness, not its agreement (or not) with the solution (b) above. But you should briefly rationalize your plot.

2D



given:

$$\begin{cases} m = .02\ \text{Kg} \\ c = .1\ \text{N/(m/s)} \\ g = 10\ \text{N/Kg} \\ \underline{v}_0 = v_0 \underline{\lambda}_0 \\ \quad = 50\ \text{m/s}\left(\cos 60^\circ \underline{i} + \sin 60^\circ \underline{j}\right) \end{cases}$$

LMB $\quad \sum \underline{F}_i = m\underline{a}$

$$\{ -c\underline{v} - mg\underline{j} = m(\ddot{x}\underline{i} + \ddot{y}\underline{j}) \}$$

$$\underline{v} = \dot{x}\underline{i} + \dot{y}\underline{j}$$

$$\{\}\cdot\underline{i} \Rightarrow \quad -c\dot{x} = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = \frac{-c}{m}\dot{x} \qquad (1)$$

$$\{\}\cdot\underline{j} \Rightarrow \quad -c\dot{y} - mg = m\ddot{y} \Rightarrow \quad \ddot{y} = \frac{-c}{m}\dot{y} - g \qquad (2)$$

Define $v_x = \dot{x}, \quad v_y = \dot{y}$

(1), (2) $\Rightarrow$

$$\boxed{\begin{aligned} \dot{x} &= v_x & (3) \\ \dot{y} &= v_y & (4) \\ \dot{v}_x &= -(c/m)v_x & (5) \\ \dot{v}_y &= -(c/m)v_y - g & (6) \end{aligned}}$$

4 coupled 1st order ODEs

I.C.'s

$$\boxed{\begin{aligned} x_0 &= 0 \\ y_0 &= 0 \\ v_{x0} &= 50\cos 60^\circ\ \text{m/s} \\ v_{y0} &= 50\sin 60^\circ\ \text{m/s} \end{aligned}}$$

(prob 3 cont'd)

Matlab solution (a)

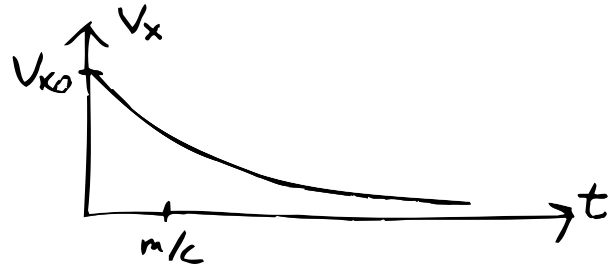
```
% Ruina trajectory soln., assume consistent units
x0=0; y0=0;
vx0 = 50*cos(60*pi/180); vy0= 50*sin(60*pi/180);
z0 = [x0 y0 vx0 vy0];
tspan = linspace[0, 10, 101];
[t z]= ode23('myrhs', tspan, z0);
x= z(:,1); y= z(:,2);
plot (x, y); xlabel('x'); ylabel('y'); title('trajectory');
axis ('equal')
```

in file myrhs.m

```
function zdot = myrhs(t, z)
x= z(1); y= z(2); vx= z(3); vy= z(4);
c=.1; m=.02; g=10;
xdot = vx;
ydot = vy;
vxdot = -(c/m)*vx;
vydot = -(c/m)*vy -g;
zdot=[xdot ydot vxdot vydot]';
```

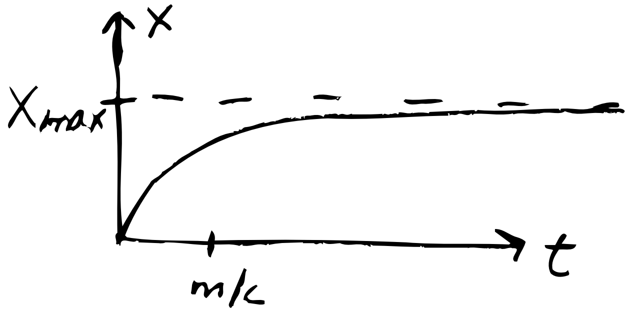
Analytic Soln. (prob 3 cont'd)

5 $\Rightarrow$ $$V_x = V_{xo}\, e^{-(c/m)\,t}$$ [See comments on pgs. 9-13 about ODE solutions.]



3 $\Rightarrow$ $$x = x_o + \int_0^t V_x(t')\,dt' = 0 + \int_0^t V_{xo}\, e^{-(c/m)\,t'}\,dt'$$

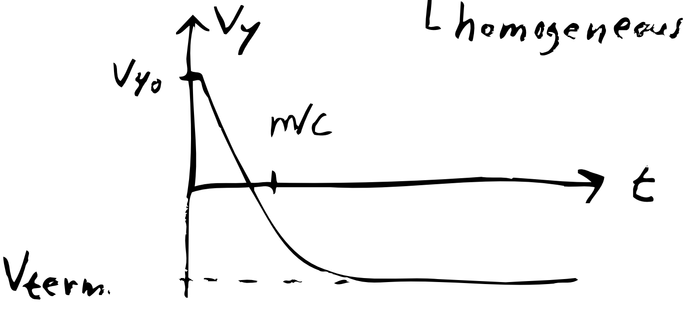
$$= -\frac{mV_{xo}}{c}\, e^{-(c/m)\,t'}\Big|_0^t = \underbrace{\frac{mV_{xo}}{c}\left(1 - e^{-(c/m)t}\right)}_{x(t)}$$



$$x_{max} \overset{7}{=} \frac{mV_{xo}}{c} = \frac{.02\,(kg)\ 25\,(m/s)}{.1\,N/(m/s)}$$

$$= \underbrace{5\,m}$$ [x expontially approaches 5 m w/ time

6 $\Rightarrow$ $$V_y = \underbrace{\left(V_{yo} + \frac{gm}{c}\right) e^{-\frac{c}{m}t}}_{\text{homogeneous soln., constant picked to sat. I.C.}} \underbrace{-\, gm/c}_{\text{partic. soln. from inspection (terminal vel.)}}$$



V_{term} = terminal vel.
$= -gm/c$ (drag balances weight)
$= -10 \cdot .02/.1$ (m/s)
$= 2$ m/s

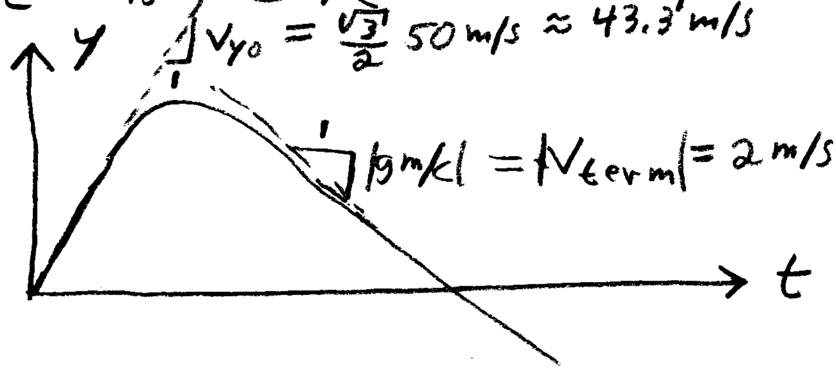
4 $\Rightarrow$ $$y = y_o + \int_0^t v_y(t')\,dt'$$

$$= 0 + \int_0^t \left[\left(V_{yo} + \frac{gm}{c}\right) e^{-(c/m)\,t'} - gm/c\right] dt'$$

(prob (3) cont'd)

$$y = \left[-\frac{m}{c}\left(V_{y_0} + \frac{gm}{c}\right)e^{-(c/m)t'} - \frac{gm}{c}t'\right]_0^{t'}$$

$$= \frac{m}{c}\left(V_{y_0} + \frac{gm}{c}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c}t$$



$$\underline{r}(t) = x\,\underline{i} + y\,\underline{j}$$

$$= \frac{mV_{x0}}{c}\left(1 - e^{-(c/m)t}\right)\underline{i}$$

$$+ \left[\left(\frac{mV_{y0}}{c} + \frac{gm^2}{c^2}\right)\left(1 - e^{-(c/m)t}\right) - \frac{gm}{c}t\right]\underline{j}$$

$c/m = (.1/.02)\,s^{-1} = 5/s$, $\quad \frac{gm^2}{c^2} = 10 \cdot \frac{(.02)^2}{(.1)^2}\,m = .4\,m$

$\frac{mV_{x0}}{c} = 5\,m$ (see 7)

$\frac{mV_{y0}}{c} = \frac{\sqrt{3}}{2}\cdot 5\,m \approx 8.66\,m$

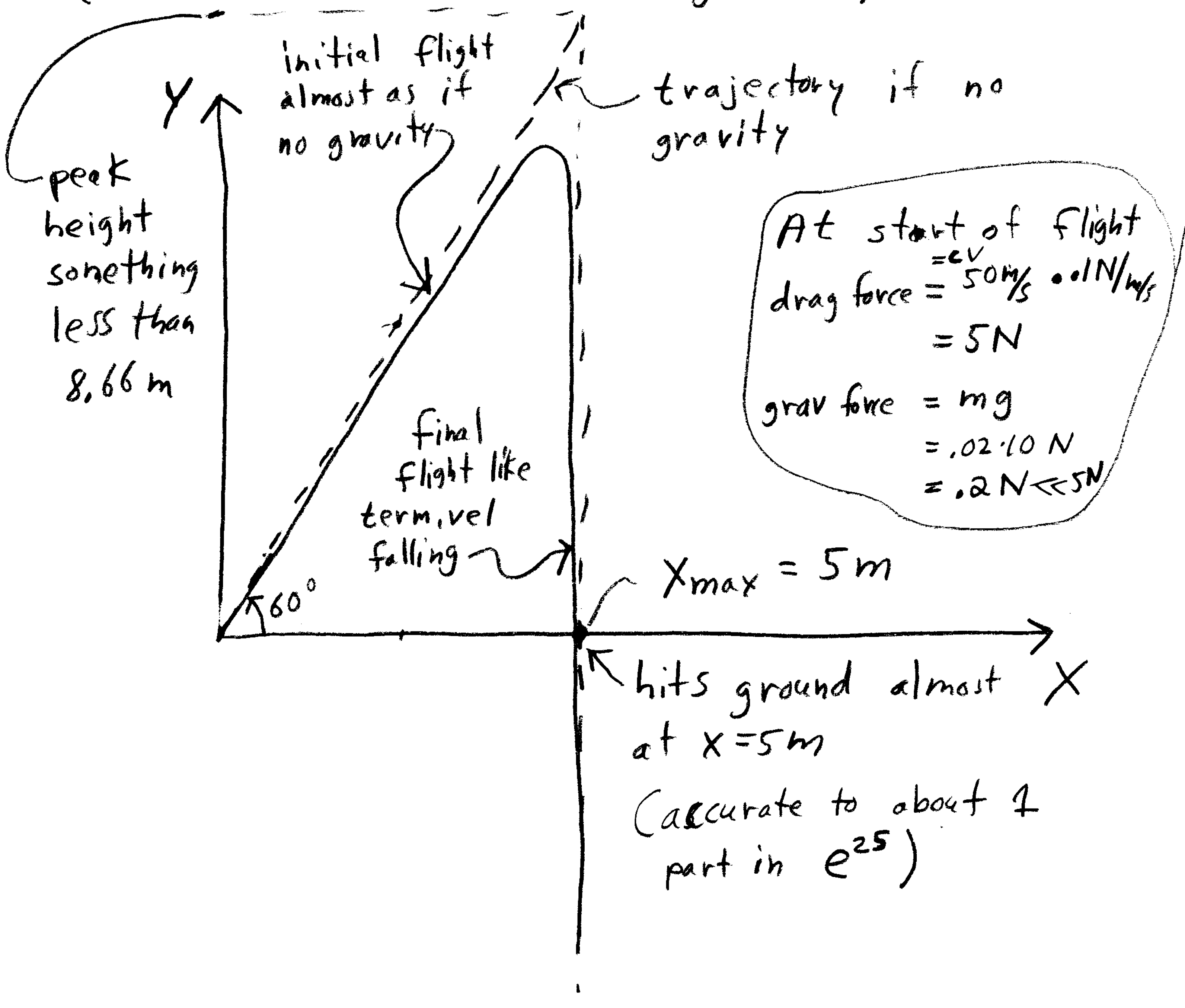
$\frac{gm}{c} = 2\,m/s$

(b)

$$\boxed{\underline{r}(t) = 5m\left(1 - e^{-5t/s}\right)\underline{i} + \left[\left(\frac{5\sqrt{3}}{2} - .4\right)\left(1 - e^{-5t/s}\right)m - 2(m/s)\,t\right]\underline{j}}$$

(prob. 3, cont'd)

c) Plot. All we need from anal. soln. is $\underbrace{\text{eqn. } 7}_{\text{i.e. eqn for } x(t)}$.
(No need for soln. of inhomog. soln.)



Some comments on ODE solns.

Solve $\dot{V}_x = -(c/m)V_x$

method 1: guess $V_x = e^{rt}$

plug in $r\cancel{e^{rt}} = -(c/m)\cancel{e^{rt}}$

$$r = -c/m$$

$$\Rightarrow V_x = c_1 e^{-(c/m)t}$$

arb. const., pick to sat. I.C.

method 2:
(Edwards & Penny 1.4)

$$\frac{dV_x}{dt} = -(c/m)V_x$$

$$\Rightarrow \frac{dV_x}{V_x} = -(c/m)dx$$

$$\Rightarrow \int \frac{dV_x}{V_x} = -\int c/m\, dt$$

$$\Rightarrow \ln V_x = -\frac{c}{m}t + c_1'$$

$$\Rightarrow V_1 = e^{-(c/m)t + c_1'}$$

$$V_x = c_1 e^{-(c/m)t} \qquad (c_1 = e^{c_1'})$$

(again)

ODEs (cont'd)

method 3: Integrating factor (Edwards & Penny 1.5)

$$\frac{dV_x}{dt} + \frac{c}{m} V_x = 0$$

$e^{c/m\, t}$ is the integrating factor.

$$\Rightarrow \quad e^{\frac{c}{m}t}\frac{dV_x}{dt} + e^{(c/m)t}\frac{c}{m}V_x = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{(c/m)t}\,V_x\right) = 0$$

$$\Rightarrow \quad e^{(c/m)t}\,V_x = C_1$$

$$\Rightarrow \quad V_x = C_1 e^{-(c/m)t}$$

(again)

Solve $\ddot{x} + c/m\,\dot{x} = 0$

method 1: guess $x = e^{rt}$

$$\Rightarrow \quad r^2 \cancel{e^{rt}} + r\frac{c}{m}\cancel{e^{rt}} = 0$$

$$\Rightarrow \quad r\left(r + \frac{c}{m}\right) = 0 \quad \Rightarrow \quad r = 0, -c/m$$

$$\Rightarrow \quad x(t) = C_1 e^{-\frac{c}{m}t} + C_2 e^{-0t}$$

$$= C_1 e^{-(c/m)t} + C_2$$

find using ICs.

This is $x(t)$ as found.

<u>**Solve**</u> $\dot{V}_y + (c/m)\, V_y = -g$

<u>method 1</u>: a) find homog. soln.

$$\dot{V}_y + (c/m) V_y = 0.$$

This is identical to prev. problem which we solved to get

$$\underbrace{V_{y_h}}_{\text{homog. soln.}} = c_1 e^{-(c/m)t}$$

b) Find <u>any</u> "particular" soln. of

$$\dot{V}_y + (c/m) Y = -g.$$

As for spring-mass problem. Easiest guess is a constant. In this case you can get this physically by thinking of falling at terminal velocity.

guess $V_{y_p} = c_2$

$$\cancelto{0}{\dot{c}_2} + (c/m)\, c_2 = g$$

$$c_2 = gm/c$$

Solution is

$$V_y = V_{yh} + V_{yp}$$

$$= c_1 e^{-(c/m)t} + gm/c$$

pick to match I.C.

method 2; (E 8P 1.4) (seperable eqn.)

$$\frac{dV_y}{dt} + \frac{c}{m} V_y = -g$$

$$\Rightarrow \frac{dV_y}{dt} = -\frac{c}{m} V_y - g$$

$$\Rightarrow \frac{dV_y}{V_y + \frac{gm}{c}} = -\frac{c}{m}\, dt$$

$$\Rightarrow \int \frac{dV_y}{V_y - \frac{gm}{c}} = -\int \frac{c}{m}\, dt$$

$$\Rightarrow \ln\left(V_y + \frac{gm}{c}\right) = -\frac{c}{m} t + c_1''$$

$$\Rightarrow V_y + \frac{gm}{c} = e^{-\frac{c}{m}t + c_1'}$$

$$\Rightarrow V_y = c_1 e^{-(c/m)t} - gm/c$$

$c_1 = e^{c_1'}$, pick to sat. I.C.

(again)

method 3; (E 8P 1.5) linear ODE using integrating factor

$$\frac{dV_y}{dt} + \frac{c}{m} V_y = -g$$

define $g(t) = e^{(c/m)t}$

integrating factor

mult. through by g

$$\frac{dV_y}{dt} e^{(c/m)t} + \frac{c}{m} V_y e^{(c/m)t} = -g e^{(c/m)t}$$

$$\frac{d}{dt}\left(v_y e^{(c/m)t}\right) = -g e^{(c/m)t}$$

$$\Rightarrow \quad d\left[v_y e^{(c/m)t}\right] = -g e^{(c/m)t} dt$$

$$\Rightarrow \quad \int d\left(v_y e^{(c/m)t}\right) = -\int g e^{c/m\, t} dt$$

$$\Rightarrow \quad v_y e^{(c/m)t} = -\frac{gm}{c} e^{(c/m)t} + c_1$$

$$\Rightarrow \quad v_y = \frac{-gm}{c} + c_1 e^{-(c/m)t}$$

(again)

pick to sat. I.C.

SOLUTIONS

Your Name: ____________________

TA name and section time: __________

T&AM 203 Final Exam

Friday May 19, 2006, 2-4:30 PM

Draft May 15, 2006

5 problems, 25^+ points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A} + \underline{\mathbf{v}}_{rel}$$

$$\underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\boldsymbol{\omega}} \times (\underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A}) + \underline{\dot{\boldsymbol{\omega}}} \times \underline{\mathbf{r}}_{B/A} + 2\underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}}_{rel} + \underline{\mathbf{a}}_{rel}$$

$$\frac{1}{\rho} = \frac{y''}{(1+y'^2)^{3/2}}$$

Problem	0:	/-125
Problem	1:	/25
Problem	2:	/25
Problem	3:	/25
Problem	4:	/25
Problem	5:	/25
TOTAL:		

0) -125 pt In order to *not* get -125 points you need to sign your name below. If you do not sign your name you get *negative* 125 points. Whether or not you sign, any violations of the pledge below will be fully prosecuted under the Cornell policies concerning academic integrity. I (Andy) have prosecuted many such cases and no student I have accused has ever been found innocent or had a decision reversed on appeal.

Pledge: I realize that the regularly scheduled final might be identical to this test. No student taking the late final should have any more foreknowledge of the test than have students taking this early final now. Between now and 3 PM Friday May 19 I promise not to discuss any aspect of this test with anyone, or within earshot of anyone, with the exception of TAM 203 staff and other TAM 203 students who also took this early test (assuming I know and recognize them and saw them taking this test). That is, there should be *no possible means* by which any student in TAM 203 who is not taking the test with me now could learn by any direct or indirect way from me (for example though a third person overhearing me or reading my email or through my parents talking to their friends etc) anything about this test. For example, and these are only examples, no-one will get in any direct or indirect way from me the answers to any of these questions:

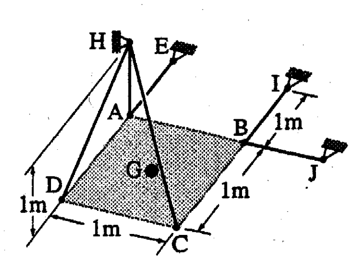
- Did I think the test was easy or hard, fair or unfair?
- Was there a Matlab question on the test?
- Did the test have a statics problem, a problem from the lab, a problem involving pulleys, etc?
- How many questions were on the test?
- Were any formulas given on the test?
- Did the test include material from the final homework?
- How well did I think I did on the test?

If anyone asks me any such questions or tries to get such information from me I will say that I am not allowed to even hint at the answers. If pressed further I will tell the person asking that such pressure is a violation of the rules of academic integrity. If pressed further I will tell 203 staff who was asking. If I know of any violations of this pledge I will promptly inform TAM 203 staff. By signing below I indicate that I understand and agree to the text above on this page.

Signed ______________________________
(sign clearly and legibly)

---

1) (25 pt) A uniform square horizontal rigid plate ABCD has weight mg and is held in place by 6 negligible-mass rods. You need not write long vector formulas if you can confidently justify your answers without them. Find the tension in bar HD.



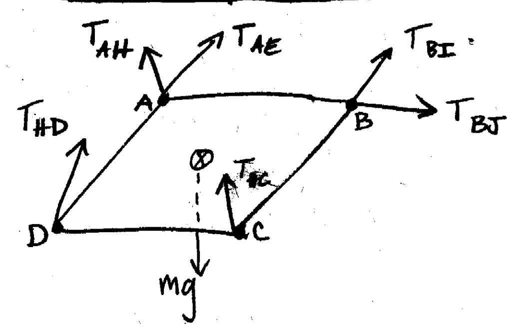
Taking moments about AC, the only force that doesn't pass through it is the force due to tension in bar HD.

Since for static equilibrium

$$\sum \underline{M}_{AC} = 0$$

$\Rightarrow$ The tension HD must be 0.

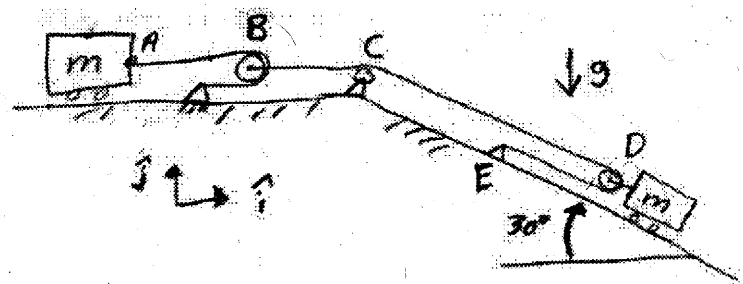
FBD of plate ABCD



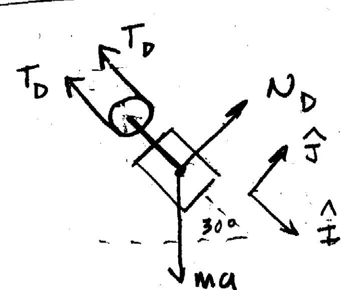
2) (25 pt) Make the usual assumptions about pulleys and the like.

a) (20 pt) In terms of some or all of m and g find $\mathbf{a}_D \cdot \hat{\mathbf{j}}$. That is, find the y component of the acceleration of point D.

b) (5 pt) Roughly speaking can you explain the answer to part (a). Hint: the answer to part (a) is a number multiplied by a symbol or symbols. That number is close to $2^{\pm n}$ where n is an integer. For example, if the answer to part (a) was $9m/g$ (it isn't) then we could say that answer was close to $2^3 m/g$ and we would have $n = 3$. Use words and/or diagrams to rationalize the appropriate value of n from part (a). That is, somehow the mechanics has in it, approximately, n factors of two. Can you identify each one of these factors. [A very good answer to this part can make up for lost points in part (a)].

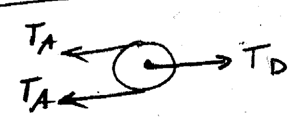


FBD for mass D



$(\Sigma F)\cdot\hat{I} = \boxed{mg\cos 60^\circ - 2T_D = m\ddot{X}_D}$

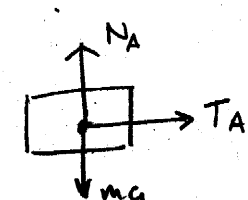
FBD for "massless" drum B



$\Sigma F = -2T_A + T_D = 0$

$\Rightarrow \boxed{T_D = 2T_A}$

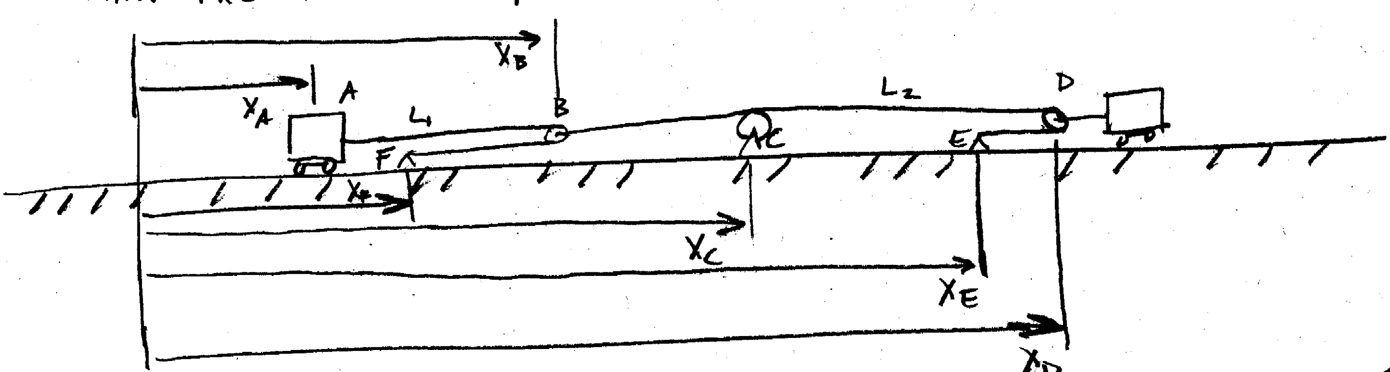
FBD for mass A



$(\Sigma F)\cdot\hat{i} = \boxed{T_A = m\ddot{X}_A}$

① $\boxed{mg\cos 60^\circ - 4m\ddot{X}_A = m\ddot{X}_D}$

To find the relationship between $\ddot{X}_A$ & $\ddot{X}_D$



$L_2 = X_D - X_E + X_D - X_C + X_C - X_B \Rightarrow \boxed{2\ddot{X}_D = \ddot{X}_B}$ ②

$L_1 = X_B - X_F + X_B - X_A \Rightarrow \boxed{2\ddot{X}_B = \ddot{X}_A} \longrightarrow \boxed{\ddot{X}_D = \frac{1}{4}\ddot{X}_A}$

Combining ① & ②

$$mg\cos 60^\circ - 4m(4\ddot{x}_D) = m\ddot{x}_D \implies \boxed{\ddot{x}_D = \frac{1}{17} g\cos 60^\circ}$$

The $\hat{j}$-component of $\underline{a_D}$ is then

$$(\underline{a_D})\cdot\hat{j} = -\frac{1}{17} g\cos 60^\circ \sin 30^\circ = -\frac{g}{17}\sin^2 30^\circ$$

$$= -\frac{g}{17}\left(\frac{1}{2}\right)^2 = \boxed{-\frac{1}{68}g \approx -2^{-6}g}$$

Thus we find that $\boxed{n=-6}$. This number can be explained as follows:

From the kinematics we found that mass $\dot{A}$ has 4 times the acceleration of mass D $\implies$ mass A has 16 times the energy & thus dominates the inertia of the 1-d.o.f. system. So basically we have the weight at D (w/ negligible mass) pulling on mass A.

1. Only half the weight at D is carried by the rope because of the slope, $\sin 30^\circ = 1/2$.
2. Only half of this weight is carried by rope BD because of the pulley at D.
3. Only half the tension of rope BD is carried by rope AB because of pulley B.

So far we have $\ddot{X}_A = T_A/m = \left(\frac{mg}{8}\right)/m = \frac{g}{8}$ from the force analysis. From kinematics we find:

4. Point B has half the acceleration of mass A because of pulley B.

5. Point D has only half the acceleration of B because of the pulley at D.

6. The $\hat{j}$-component of the acceleration of mass D is half the along-slope acceleration because of the slope, $\sin 30° = 1/2$.

Thus the acceleration of mass D is $\approx 2^6$ times smaller than it would be in free flight. It's actually smaller than that since we neglected the inertia of mass D completely, and that would slow the system down more.

3) (25 pt) A person with mass m stands still at the back of a stationary boat with mass M. Then at $t = 0$ she walks the length L of the boat over time T according to the equation

$$x_{p/b} = \frac{L(1 - \cos(\pi t/T))}{2}$$

where $x_{p/b}$ is how far she has moved relative to the boat. Then for $t > T$ she stands still in the front of the boat.

a) (5 pt) Make a plot of $x_{p/b}$ vs t (put t on the "x" axis). Label key points on the "x" and "y" axes in terms of m, M, T and L.

b) (10 pt) Make a plot of x_b vs t, labeling key points on the axis as for part (a). x_b is the absolute position of the boat relative to a fixed reference frame. Assume the boat moves frictionlessly on the water.

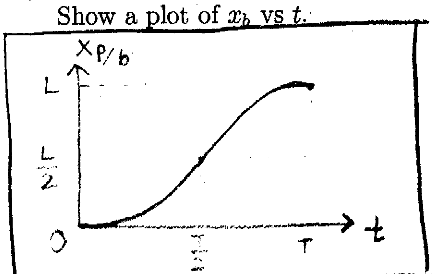
c) (5 pt) For parts (c & d) assume that the boat *has* friction with the water. The drag force is proportional to the boat speed:

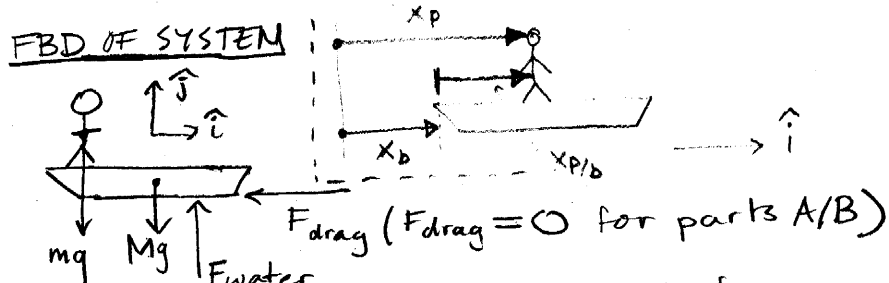
$$F_{drag} = cv_b.$$

Eventually, as $t \to \infty$, the boat speed tends to zero and the system comes to rest. What is the net impulse of the force of the water on the boat? That is, evaluate $\int_0^\infty F_{drag}\,dt$ (using basic mechanics principles this is a short calculation).

d) (5 pt) What is $x_b(\infty)$? That is, after all has come to rest how far will the boat have moved? Show a plot of x_b vs t.

a)





b) Taking man & boat as system, there is no external force on the system ⇒ Linear momentum of system is constant

$\Rightarrow \quad \vec{P}(t<0) = \vec{P}(0<t<T) = \vec{P}(t>T)$

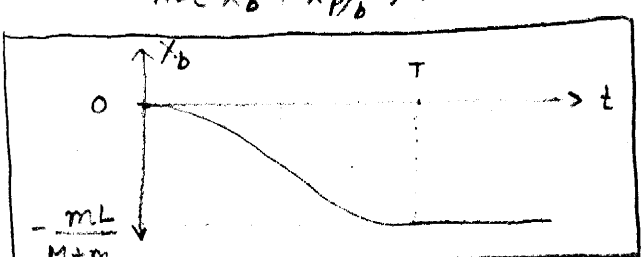
$\Rightarrow \quad 0 = m\vec{v}_p + M\vec{v}_b = 0$

note $\vec{v}_p = \vec{v}_b + \vec{v}_{p/b}$

$\Rightarrow \vec{x}_p = \vec{x}_b + \vec{x}_{p/b}$

integrating $\quad m\vec{x}_p + M\vec{x}_b = 0$

$m(\vec{x}_b + \vec{x}_{p/b}) + M\vec{x}_b = 0 \quad \Rightarrow \quad \boxed{x_b = -\frac{m}{M+m} x_{p/b}} \quad t \leq T$



for $t > T$ the system comes to rest.

c) $\int_0^\infty F_{drag}\,dt = \Delta \vec{P}_{system} = (m+M)(\overset{0}{V_{after}} - \overset{0}{V_{before}}) = \boxed{0}$

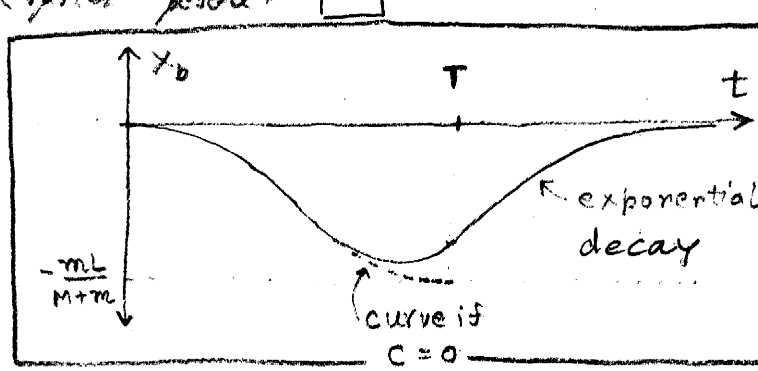
d) $X_b(\infty) = X_b(\infty) - X_b(0)$

$= \int_0^\infty V_b\,dt$

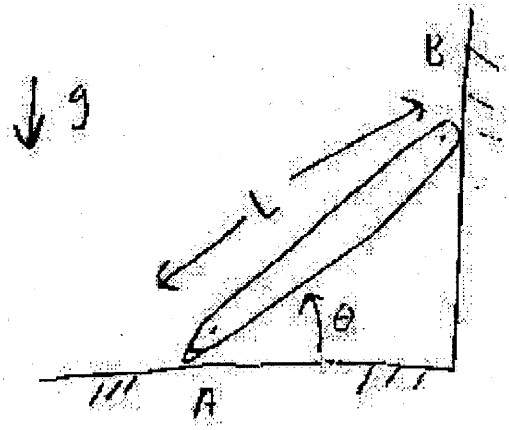
$= \frac{1}{c}\int_0^\infty cV_b\,dt$

$= \frac{1}{c}\int_0^\infty F_{drag}\,dt = \boxed{0}$

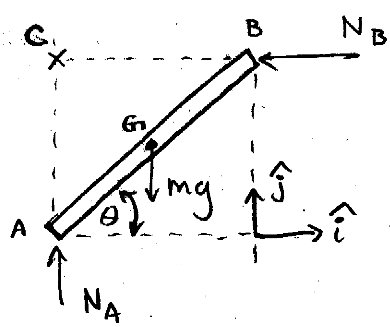
See part (c) above



4) **(25 pt)** A uniform ladder with mass m and length L slides on a slippery floor and against a slippery wall. It is released from rest at angle θ. Immediately after release find the angular acceleration of the rod. Answer in terms of some or all of θ, g, L, m, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. If you think you need I_G, I_A or I_B you can recall them or derive them or, for less credit, leave them in your final answer.



FBD of ladder AB



LMB

$$\Sigma \underline{F} = \boxed{-N_B\,\hat{i} + (N_A - mg)\,\hat{j} = m\underline{a}_G}$$

AMB about point C

$$\Sigma \underline{M} = \boxed{\underline{r}_{G/C} \times -mg\,\hat{j}}$$
$$\boxed{= I_G\,\ddot{\theta}\,\hat{k} + \underline{r}_{G/C} \times m\underline{a}_G}$$

To find $\underline{a}_G$ we use the rigidity of AB to write

$$\boxed{\underline{a}_G = \underline{a}_A + \underline{a}_{G/A} = \underline{a}_A + \underline{\alpha} \times \underline{r}_{G/A} + \underline{\omega} \times \underline{\omega} \times \underline{r}_{G/A}}$$

To find $\underline{a}_A$ we use rigidity & the fact that ends A & B are constrained to move along their respective walls.

$$\underline{a}_B = a_B\,\hat{j} = \underline{a}_A + \underline{a}_{B/A} = a_A\,\hat{i} + \underline{\alpha} \times \underline{r}_{B/A} + \underline{\omega} \times \underline{\omega} \times \underline{r}_{B/A}$$

$$\underline{\alpha} \times \underline{r}_{B/A} = \ddot{\theta}\,\hat{k} \times (L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}) = -L\ddot{\theta}\sin\theta\,\hat{i} + L\ddot{\theta}\cos\theta\,\hat{j}$$

$$\underline{\omega} \times \underline{\omega} \times \underline{r}_{B/A} = \dot{\theta}\hat{k} \times \dot{\theta}\hat{k} \times (L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j})$$
$$= -L\dot{\theta}^2\cos\theta\,\hat{i} - L\dot{\theta}^2\sin\theta\,\hat{j}$$

$$\{\ \}\cdot\hat{i} \Rightarrow 0 = a_A - L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta$$

$$\Rightarrow \boxed{a_A = L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta}$$

Thus we have

$$\boxed{\underline{a}_G = (L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta)\hat{\imath} + \left(-\frac{L\ddot{\theta}}{2}\sin\theta\,\hat{\imath} + \frac{L\ddot{\theta}}{2}\cos\theta\,\hat{\jmath}\right) + \left(-\frac{L\dot{\theta}^2}{2}\cos\theta\,\hat{\imath} - \frac{L\dot{\theta}^2}{2}\sin\theta\,\hat{\jmath}\right)}$$

$$\boxed{= \left(\frac{L\ddot{\theta}}{2}\sin\theta + \frac{L\dot{\theta}^2}{2}\cos\theta\right)\hat{\imath} + \left(\frac{L\ddot{\theta}}{2}\cos\theta - \frac{L\dot{\theta}^2}{2}\sin\theta\right)\hat{\jmath}}$$

Plugging this back into AMB we get

$$-\frac{mgL}{2}\cos\theta\,\hat{k} = \frac{1}{12}mL^2\hat{k} + \left(\frac{L}{2}\cos\theta\,\hat{\imath} - \frac{L}{2}\sin\theta\,\hat{\jmath}\right) \times m\underline{a}_G$$

$$\{\ \}\cdot\hat{k} \Rightarrow -\frac{mgL}{2}\cos\theta = \frac{1}{12}mL^2\ddot{\theta} + \frac{1}{4}mL^2\ddot{\theta}$$

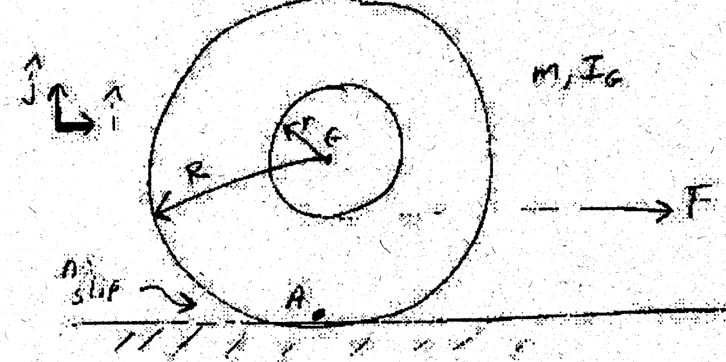
$$\frac{1}{3}mL^2\ddot{\theta}$$

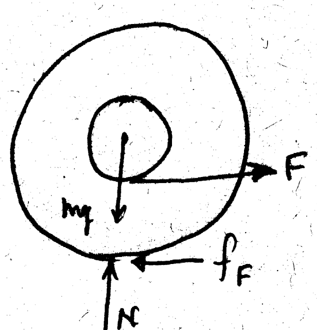
NOTE: The terms with $\dot{\theta}^2$ cancel out after the cross-product.

$$\Rightarrow \ddot{\theta} = -\frac{3g}{2L}\cos\theta \quad \Rightarrow \quad \boxed{\underline{\alpha} = -\frac{3g}{2L}\cos\theta\,\hat{k}}$$

NOTE #1 - An alternative method would be to use conservation of energy. Set $T + V = E$ (constant) and differentiate w.r.t. time.

5) (25 pt) A spool (like the movie *Heat Treatment of Aluminum* shown in lecture), with outer radius R rolls without slip on a flat horizontal surface. The film is at a radius r and is being pulled with a horizontal force F. At the moment in question the velocity of the middle of the spool is $v\hat{\mathbf{i}}$. The mass of the spool is m and its moment of inertia about its center of mass is I_G. What is the acceleration of point A on the spool which is, at the instant in question, touching the ground. Answer in terms of some or all of m, I_G, r, R, g, v and F.





$\underline{V}_A = \underline{V}_A$ no slip

$\Rightarrow \overset{0}{\cancel{\underline{V}_{Ground}}} + \overset{0}{\cancel{\underline{V}_{A/Ground}}} = \underline{V}_G + \underline{V}_{A/G}$

$\Rightarrow \underline{0} = \underline{V}_G + \underline{\omega} \times \underline{r}_{A/G}$

$\Rightarrow \underline{0} = v\hat{i} + \omega\hat{k} \times (-R\hat{j})$

$\Rightarrow \underline{0} = v\hat{i} + \omega R\hat{i}$

$\{\ \}\cdot\hat{i} \Rightarrow \omega = -\frac{v}{R} = \dot{\theta}$

$\underline{a}_G = -R\ddot{\theta}\,\hat{i}$

$\underline{a}_A = \underline{a}_G + \underbrace{(\ddot{\theta}\hat{k})}_{\alpha} \times \underbrace{\underline{r}_{A/G}}_{-R\hat{j}} + \underbrace{\underline{\omega} \times (\underline{\omega} \times \underline{r}_{A/G})}_{-\omega^2 \underline{r}_{A/G}}$

$\Rightarrow \underline{a}_A = \cancel{-R\ddot{\theta}\hat{i}} + \cancel{R\ddot{\theta}\hat{i}} + \omega^2 R\hat{j}$

$\Rightarrow \boxed{\underline{a}_A = \frac{v^2}{R}\hat{j}}$

Hence the answer doesn't depend on using momentum balance.

"Solutions"

Your Name: ANDY RUINA

TA name and section time: —

T&AM 203 Prelim 1

Tuesday February 28, 2006

Draft February 26, 2006

3 problems, 25 points each, and 90^{+} minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot\theta_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	1:	/25
Problem	2:	/25
Problem	3:	/25

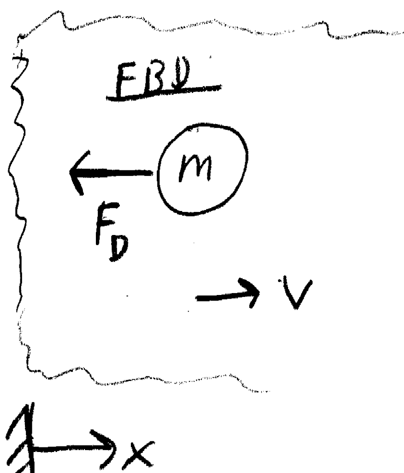
1) **(25 pt)** A shot pellet mass m hits a bird or person's skin at speed v_0. Neglect gravity. Assume m, v_0 and c (below) are given. Assume one dimensional motion in, say, the x direction.

a) (15 points) Assume that the force of the flesh on the pellet is $-c\mathbf{v}$, that is the drag force resists motion and is proportional to the speed. How far does the pellet go before it comes to rest? (Please re-read the rules at the front of the exam.) $\Rightarrow F_D = cV$

b) (7 points) Assume that the force of the flesh on the pellet is $-c|\mathbf{v}|\mathbf{v}$, that is the drag force resists motion and is proportional to the speed squared. How far does the pellet go before it comes to rest (the answer is perhaps surprising). $\Rightarrow F_D = cV^2$

(assume $V \geq 0$)

c) (3 points) Given that quadratic drag (b above) is a much more accurate model than linear drag (a above) for fast moving things in air, water and flesh, why does the calculation in b give a patently ridiculous answer? How could you change the calculation to make it more accurate? (It might be possible to get this problem right without getting b right.)



$$V = \dot{x}$$

$$\sum F = m\underline{a} \Rightarrow \boxed{m\ddot{x} = -F_D} \quad (1)$$

a) $m\ddot{x} = -cV$

$$\Rightarrow \ddot{x} + \tfrac{c}{m}\dot{x} = 0$$

$$\Rightarrow \dot{V} + \tfrac{c}{m}V = 0$$

$$\Rightarrow V = V_0 e^{-(c/m)t}$$

$$\Rightarrow x = -V_0 \tfrac{m}{c} e^{-(c/m)t} + C_1$$

$$x(0) = 0 \Rightarrow C_1 = \frac{mV_0}{c}$$

$$\Rightarrow x = \frac{mV_0}{c}\left(1 - e^{-(c/m)t}\right)$$

$$\boxed{\underset{t\to\infty}{x =} \quad mV_0/c} \quad (a)$$

b) (1) $\Rightarrow m\dot{V} = -cV^2$

$$\Rightarrow \frac{dV}{V^2} = \frac{-c}{m}dt$$

$$-1/V = -\tfrac{c}{m}t + C_1$$

$$V(0) = V_0 \Rightarrow C_1 = -1/V_0$$

$$\Rightarrow V = \frac{1}{\frac{1}{V_0} + \frac{c}{m}t}$$

$$\Rightarrow dx = \frac{V_0\,dt}{1 + cV_0t/m}$$

$$x = \tfrac{m}{c}\ln(1 + cV_0t/m) + C_1$$

$$x(0) = 0 \Rightarrow \quad C_1 = 0$$

$$\Rightarrow x = m/c \ln(1 + cV_0t/m)$$

$$\boxed{\underset{t\to\infty}{x =} \quad \infty} \quad (b)!$$

c) because eventually the pellet goes slowly. other neglected forces then dominate the quadratic drag. Some fixes

i) $F_D = C_1V^2 + C_2V$

or ii) $F_D = C_1V^2 + C_2$

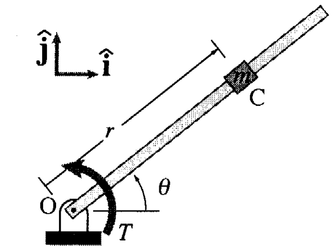
Both of these slow the pellet down enough at low speeds to predict finite (and more accurate) penetration

2) (25 pt) A torque T varies in time as it must in order to rotate a rigid rod at constant rate of $\dot{\theta} = 2$ rad/s. A bead slides on the rod. At the start $t = 0$, $\theta = 0$, $r = 1$m and $\dot{r} = 0$. Neglect gravity and friction.

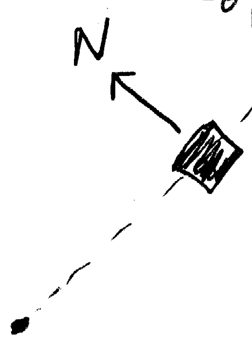
a) (15 points) What is the radius when $\theta = 2\pi$?

b) (5 points) What is the speed $|\underline{\mathbf{v}}|$ when $\theta = 2\pi$?

c) (5 points) When $\theta = 9\pi/4$ what is the direction of $\underline{\mathbf{v}}$. A *very* simple answer is desired which is not exact, but is accurate to within a degree or less.



FBD



$\underline{F} = m\underline{a}$

$$\Rightarrow \{N\hat{e}_\theta = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\cancel{\ddot{\theta}}^{\,0} + 2\dot{r}\dot{\theta})\hat{e}_\theta\}$$

$$\{\ \}\cdot\hat{e}_r \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = 0$$

a constant,

$$\Rightarrow \quad r = c_1 e^{\dot{\theta}t} + c_2 e^{-\dot{\theta}t}$$

$$\dot{r} = c_1\dot{\theta}e^{\dot{\theta}t} - c_2\dot{\theta}e^{-\dot{\theta}t}$$

note: $\dot{\theta}t = \theta$ const.

$\dot{r}(0) = 0 \Rightarrow c_1 = c_2 \Rightarrow$ $\quad r = r_0(e^{\theta} + e^{-\theta})/2$

$r(0) = r_0 \Rightarrow c_1 + c_2 = r_0 \Rightarrow$ $\quad \dot{r} = r_0\dot{\theta}(e^{\theta} - e^{-\theta})/2$

a) $r(2\pi) = \underbrace{r_0}_{1m}(e^{2\pi} + e^{-2\pi})/2 = \boxed{[(e^{2\pi} + e^{-2\pi})/2\,]\,m \approx \frac{e^{2\pi}}{2}\ m}$ (a)

b) $\underline{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = \overbrace{r_0\dot{\theta}\frac{(e^{\theta} - e^{-\theta})}{2}}^{V_r}\hat{e}_r + \overbrace{r_0\dot{\theta}\frac{(e^{\theta} + e^{-\theta})}{2}}^{V_\theta}\hat{e}_\theta$

$$|\underline{V}| = \sqrt{V_r^2 + V_\theta^2} = \frac{r_0\dot{\theta}}{2}\sqrt{(e^{2\theta} - 2 + e^{-2\theta}) + (e^{2\theta} + 2 + e^{-2\theta})}$$

$$= \frac{r_0\dot{\theta}}{2}\sqrt{2(e^{2\theta} + e^{-2\theta})}$$

$$= \frac{2}{2}\sqrt{2e^{4\pi} + 2e^{-4\pi}}\ m/s = \boxed{2\sqrt{\cosh(4\pi)}\ m/s} \quad (b)$$

c) for large θ, $e^{-\theta}/e^{\theta} \Rightarrow \underline{V} \approx \frac{r_0\dot{\theta}}{2}e^{\theta}(\hat{e}_r + \hat{e}_\theta)$ (c)

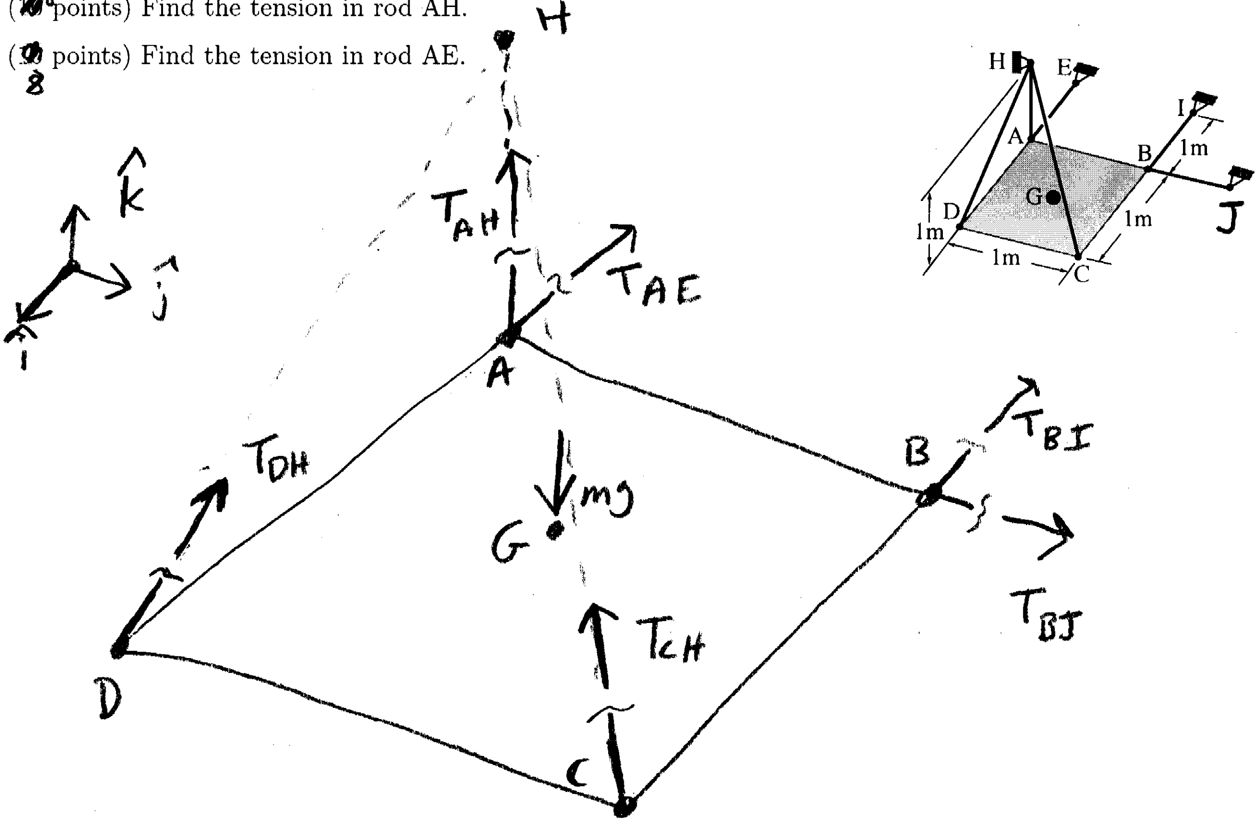
at $\theta = 9\pi/4$ "=" $\pi/4$ $\quad \hat{e}_r + \hat{e}_\theta = \uparrow$ ($\hat{e}_\theta$, $\hat{e}_r$) $= \boxed{\underline{V} \text{ in } \hat{\jmath} \text{ direction}}$ (approx)

3) (25 pt) A uniform square horizontal rigid plate ABCD has weight mg and is held in place by 6 negligible-mass rods. You need not write long vector formulas if you can confidently justify your answers without them. 9

a) (~~10~~ points) Use moment balance about axis AH to find the tension in rod BI.

b) (~~10~~ 8 points) Find the tension in rod AH.

c) (~~5~~ points) Find the tension in rod AE. 8



a) All forces have lines of action $\parallel$ to or intersecting AH except T_{BI} $\Rightarrow$ $T_{BI}\cdot(1m)=0 \Rightarrow \boxed{T_{BI}=0}$ (a)

b) $\sum M_{CD}=0$: only T_{AH} and mg contribute. T_{AH} has twice the lever arm $\Rightarrow$ $\boxed{T_{AH}=mg/2}$ (b)

c) $\sum M_{BH}=0$: only mg & T_{AE} contribute

$$\{\sum \underline{M}_{/B}\}\cdot \underline{r}_{BH}=0 \Rightarrow 0=\{\underbrace{\underline{r}_{BG}}_{\frac{\hat i-\hat j}{2}m}\times -mg\hat k+\underbrace{\underline{r}_{BA}}_{-\hat j m}\times T_{AE}(-\hat i)\}\cdot(-\hat j+\hat k)$$

$$\Rightarrow 0=\left[mg\left(\frac{\hat j-\hat i}{2}\right)+T_{AE}\hat k\right]\cdot(-\hat j+\hat k)=\frac{mg}{2}+T_{AE} \Rightarrow \boxed{T_{AE}=-\frac{mg}{2}}$$ (c)

Your Name: STAFF

"SOLUTIONS"

TA name and section time: ____________

T&AM 203 Prelim 2

Tuesday April 18, 2006, 2006

Draft April 18, 2006

3 problems, 25 points each, and 90^{+} minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems ***poorly defined***;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot\theta_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem	4:	/25
Problem	5:	/25
Problem	6:	/25

$$\underline{\mathbf{v}}_B = \underline{\mathbf{v}}_A + \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A} + \underline{\mathbf{v}}_{rel} \qquad \underline{\mathbf{a}}_B = \underline{\mathbf{a}}_A + \underline{\boldsymbol{\omega}} \times (\underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}}_{B/A}) + \dot{\underline{\boldsymbol{\omega}}} \times \underline{\mathbf{r}}_{B/A} + 2\underline{\boldsymbol{\omega}} \times \underline{\mathbf{v}}_{rel} + \underline{\mathbf{a}}_{rel}$$

$$\frac{1}{\rho} = \frac{y''}{(1+y'^2)^{3/2}}$$

ρ = radius of curvature

1) (29 pt) A car with mass m_1 moving at v_1 crashes into the rear of a stationary car with mass m_2 and sticks to it. The duration of the crash is Δt after which the cars move together. Give all answers in terms of m_1, m_2, v_1 and Δt. [Later work may not be graded if it depends on incorrect earlier work].

a) (15 points) Please re-read the rules at the front of the exam. How fast are the cars moving after the crash?

b) (5 points) What is F (the force of car 2 on car 1) during the crash, assuming F is constant in this time interval.

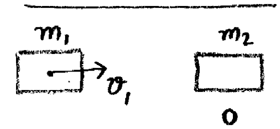
c) (3 points) Given v_1 and m_1 consider a range of cars that might be hit by car 1. For what mass m_2 car is its acceleration during the crash the biggest compared to all other possible cars? (Answers of the form $m_2 \to 0$ or $m_2 \to \infty$ are acceptable.)

d) (3 points) Like part (c), for what mass m_2 car is its final kinetic energy maximum (that is, more than the kinetic energy of any other car with a different m_2)?

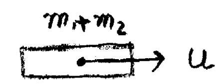
e) (3 points) Like parts (c) and (d) for what mass m_2 car is the total crash energy dissipation maximum (that is, more dissipation than for all other m_2)?

a) Taking m_1, m_2 as system

before crash

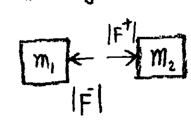


After crash



momentum before crash = momentum after crash

during crash



$$\Rightarrow \quad m_1\upsilon_1 + 0 = (m_1+m_2)u \quad \Rightarrow \quad \boxed{u = \frac{m_1\upsilon_1}{m_1+m_2}}$$

b) For mass m_1

$$I = F\Delta t = \Delta P = m_1 u - m_1\upsilon_1 = \frac{m_1^2\upsilon_1}{m_1+m_2} - m_1 V_1 = -\frac{m_1 m_2 V_1}{m_1+m_2}$$

$$\boxed{F^- = -\frac{m_1 m_2 V_1}{(m_1+m_2)\Delta t}}$$

$\because |F^-| = |F^+|$

c) acceleration of m_2 during crash $= \dfrac{F^+}{m_2} = \dfrac{m_1 V_1}{(m_1+m_2)\Delta t}$ this is max when m_1+m_2 is min

$\Rightarrow \boxed{m_2 \to 0}$

d) Final KE of $m_2 = \dfrac{m_2 u^2}{2} = \dfrac{1}{2} m_2 \left(\dfrac{m_1\upsilon_1}{m_1+m_2}\right)^2 = \dfrac{m_1^2\upsilon_1^2}{2}\left[\dfrac{m_2}{(m_1+m_2)^2}\right]$

for max KE $\dfrac{d(KE)}{dm_2} = 0 \quad \Rightarrow \quad (m_1+m_2)^2 - 2(m_1+m_2) = 0$

$\Rightarrow \quad (m_1+m_2)(m_2-m_1) = 0$

$\Rightarrow \quad \boxed{m_2 = m_1}$

e) Total crash energy dissipation $= \text{Initial}_{KE} - \text{Final}_{KE} = \frac{1}{2}m_1 v_1^2 - \frac{1}{2}(m_1+m_2)u^2$

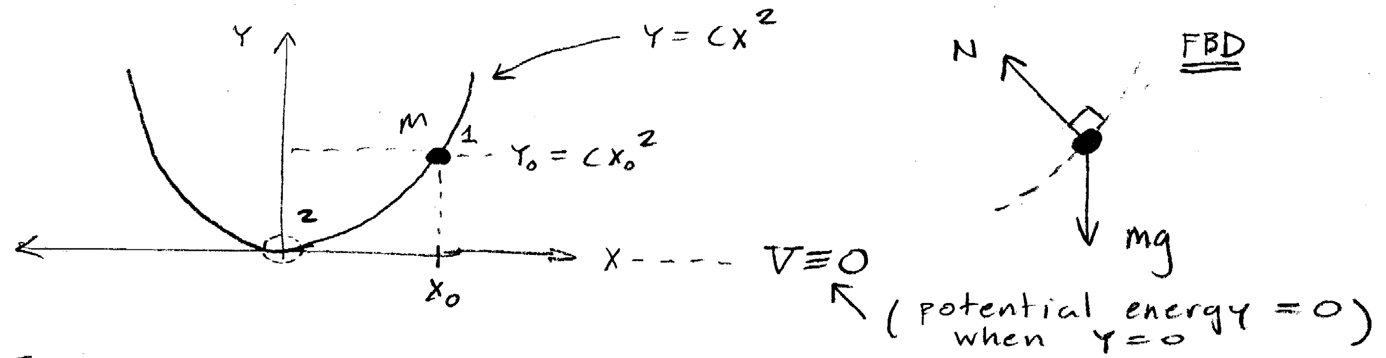
$= \frac{1}{2}m_1\upsilon_1^2 - \frac{1}{2}(m_1+m_2)\left(\dfrac{m_1\upsilon_1}{m_1+m_2}\right)^2$

for max dissipation $\dfrac{1}{2}\dfrac{m_1^2\upsilon_1^2}{m_1+m_2}$ is min $\Rightarrow$ m_1+m_2 is max

$\Rightarrow \boxed{m_2 \to \infty}$

2) (27 pt) A particle with mass m slides with no friction in a parabolic trough that is described with the equation $y = cx^2$. Equivalently you could think of a bead on a wire. Gravity g points in the negative y direction. The bead is released from rest at $x = x_0$. Find the force of the trough/wire on the mass/bead when it reaches $x = y = 0$. Answer in terms of some or all of x_0, c, g, m, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.

The problem is set-up as

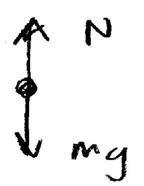


There is NO FRICTION. Use CONSERVATION OF ENERGY.

$$T_1 + V_1 = T_2 + V_2 \Rightarrow 0 + \cancel{m}g(cx_0^2) = \frac{1}{2}\cancel{m}V_2^2$$

$(x,y) = (x_0, cx_0^2)$ $\quad$ $(x,y) = (0,0)$ $\Rightarrow$ $\boxed{V_2 = x_0\sqrt{2gc}}$

When $x = 0, y = 0$ the FBD is



$$\Rightarrow (\Sigma F)\cdot\hat{n} = \boxed{N - mg = m\frac{V_2^2}{\rho|_{x=0}}}$$

Solving for N gives

$$N = mg + m\frac{2gcx_0^2}{\rho|_{x=0}} = \boxed{mg\left(1 + \frac{2cx_0^2}{\rho|_{x=0}}\right)}$$

Finally from the equation on the front page of the test

$$\rho|_{x=0} = \left.\frac{(1+y'^2)^{3/2}}{y''}\right|_{x=0} = \frac{(1+0)^{3/2}}{2c} = \boxed{\frac{1}{2c}}$$

Thus

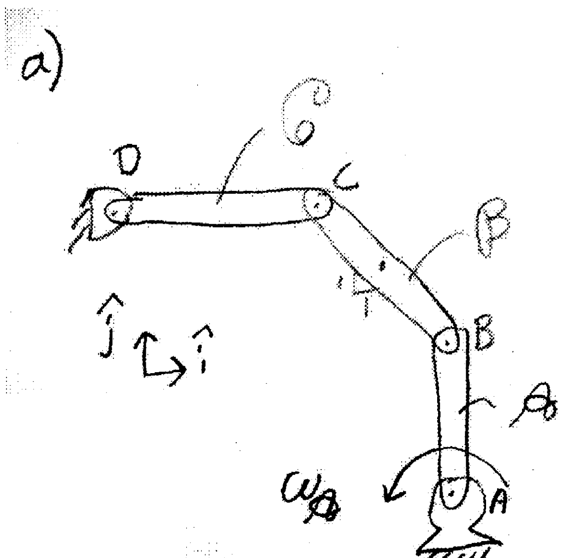
$$\boxed{N = mg\left(1 + 4c^2x_0^2\right)}$$

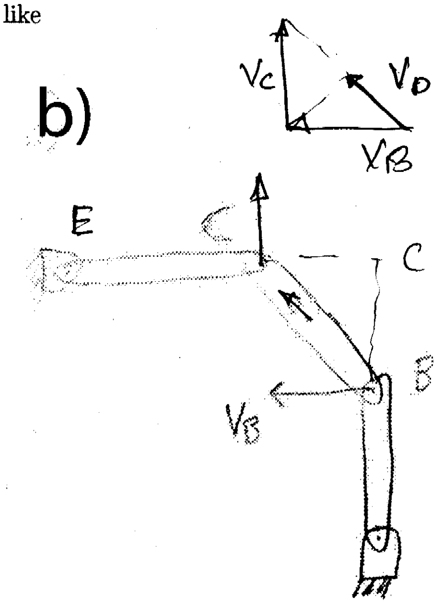
3) (27 pt) A motor drives link $\mathcal{A}$ at given constant $\omega_{\mathcal{A}}$. All three links are equal length ℓ. All questions concern the velocities and accelerations when the system is passing through the configuration shown.

a) (9 points) What is the angular velocity of link $\mathcal{B}$.

b) (8 points) On figure (b) draw in, as accurately as you can, the velocities of points B, C and D. The velocity of point B is drawn for you. This problem will be graded independently of problem (a) and your reasoning can be based on equations or any thing else.

c) (8 points) Write out, but do not solve, one or more vector equations from which you could find the angular acceleration of $\mathcal{B}$. Clearly indicate which terms are known and which unknown in your equation(s) and explain how the number of equations match the number of unknowns. Expressions like $\underline{\mathbf{r}}_{B/A}$ should be evaluated in terms of ℓ, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.





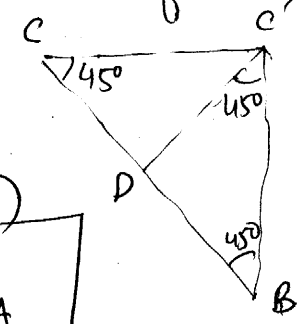
Method 2

V_B is given to be towards left.

C can only move up or down. if it moves down the rod CB needs to compress or buckle which is not possible (rigid body).

Hence V_C is upward.

Drawing perpendiculars to $\underline{V}_C$ and $\underline{V}_B$ we get c' to be the instantaneous center of rotation



$r_{cc'} = r_{c'B} = \ell/\sqrt{2}$

$V_c = \omega_B r_{cc'} = \omega_B r_{c'B} = V_B$

Part (a)

$$\Rightarrow \boxed{\omega_B = \frac{V_B}{r_{c'B}} = \frac{\omega_A \ell}{\ell/\sqrt{2}} = \sqrt{2}\,\omega_A}$$

$\boxed{v_B = \omega_A l}$ (rotation about point A)

v_D needs to be perpendicular to ~~$r_{C/D}$~~ $\vec{r}_{C/D}$
hence it is along the rod BC.

$$v_D = \omega_B r_{C/D} = \sqrt{2}\,\omega_A \cdot \frac{l}{2}$$

$$\boxed{v_D = \frac{\omega_A l}{\sqrt{2}}}$$

$$\boxed{v_C = v_B = \omega_A l}$$

~~a_E ... $= a_A + a_{C/E} + a_{B/C} + a_{B/A}$~~

$$\underline{a}_E \Rightarrow \underline{0} = \cancelto{0}{\underline{a}_A} + \underline{a}_{B/A} + \underline{a}_{C/B} + \underline{a}_{E/C}$$

$\underline{\alpha}_A \times \underline{r}_{B/A} + \underline{\omega}_A \times (\underline{\omega}_A \times \underline{r}_{B/A})$

$\underline{\alpha}_B \times \underline{r}_{B/A} + \underline{\omega}_B \times (\underline{\omega}_B \times \underline{r}_{C/B})$

to be determined

$\underline{\alpha}_C \times \underline{r}_{E/C} + \underline{\omega}_C \times (\underline{\omega}_C \times \underline{r}_{E/C})$

$\underline{r}_{B/A} = l\hat{j}$

$\underline{r}_{C/B} = \frac{l}{\sqrt{2}}(-\hat{i} + \hat{j})$

$\underline{r}_{D/C} = -l\hat{i}$

$\underline{\alpha}_A = 0$

hence we get 2 eqns for two unknowns α_B and α_C

Your TA, Section # and Section time: "SOLUTION"

Your name: ANDY RUINA

Cornell TAM 2030

No calculators, books or notes allowed.

5 Problems, 150 minutes total.

Final Exam

May 8, 2009

Directions. To ease your TA's grading and to maximize your score, please:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation.**
- √+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifiable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$\dot{\theta}_7$ `= 18`" instead of, say, "`theta7dot = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- □ Put your name on each extra sheet, fold it in, and refer to it at the relevant problem.
  Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 1: ____/25

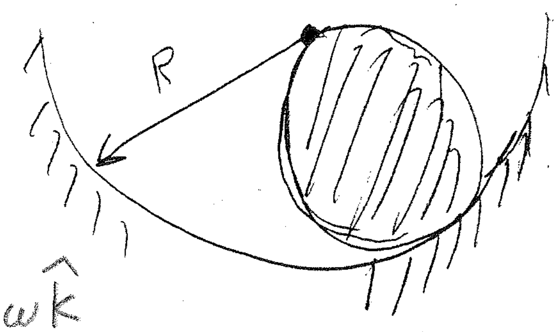
Problem 2: ____/25

Problem 3: ____/25

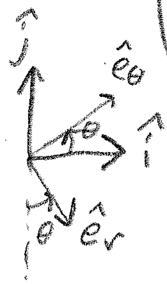
Problem 4: ____/25

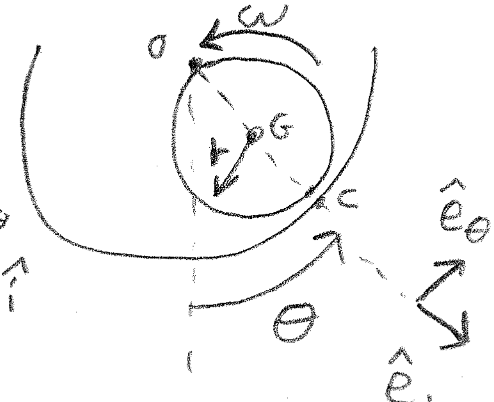
Problem 5: ____/25

1) A uniform disk with mass m and diameter R rolls back and forth in a trough with radius R. Assuming small oscillations what is the period of oscillation~~?~~ Answer in terms of some or all of R, g and m.



$\downarrow g$





$\vec{\omega} = \omega \hat{k}$

$\omega = \omega_{disk}$

$2r = R$

Kinematics

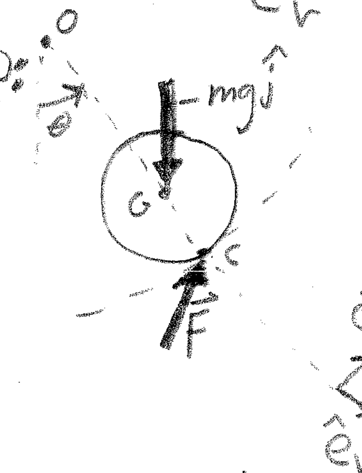
$$\vec{V}_G = \vec{V}_G$$

$$\{\dot{\theta} r \hat{e}_\theta = -\omega r \hat{e}_\theta\}$$

$$\{\}\cdot\hat{e}_\theta \Rightarrow \boxed{\dot{\theta} = -\omega} \; (1)$$

$\hat{e}_r \times \hat{j} = \sin\theta \hat{k}$

FBD:



AMB/c

$$I^G = \int \rho^2 dm = mr^2/2$$

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$\vec{r}_{G/c} \times -mg\hat{j} = \vec{r}_{G/c} \times (m\vec{a}_G) + I^G \dot{\omega} \hat{k}$$

$$\dot{\omega} = -\ddot{\theta} \quad (1)$$

$$\vec{r}_{G/c} = -r\hat{e}_r \qquad \vec{a}_G = r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

$$\Rightarrow (-r\hat{e}_r) \times (-mg\hat{j}) = (-r\hat{e}_r) \times [m(r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r)] + \tfrac{1}{2}mr^2(-\ddot{\theta}\hat{k})$$

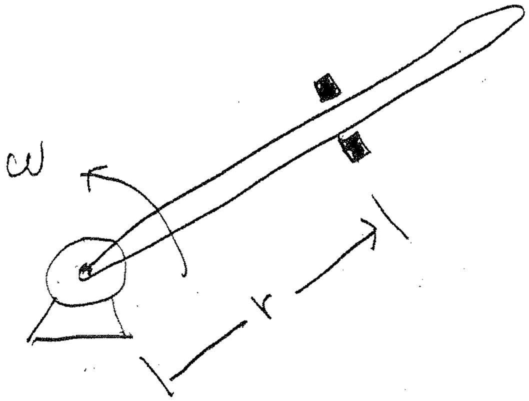
$$\{rmg\sin\theta \hat{k} = -mr^2\ddot{\theta}\hat{k} - \tfrac{1}{2}mr^2\ddot{\theta}\hat{k}\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad rmg\sin\theta = -\ddot{\theta}\,\tfrac{3}{2}mr^2$$

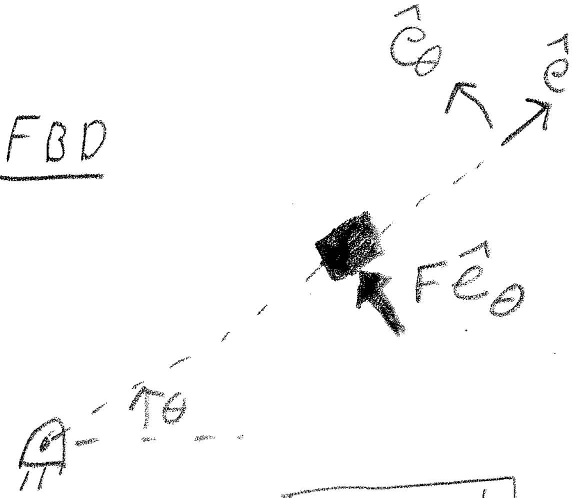
$$\theta \ll 1 \Rightarrow \sin\theta \doteq \theta \Rightarrow \ddot{\theta} + \frac{2g}{3r}\theta = 0 \Rightarrow \text{say, } \theta = \theta_0 \cos\sqrt{\frac{2g}{3r}}\, t$$

$$r = R/2, \quad t^* = 1 \text{ period} \Rightarrow \sqrt{\frac{4g}{3r}}\, t^* = 2\pi \Rightarrow \boxed{t^* = 2\pi\sqrt{\frac{3r}{4g}} = \pi\sqrt{\frac{3r}{g}}}$$

2) A small bead m slides without friction on a straight rod which is rotating at constant ω about a point on the rod. Neglect gravity. 2D. Find $\ddot{r}$ in terms of some or all of r, $\dot{r}$ and ω.



FBD



$$\boxed{\begin{gathered}\omega = \text{const} \\ \Rightarrow \dot{\omega} = 0\end{gathered}}$$

LMB ;

$$\sum \vec{F} = m\vec{a}$$

$$\left\{ F\hat{e}_\theta = m\left[(\ddot{r} - \omega^2 r)\hat{e}_r + 2\dot{r}\omega\hat{e}_\theta\right.\right.$$

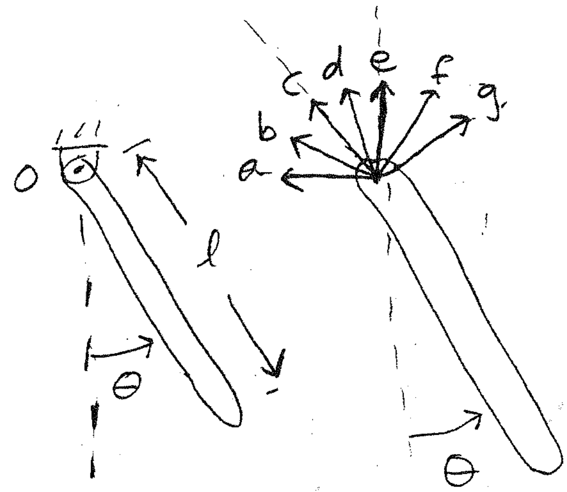
$$\{\ \}\cdot\hat{e}_r \Rightarrow \qquad 0 = \ddot{r} - \omega^2 r$$

$$\Rightarrow \qquad \boxed{\boxed{\ddot{r} = \omega^2 r}}$$

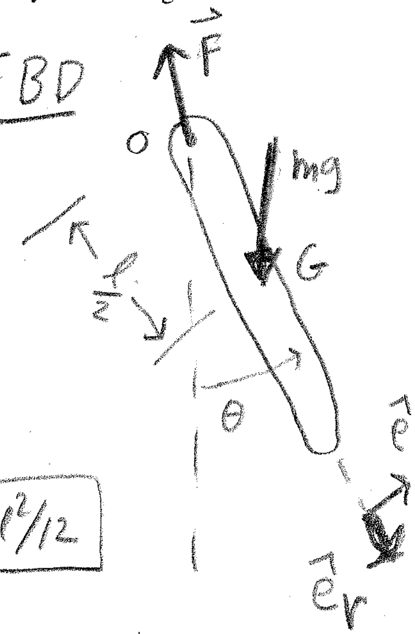
3) A uniform stick with mass m and length ℓ swings from a frictionless hinge at 0.

a) Find the equation of motion. That is, find $\ddot{\theta}$ in terms of some or all of θ, $\dot{\theta}$, ℓ, m and g.

b) The partial free body diagram shows some conceivable reaction forces at O. Which of these are you confident are in the wrong direction (a,b,c,d,e,f,g)? Justify your answer in a manner that is fully convincing.



FBD



$$\boxed{I^G = m\ell^2/12}$$

AMB/O:

$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\vec{r}_{G/O} \times mg\hat{i} = \vec{r}_{G/O} \times m\vec{a}_G + I^G \ddot{\theta}\hat{k}$$

$$\vec{r}_{G/O} = \frac{\ell}{2}\hat{e}_r \qquad \vec{a}_G = \ddot{\theta}\frac{\ell}{2}\hat{e}_\theta - \frac{\ell}{2}\dot{\theta}^2\hat{e}_r$$

$$\Rightarrow \quad \frac{\ell}{2}\hat{e}_r \times mg\hat{i} = \frac{m\ell}{2}\hat{e}_r \times \left(\ddot{\theta}\frac{\ell}{2}\hat{e}_\theta - \frac{\ell}{2}\dot{\theta}^2\hat{e}_r\right) + I\ddot{\theta}\hat{k}$$

$$\left\{ -\frac{\ell}{2}\cancel{m}g\sin\theta\,\hat{k} = \cancel{m}\frac{\ell^2}{4}\ddot{\theta}\hat{k} + \frac{\cancel{m}\ell^2}{12}\ddot{\theta}\hat{k} \right\}$$

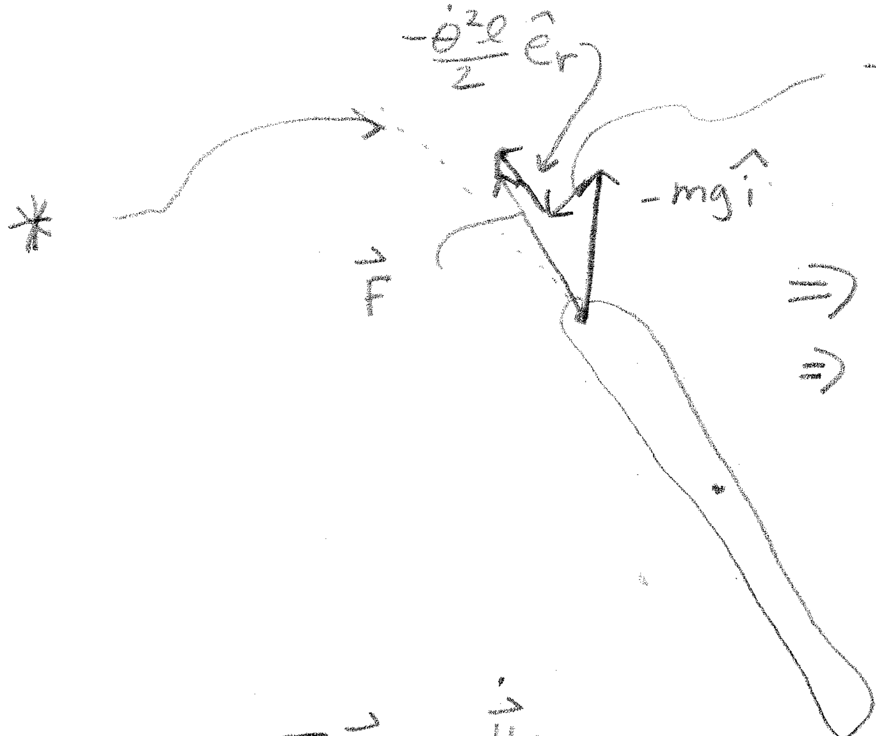
$$\{\}\cdot\hat{k} \Rightarrow \quad -\frac{\ell g}{2}\sin\theta = \frac{\ell^2}{3}\ddot{\theta}$$

$$\Rightarrow \quad \boxed{\ddot{\theta} = \frac{-3g}{2\ell}\sin\theta}$$

LMB: $\quad \sum \vec{F} = m\vec{a}$

$$\vec{F} + mg\hat{i} = m\Big(\underbrace{\ddot{\theta}}_{-\frac{3g}{2\ell}\sin\theta}\tfrac{\ell}{2}\hat{e}_\theta - \dot{\theta}^2\tfrac{\ell}{2}\hat{e}_r\Big)$$

$$\Rightarrow \quad \vec{F} = m\left[-g\hat{i} - \frac{3g}{4}\sin\theta\,\hat{e}_\theta + \left(-\dot{\theta}^2\frac{\ell}{2}\right)\hat{e}_r\right]$$



$\Rightarrow \vec{F}$ is up & to left.

$\Rightarrow \vec{F}\cdot\hat{j} < 0$

$\vec{F}\cdot\hat{i} < 0$

AMB/G: $\quad \sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$

$$\underbrace{\left(-\frac{\ell}{2}\hat{e}_r\right)}_{\vec{r}_{O/G}} \times \vec{F} = \underbrace{\ddot{\theta}}_{-\frac{3g}{2\ell}\sin\theta} I\hat{k}$$

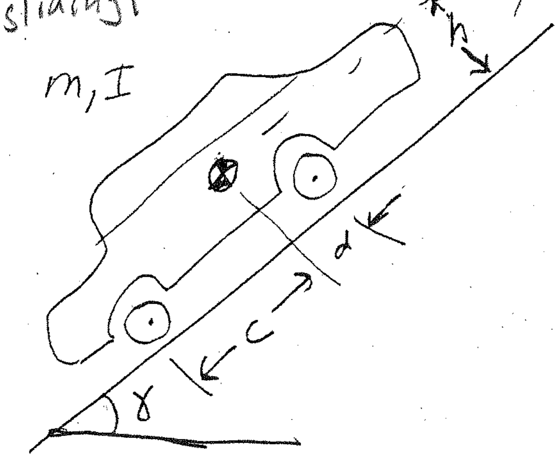
for $\theta > 0 \Rightarrow \ddot{\theta} < 0 \Rightarrow \vec{F}$ causes CW rotation

$\Rightarrow \vec{F}$ points to right of line $*$

$\boxed{\text{Only (d) is possible.}}$

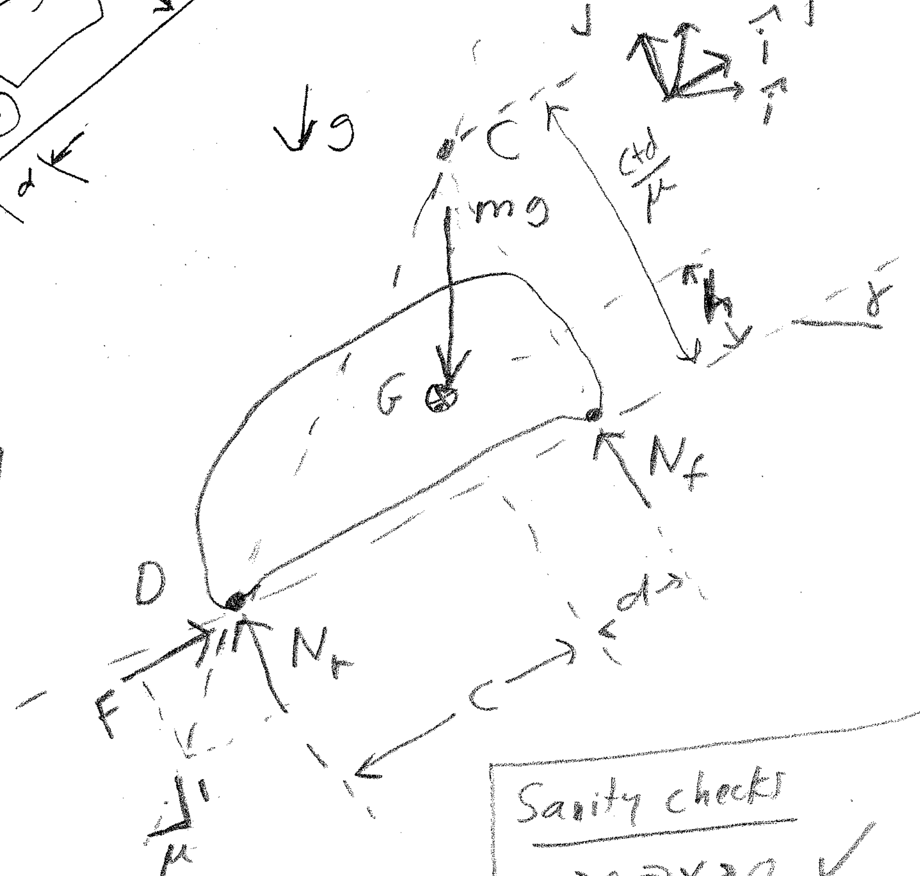
4) A rear-wheel drive car attempts to drive uphill. Assume it does not tip over. What is the steepest slope γ it can go up? Answer in terms of some or all of m, c, d, h, g, μ and the moment of inertia about the center of mass I.

without sliding



$\vec a = a\hat i'$

$\downarrow g$



Borderline case:

$\vec a = \vec 0 \Rightarrow$ statics!

AMB/C:

$$\sum \vec M_{/C} = \vec 0$$

$$\vec r_{G/C} \times (-mg\hat j) = \vec 0$$

$$\left[\left(-\frac{(c+d)}{\mu} + h\right)\hat j' - d\,\hat i'\right] \times (-mg\hat j) = \vec 0$$

$$\Rightarrow \left\{ \left(-\frac{(c+d)}{\mu}+h\right)\underbrace{\hat j'\times\hat j}_{-\sin\gamma\hat k} + d\,\underbrace{\hat i'\times\hat j}_{\cos\gamma\hat k} = \vec 0 \right\}$$

$$\{\ \}\cdot\hat k \Rightarrow \boxed{\tan\gamma = \frac{d}{\frac{c+d}{\mu} - h}}$$

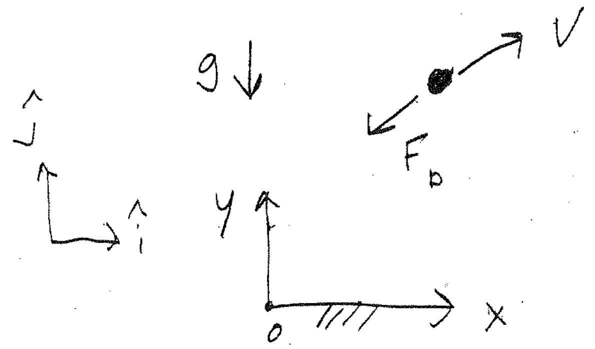
Sanity checks

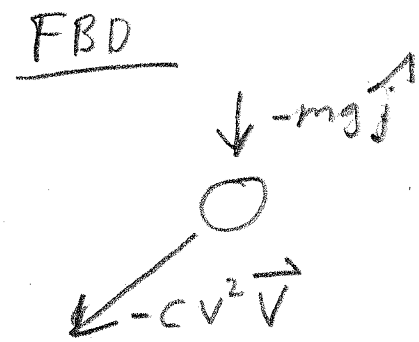
$\mu \to 0 \Rightarrow \gamma \to 0$ ✓

$c=h=0 \Rightarrow \tan\gamma = \mu$ ✓

$d=0 \Rightarrow \gamma=0$ ✓ (interesting)

5) A particle m is acted on by gravity and a cubic drag force $F_D = cv^3$ that opposes its motion. Find $\ddot{x}$ in terms of some or all of $x, y, \dot{x}, \dot{y}, m, g$ and c.





LMB

$$\sum \vec{F} = m\vec{a}$$

$$\{-mg\hat{j} - cv^2\vec{v} = m(\ddot{x}\hat{i} + \ddot{y}\hat{j})\}$$

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$$

$$\dot{x}^2 + \dot{y}^2$$

$$\{\}\cdot\hat{i} \Rightarrow \quad \ddot{x} = \frac{-c}{m} v^2 \dot{x}$$

$$\boxed{\ddot{x} = \frac{-c}{m}(\dot{x}^2 + \dot{y}^2)\dot{x}}$$

"SOLUTIONS"

Your Name: RUINA

TA's name, Section # and Section time: ____________

T&AM 203 Prelim 1

Tuesday February 23, 2009

Draft February 24, 2009

3 problems, 25^+ points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in this problem book at the location of the relevant problem.

b) Full credit if

- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance are used;
- $\vec{\bullet}$ correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given;
- you clearly state any reasonable assumptions if a problem seems *poorly defined*;
- work is I.) neat,
  II.) clear, and
  III.) well organized;

- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ `Matlab code`, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "`theta7dot = 18`". You will be penalized slightly for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve. Unless specifically stated otherwise, substantial partial credit if you provide Matlab code which would generate the desired answer.

Problem 1:	25 /25
Problem 2:	25 /25
Problem 3:	25 /25

1) **(25 pt)** A mass m moves on a frictionless plane. There are two forces on it

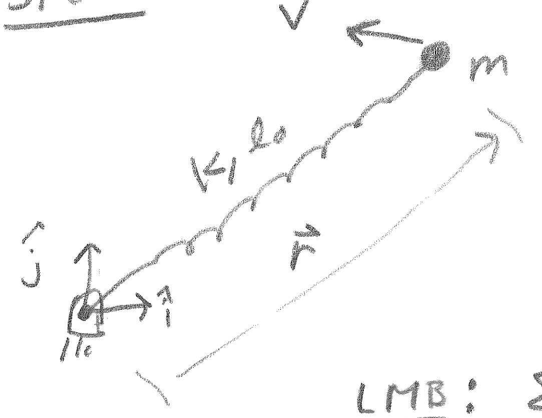
i) $\vec{F}_s$ the force of a linear spring. One end of the spring is anchored at the origin, the other is attached to the mass. The spring has rest length ℓ_0 and spring constant k.

ii) $\vec{F}_d$ the force of a linear viscous drag. The drag force opposes the motion and is proportional, with constant c, to the speed.

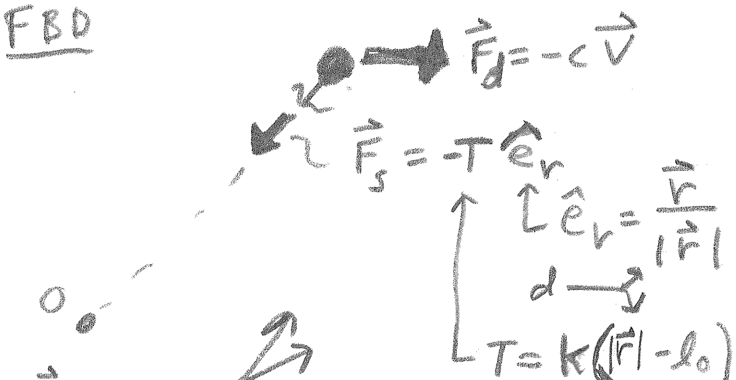
The Matlab code below is intended to give the differential equation to a Matlab solver like ODE45. It is incomplete. On the opposite page, provide the missing lines and box them in.

```
function zdot = rhs(t,z,m,k,Lo,c)
r = z(1:2); % two components of position
v = z(3:4); % two components of velocity
d = sqrt(r(1) ^ 2 + r(2) ^ 2); % radius, use if convenient
s = sqrt(v(1) ^ 2 + v(2) ^ 2); % speed, use if convenient
(*) →
% Missing lines of code here.  You can use vectors or scalars
% so long as the final code would run properly.
% Please define intermediate variables, like d and s above,
% to simplify the readability of your code.

zdot = [rdot;vdot]';
end
```

Sketch



FBD

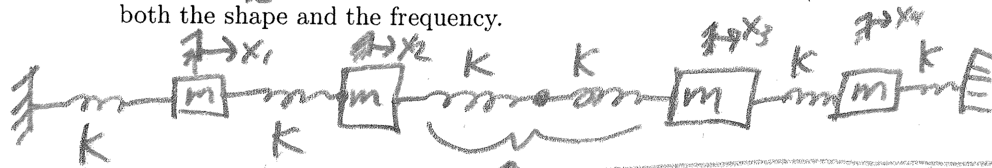


LMB: $\sum \vec{F} = m\vec{a}$

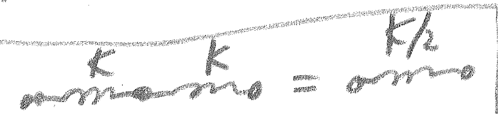
$$\boxed{\vec{F}_s + \vec{F}_d = m\dot{\vec{v}}}$$

```
eR = r/d;  % unit vector in r dir,
T = k*(d-Lo);  % spring tension
Fs = -T*eR;     % spring force, 2 element vector
Fd = -c*v;    % drag force, 2 element vector
vdot = (Fs + Fd)/m %  the main ODE,
rdot = V;          %  r' = v  (easy)
```
(*)

2) (25 pt) Four equal masses m are in a line between two walls. There are 5 springs separating the masses from each other and the walls. All springs have constant k but for the middle spring which has constant ~~$2k$~~ $k/2$. Clearly describe one normal mode of vibration (with words and equations), giving both the shape and the frequency.



$k_{eff} = k/2$



By inspection: if two masses on left go right by Δ
and " " " right " left " "
then each mass has restoring force $k\Delta$.

$\Rightarrow$ mode shape $= \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ & $\lambda = \sqrt{k/m}$.

$$\Rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left[A\cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}\, t\right)\right] \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \quad (*)$$

done. (good enough)

FBDs

kx_1 $\quad k(x_2-x_1)$ $\quad \frac{k}{2}(x_3-x_2)$ $\quad k(x_4-x_3)$ $\quad -kx_4$

To find all 4 modes one needs eigenvalues & eigenvectors. Too hard to solve 4th order polynomial on exam!

LMB

$m\ddot{x}_1 = -2kx_1 + kx_2$

$m\ddot{x}_2 = kx_1 - \frac{3k}{2}x_2 + \frac{k}{2}x_3$

$m\ddot{x}_3 = \frac{k}{2}x_2 - \frac{3k}{2}x_3 + kx_4$

$m\ddot{x}_4 = kx_3 - kx_4$

$$\Rightarrow m\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} = -k\underbrace{\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3/2 & -1/2 & 0 \\ 0 & -1/2 & 3/2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (**)$$

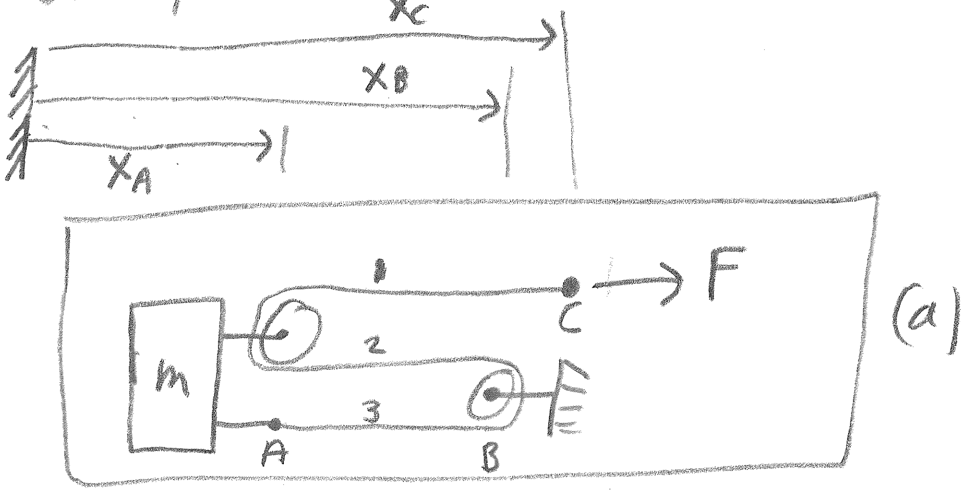
Note: $*$ solves $**$ because $A\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

Alt. Soln: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

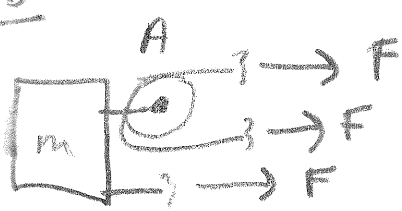
$\lambda = \sqrt{3k/m}$, $A\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

3) (25 pt) A mass m is to be accelerated using pulleys and ropes and a force F. Design a pulley system (using as many pulleys as you like) so that the acceleration of the mass is $3F/m$. For this pulley system, what is the acceleration at the point where the force is applied?

Need rope tension on mass 3 times over:



FBD



tension = const along rope

LMB:

$$3F = m a_A$$

$$\boxed{a_A = 3F/m}$$

L = String length = const = $(x_C - x_A) + 2(x_B - x_A) + 2(\pi R)$

$$0 = \ddot{L} = \ddot{x}_C - \ddot{x}_A + \cancel{2\ddot{x}_B}_{\to 0} - 2\ddot{x}_A + 0$$

$$\boxed{\ddot{x}_C = 3\ddot{x}_A = 9F/m}\ \text{(b)}$$

"SOLUTIONS"

Your TA, Section # and Section time:

Your name:

ANDY RUINA

Cornell TAM 2030

Prelim 2

No calculators, books or notes allowed.

3 Problems, 90$^+$ minutes total.

March 24, 2009

Directions. To ease your TA's grading and to maximize your score, please:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation.**
- ✓+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifiable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$\dot{\theta}_7$ `= 18`" instead of, say, "`theta7dot = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems ***poorly defined***, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- Put your name on each extra sheet, fold it in, and refer to it at the relevant problem.
  Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 4: ______ /25

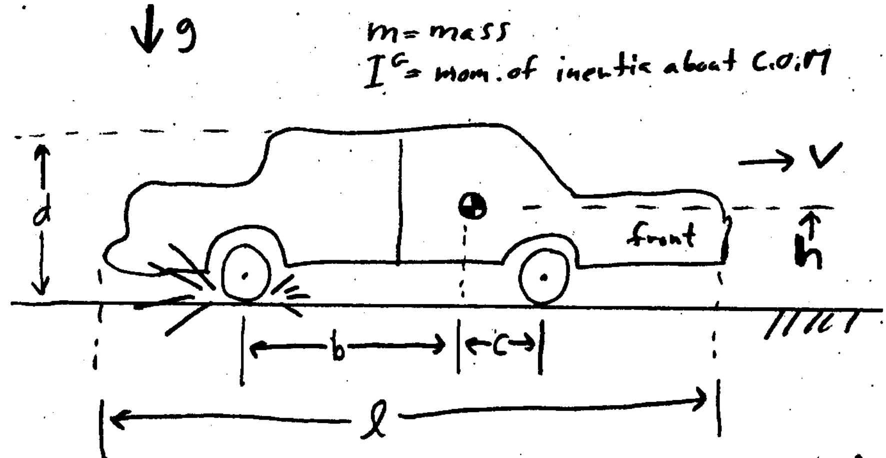
Problem 5: ______ /25

Problem 6: ______ /25

4) A car, moving to the right in the figure below, screeches to a stop, skidding the rear wheels (coefficient of friction $=\mu$, friction angle $=\phi$, with $\tan\phi=\mu$). The brakes are not applied to the light front wheels which roll easily.

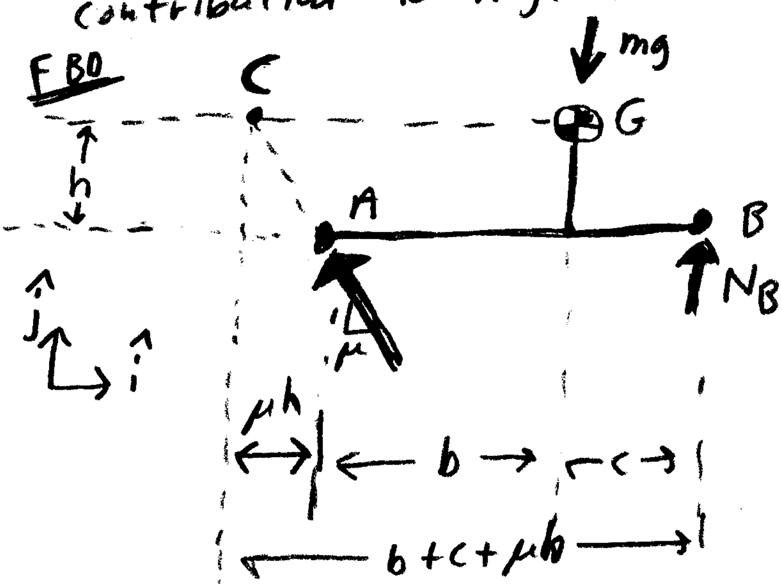
What is the vertical force from the ground on the front wheels?

Answer in terms of some or all of the variables on this page. Extra credit for showing that your answer agrees with one or more special cases that you can evaluate more simply.



Assume car translation w/ no rotation,

" wheels & other moving parts have negligible contribution to Ang. mom.



AMB/C

$$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

$$\vec{r}_{B/C}\times N_B\hat{j} + \vec{r}_{G/C}\times(-mg\hat{j}) = \underbrace{\vec{r}_{G/C}}_{(\mu h+b)\hat{i}}\times m\underbrace{\vec{a}}_{a\hat{i}} + \overbrace{I\alpha\hat{k}}^{\vec{0}}$$

$$\Rightarrow [(\mu h+b+c)N_B - mg(\mu h+b)]\hat{k}=\vec{0}$$

$$\Rightarrow \boxed{N_B = mg\frac{\mu h+b}{\mu h+b+c}} \ *$$

Special cases

1) $\mu=0 \Rightarrow$ statics $\Rightarrow N_B = mg\dfrac{b}{b+c}$ ✓

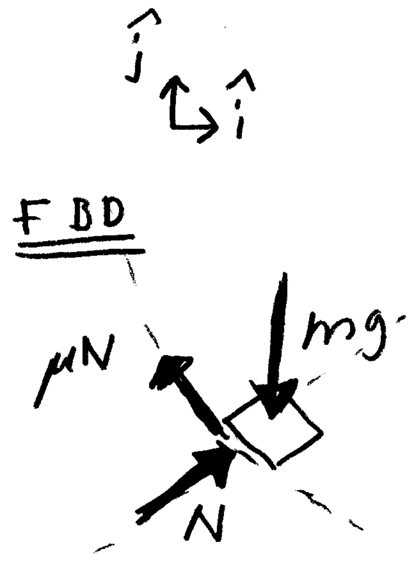
2) $h=0 \Rightarrow$ no tipping from decelation $\Rightarrow$ same N_B as for statics ✓

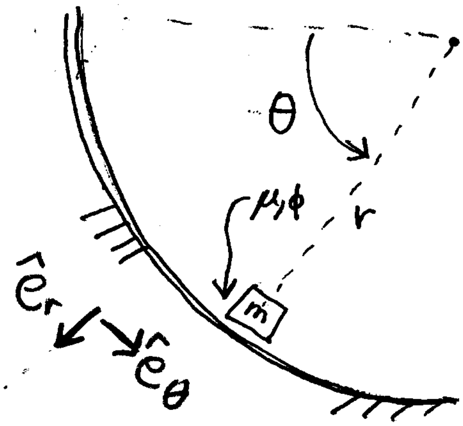
3) $h=0,\ b=0 \Rightarrow$ No load on front wheel ever ✓

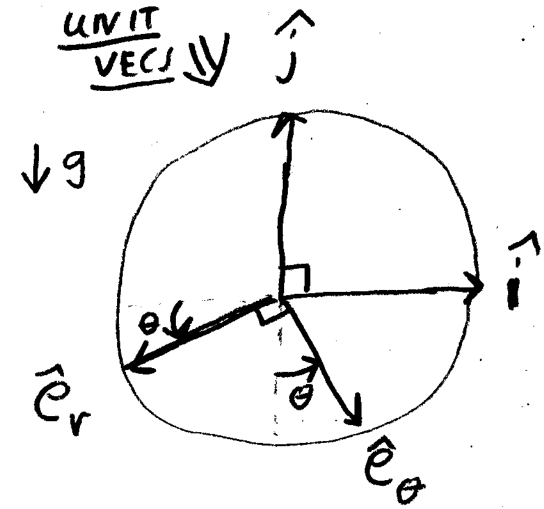
4) $h=0,\ c=0 \Rightarrow$ All load on front wheels $\Rightarrow N_B = mg$ ✓

5) A small block slides down a circular chute. You are given $\dot{\theta}$ and the other variables shown. *Find* $\ddot{\theta}$.

Extra credit for showing that your answer agrees with one or more special cases that you can evaluate more simply.







LMB

$$\sum \vec{F} = m\vec{a}$$

$\vec{a}$ for circular motion

$$\left\{ -mg\hat{j} - N\hat{e}_r - \mu N\hat{e}_\theta = m\left[r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r\right] \right\}^* \quad ①$$

$$\{①\}\cdot\hat{e}_r \Rightarrow -mg\underbrace{\hat{j}\cdot\hat{e}_r}_{-\sin\theta} - N = -mr\dot{\theta}^2 \Rightarrow N = mr\dot{\theta}^2 + mg\sin\theta \quad ②$$

$$\{①\}\cdot\hat{e}_\theta \Rightarrow -mg\underbrace{\hat{j}\cdot\hat{e}_\theta}_{-\cos\theta} - \mu N = mr\ddot{\theta} \quad ③$$

N: $mr\dot{\theta}^2 + mg\sin\theta$ from ②

Solve ③ for $\ddot{\theta}$ $\Rightarrow$ $$\boxed{\ddot{\theta} = \frac{g}{r}\cos\theta - \frac{\mu}{r}\left[r\dot{\theta}^2 + g\sin\theta\right]}$$

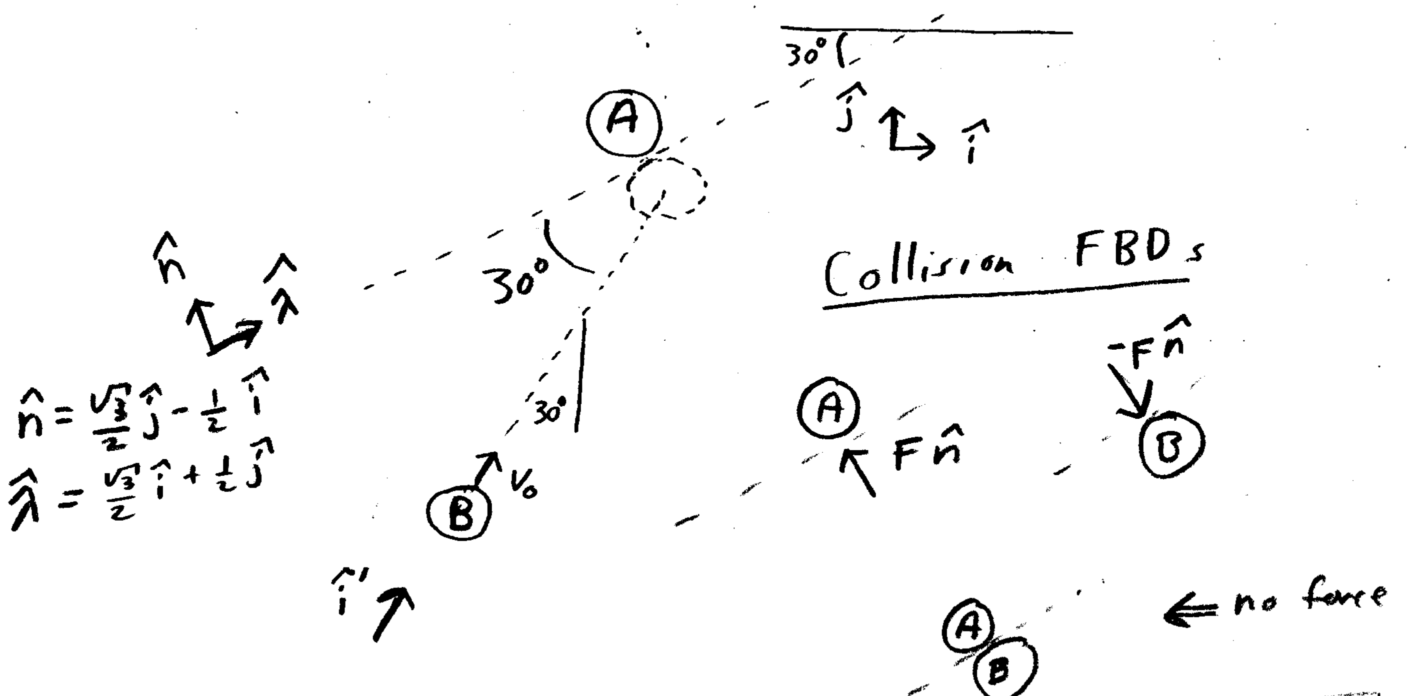
Special cases
1) $\mu = 0 \Rightarrow$ simple pendulum ($\cos\theta$ is like the usual $\sin\theta$) ✓
2) $g = 0 \Rightarrow \ddot{\theta} = -\mu\, r\dot{\theta}^2$ (only the expected centrip term) ✓
3) $\dot{\theta} = 0, \theta = \pi/2$, $\ddot{\theta} = \mu\, g/r$ ($a_\theta = r\ddot{\theta}$ is like sliding on level) ✓

*To solve with only one eqn. in one unknown: $\{①\}\cdot(\mu\hat{e}_r - \hat{e}_\theta)$
This "kills" N & μN.

6) Two balls on a plane have equal mass m. One is initially still and the other is moving at speed v_0 in the direction shown. They have a frictionless collision with coefficient of restitution $e = 1$.

Find the velocity (a vector) of either ball (your choice) after the collision.

Answer in terms of some or all of $m, v_0, \hat{\imath}, \hat{\jmath}$ and various numbers. [Note: $\sin 30° = 1/2$ and $\cos 30° = \sqrt{3}/2$.]



$$\{\text{LMB block A}\}\cdot\hat{\lambda} \Rightarrow \cancel{m}\vec{V}_A^{+}\cdot\hat{\lambda} = \cancel{m}\underbrace{\vec{V}_A^{-}}_{=\vec{0}}\cdot\hat{\lambda} \Rightarrow \boxed{\vec{V}_A^{+}\cdot\hat{\lambda} = 0} \quad (1)$$

$$\{\text{LMB block B}\}\cdot\hat{\lambda} \Rightarrow \cancel{m}\vec{V}_B^{+}\cdot\hat{\lambda} = \cancel{m}\underbrace{\vec{V}_B^{-}}_{V_0\hat{\imath}'}\cdot\hat{\lambda} \Rightarrow \vec{V}_B^{+}\cdot\hat{\lambda} = V_0\cos 30° \quad \boxed{\vec{V}_B^{+}\cdot\hat{\lambda} = V_0\sqrt{3}/2} \quad (2)$$

$$\{\text{LMB system}\}\cdot\hat{n} \Rightarrow \cancel{m}(\vec{V}_A^{+} + \vec{V}_B^{+})\cdot\hat{n} = \cancel{m}(\vec{V}_A^{-} + \vec{V}_B^{-})\cdot\hat{n}$$

$$\boxed{\vec{V}_A^{+}\cdot\hat{n} + \vec{V}_B^{+}\cdot\hat{n} = V_0/2} \quad (3)$$

$$\{\text{Restitution eqn.}\} \quad (\vec{V}_A^{+} - \vec{V}_B^{+})\cdot\hat{n} = -(\underbrace{\vec{V}_A^{-}}_{\vec{0}} - \vec{V}_B^{-})\cdot\hat{n} \quad (e=1)$$

$$\Rightarrow \boxed{\vec{V}_A^{+}\cdot\hat{n} - \vec{V}_B^{+}\cdot\hat{n} = V_0/2} \quad (4)$$

$(3)-(4) \Rightarrow \boxed{\vec{V}_B^{+}\cdot\hat{n} = 0}$ (5)

(3) or $(4) \Rightarrow \boxed{\vec{V}_A^{+}\cdot\hat{n} = V_0/2}$ (6)

$\hat{n}$ & $\hat{\lambda}$ comps of $\vec{V}_A^{+}$ & $\vec{V}_B^{+}$

(1),(2),(5),(6) $\Rightarrow$ all the final velocities

For example: $\vec{V}_A^{+}\cdot\hat{\lambda}$ is $\hat{\lambda}$ comp. of $\vec{V}_A^{+}$

$\Longrightarrow$ next page

6) (cont'd)

$$\vec{V}_B^+ = \frac{\sqrt{3}}{2} V_0 \, \hat{\lambda}$$

$$\hat{\lambda} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

No change of $\vec{V}$ in $\hat{\lambda}$ dir., Dead stop in $\hat{n}$ direction.

B: $$\boxed{\vec{V}_B^+ = \frac{\sqrt{3}}{2} V_0 \left[\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} \right] \qquad \vec{V}_B^+ = V_0 \left[\frac{3}{4}\hat{i} + \frac{\sqrt{3}}{4}\hat{j} \right]}$$

$$\vec{V}_A^+ = \frac{V_0}{2} \, \hat{n}$$

$$\hat{n} = \frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i}$$

A: $$\boxed{\vec{V}_A^+ = \frac{V_0}{2} \left[\frac{\sqrt{3}}{2}\hat{j} - \frac{1}{2}\hat{i} \right] \qquad \vec{V}_A^+ = V_0 \left[-\frac{1}{4}\hat{i} + \frac{\sqrt{3}}{4}\hat{j} \right]}$$

only gets pushed in $\hat{n}$ dir. so only goes in $\hat{n}$ dir.

<u>Note</u>: You can do this problem in your head. In $\hat{\lambda}$ direction B keeps its velocity ($\frac{\sqrt{3}}{2}V_0$) and A doesn't pick up any. In $\hat{n}$ direction B gives up its velocity of $V_0/2$ and gives it to A. In $\hat{n}$ dir. its an elastic collision between balls of equal mass $\Rightarrow$ balls trade velocities.

Your TA, Section # and Section time:

"SOLUTIONS"

Your name:

Andy Ruina

Cornell TAM 2030

Prelim 3

No calculators, books or notes allowed.

3 Problems, 90^+ minutes total.

April 14, 2009

Directions.

To ease your TA's grading and to maximize your score, please:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- ✓+ Be (I) neat, (II) clear and (III) well organized.
- ☐ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifiable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct. You can use shortcut notation like "$\dot\theta_7$ = 18" instead of, say, "`theta7dot = 18`". Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat\imath, \hat\jmath, \hat e_r, \hat e_\theta, \hat{\boldsymbol\lambda}, \hat{\boldsymbol n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- ☐ Put your name on each extra sheet, fold it in, and refer to it at the relevant problem. Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Includes alternate problem (8b)

Student picks 8 or Alternate 8b

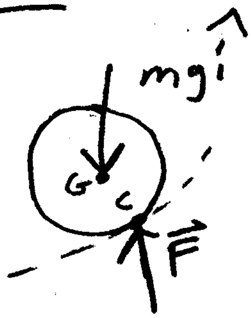
Problem 7: ______ /25

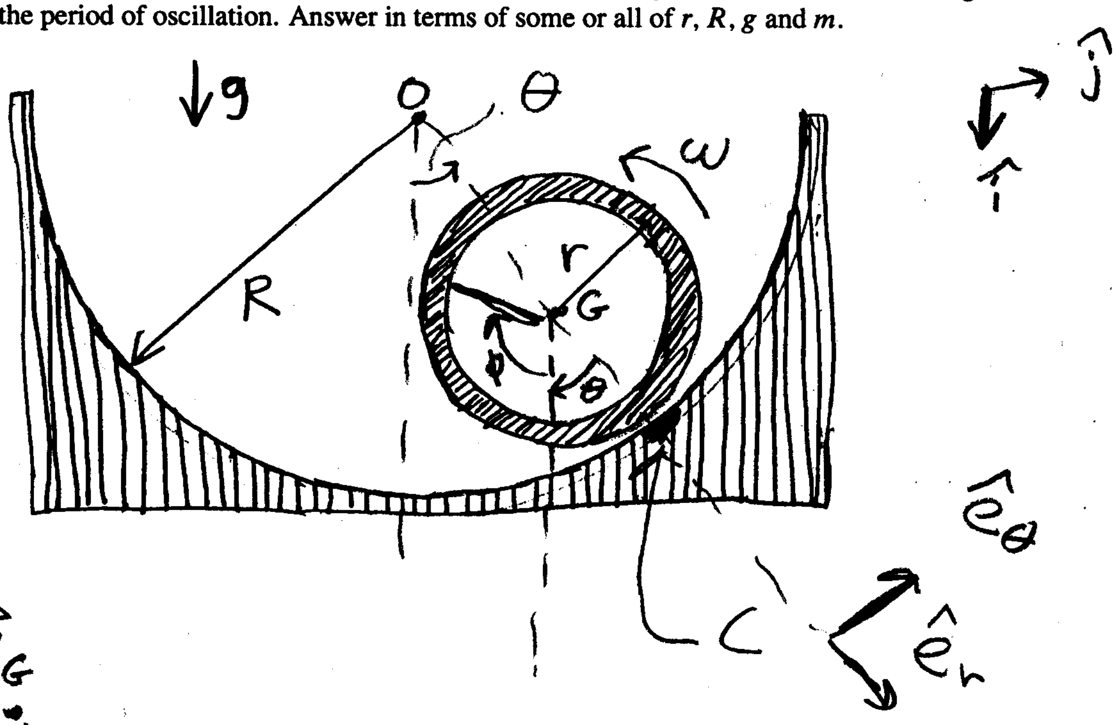
Problem 8: ______ /25

Problem 9: ______ /25

7) A thin-walled pipe with mass m and radius r rolls back and forth in a trough with radius R. Assuming small oscillations what is the period of oscillation. Answer in terms of some or all of r, R, g and m.

FBD





Kinematics

$$\vec{V}_G = \vec{V}_G$$

$$\dot{\theta}(R-r) = \dot{\phi}\, r \qquad \dot{\phi} = -\omega = -\omega_{pipe}$$

$$\Rightarrow \ddot{\theta} = \frac{-\dot{\omega}_{pipe}\, r}{R-r} \quad (1)$$

AMB$_{/c}$:

$$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

$$I^G = mr^2$$

$$mgr\sin\theta\,\hat{k} = \vec{r}_{G/C} \times m\vec{a}_G + I^G\dot{\omega}\hat{k}$$

$$\vec{r}_{G/C} = -r\hat{e}_r \qquad \vec{a}_G = -(R-r)\dot{\theta}^2\hat{e}_r + (R-r)\ddot{\theta}\hat{e}_\theta$$

$$mgr\sin\theta\,\hat{k} = -r(R-r)m\ddot{\theta}\hat{k} + mr^2\dot{\omega}\hat{k}$$

$$\left\{ \cancel{m}g\cancel{r}\sin\theta\,\hat{k} = -\cancel{r}(R-r)\cancel{m}\ddot{\theta}k - \cancel{mr^2}\frac{R-r}{\cancel{r}}\ddot{\theta}\hat{k} \right\} \quad (1)$$

$$\{\}\cdot\hat{k} \Rightarrow \quad -g\sin\theta = \ddot{\theta}\, 2(R-r)$$

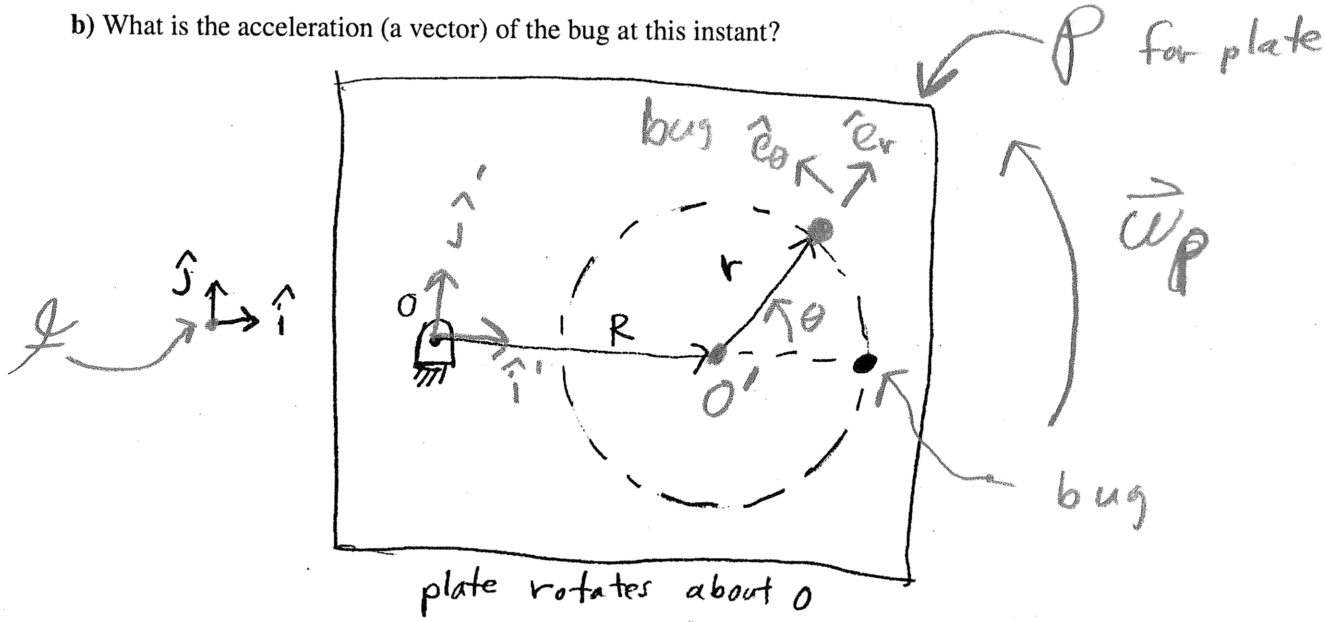
$$\theta << 1 \Rightarrow \quad \ddot{\theta} + \frac{g}{2(R-r)}\theta = 0 \Rightarrow \theta = A\cos\Big(\underbrace{\sqrt{\frac{g}{2(R-r)}}}_{2\pi}\, t - \beta\Big)$$

$$\boxed{t^* = \text{period} = 2\pi\sqrt{\frac{2(R-r)}{g}}}$$

8) A rectangular plate $\mathcal{P}$ rotates with constant counter-clockwise angular velocity $\omega_{\mathcal{P}}$ about the point O marked. A bug walks on the plate with constant speed v, relative to the plate, on the dotted circle shown (radius r, with center a distance R from O). At the instant of interest the center of the circle and the bug are both directly to the right of O.

a) What is the velocity (a vector) of the bug at this instant?

b) What is the acceleration (a vector) of the bug at this instant?



Lots of (legitimate) choices!

$$\underbrace{\vec V_b}_{=\vec V_{bug}} = \underbrace{\vec V_{O'/\mathcal{F}}}_{=\omega_P R\hat J} + \underbrace{\vec V_{/P}}_{\vec V_{/P} = v\hat e_\theta = v\hat J} + \underbrace{\vec\omega \times \vec r_{/O'}}_{\omega_P r\hat J}$$

$$\boxed{= (\omega_P(R+r)+v)\hat J = \vec V_{bug}} \quad (a)$$

$$\vec a_b = \underbrace{\vec a_{O'/O}}_{-\omega^2 R\hat i} + \underbrace{\vec a_{/P}}_{-\frac{v^2}{r}\hat i} - \omega_P^2 \underbrace{\vec r_{/O'}}_{r\hat i} + \underbrace{\cancel{\dot{\vec\omega}}}_{\vec 0}\times \vec r_{/O'} + 2\underbrace{\vec\omega_P}_{=\omega_P\hat K}\times \underbrace{\vec v_{/P}}_{=v\hat J}$$

$$\boxed{\vec a_b = \left[-\omega_P^2(R+r) - \frac{v^2}{r} - 2\omega_P v\right]\hat i} \quad (b)$$

Sanity check: when $R=0 \Rightarrow \vec a_b = -\underbrace{\left(\omega_P + \frac{v}{r}\right)}_{= \text{the net } \dot\theta}^2 r\,\hat i$ ✓

8b) Assume r and θ are measured in the standard way relative to an xy coordinate system. A particle motion is described with polar coordinates with

$$r = r_0 \cos\theta \quad \text{and} \quad \dot{\theta} = \omega = \text{constant}.$$

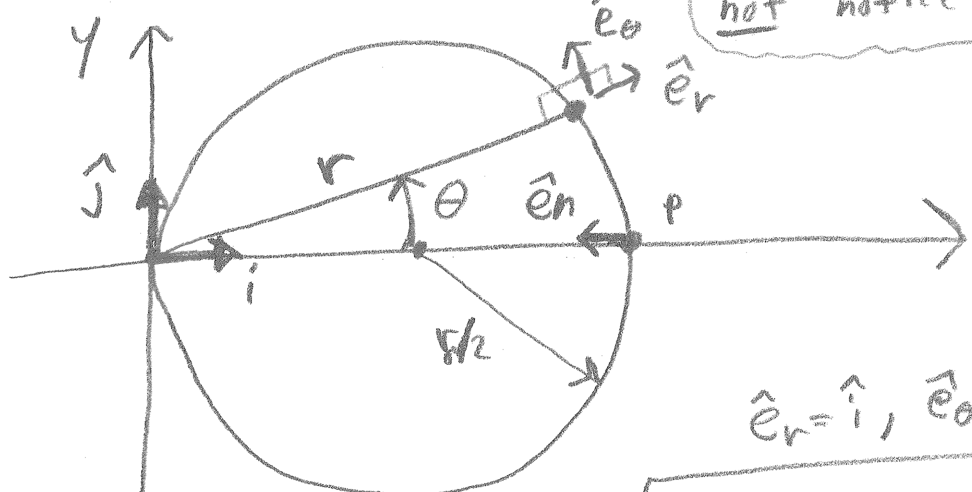
We are interested in the instant that the particle passes through the x axis at $\vec{r} = r_0 \hat{e}_r = r_0 \hat{\imath}$. Answer in terms of some or all of $r_0, \omega, \hat{\imath}$ and $\hat{\jmath}$.

a) What is the velocity of the particle at this instant?

b) What is the acceleration of the particle at this instant?

c) What is the the radius of curvature of the particle path at this instant?

Happens to be polar coord. formula for a circle. But you need <u>not</u> notice this.



$\hat{e}_r = \hat{\imath}, \; \hat{e}_\theta = \hat{\jmath}, \; \theta = 0$

Take $\theta = \omega t$ so at $t = 0$ p is on x axis

$$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = \boxed{r_0 \omega \hat{\jmath} = \vec{V}(t=0)} \text{ (a)}$$

$\dot{\theta} = \omega$

$r = r_0 \cos\theta = r_0$

$\dot{r} = \frac{d}{dt}(r_0 \cos\theta) = -r_0 \cancel{\sin\theta}^{\,0}\, \dot{\theta}$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} - 2\dot{r}\dot{\theta})\hat{e}_\theta = \boxed{-2 r_0 \omega^2 \hat{\imath} = \vec{a}} \text{ (b)}$$

$\dot{\theta}^2 = \omega^2$, $\ddot{\theta} = 0$, $\dot{r} = 0$

$\ddot{r} = \frac{d}{dt} r_0 (\sin\theta)\omega = -r_0 \underbrace{\cos\theta}_{1}\, \omega^2$

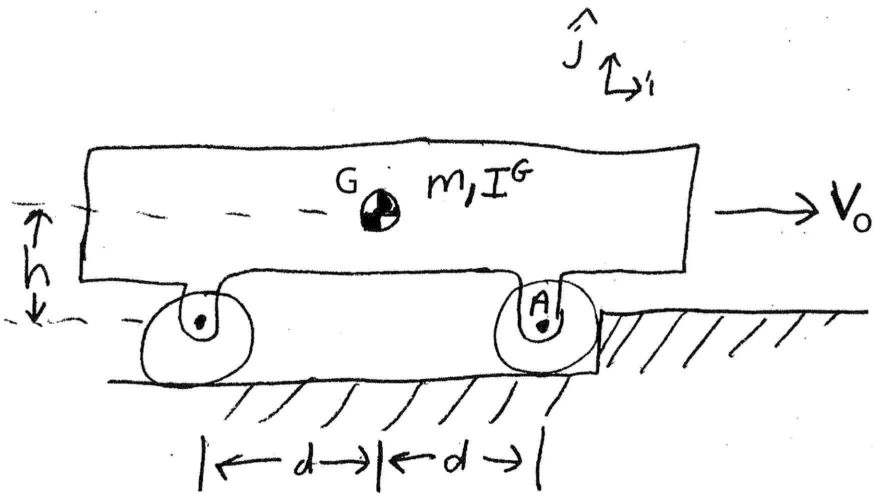
Check: $|\vec{a}| = v^2/(r_0/2)$

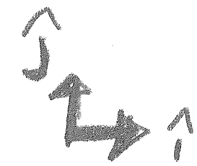
(consistent w/ circular shape)

$$\vec{a} = \frac{v^2}{\rho}\hat{e}_n = \frac{(r\omega_0)^2}{\rho}(-\hat{\imath}) \; *$$

$$\vec{a} = \vec{a} \Rightarrow \frac{(r_0\omega)^2}{\rho} = 2 r_0 \omega^2 \Rightarrow \frac{r_0^{\cancel{2}}\cancel{\omega^2}}{\rho} = 2\cancel{r_0}\cancel{\omega^2} \Rightarrow \boxed{\rho = \frac{r_0}{2}} \text{ (c)}$$

9) A rigid cart (mass m, moment of inertia I^G) with light well-lubricated wheels is rolling on level ground at constant speed v_0 when the front wheel suddenly gets completely stuck against a curb. Just after this collision what is the velocity of G? Answer in terms of some or all of v_0, m, I^G, d, h and g.



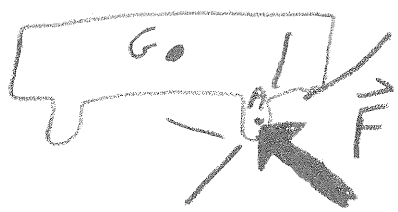


$-$ = just before collision: $\vec{\omega}^- = \vec{0}$

$+$ = just after collision: $\vec{V}_A^+ = \vec{0}$

Collisional FBD

neglects non-colli forces



AMB$_{/A}$ $\quad \Delta \vec{H}_{/A} = \int \vec{M}_{/A}\, dt = \vec{0}$

$$\Rightarrow \vec{H}_{/A}^{+} = \vec{H}_{/A}^{-}$$

$$\Rightarrow \vec{r}_{G/A} \times m\vec{V}_G^{+} + I^G \vec{\omega}^{+} = \vec{r}_{G/A} \times m \vec{V}_G^{-}$$

$\vec{r}_{G/A} = -d\hat{i} + h\hat{j}$, $\quad \vec{V}_G^{-} = V_0\hat{i}$, $\quad \vec{\omega}^+ = \omega^+\hat{k}$, $\quad \vec{V}_G^+ = \omega^+\hat{k} \times \vec{r}_{G/A}$

$$\Rightarrow \{ m(h^2+d^2)\omega^+\hat{k} + I^G\omega^+\hat{k} = -hmV_0\hat{k} \}$$

$\{\}\cdot\hat{k} \Rightarrow$

$$\omega^+ = \frac{-hmV_0}{I^G + m(h^2+d^2)}$$

$$\vec{V}_G^{+} = \vec{\omega}^+ \times \vec{r}_{G/A} = \frac{-hmV_0}{I_G + m(h^2+d^2)}\hat{k} \times (-d\hat{i} + h\hat{j})$$

$$\boxed{\vec{V}_G^{+} = \frac{mV_0 h}{I_G + m(h^2+d^2)} (h\hat{i} + d\hat{k})}$$

Sanity checks:

$h = 0 \Rightarrow \vec{V}_G^+ = 0$ ✓

$d = 0$ & $I^G = 0 \Rightarrow \vec{V}_G^+ = V_0\hat{i}$ ✓

$I^G \to \infty \Rightarrow \vec{V}_G^+ \to 0$ ✓

Units match: [V] on both sides ✓

Your TA, Section # and Section time:

"SOLUTIONS"

Your name:

ANDY RUINA

Cornell TAM/ENGRD 2030

Final Exam

May 12, 2011

No calculators, books or notes allowed.

5 Problems, 150 minutes (no extra time)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct. You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`". Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** Put your name on each extra sheet, fold it in, and refer to it at the relevant problem. Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 13: ____ /25

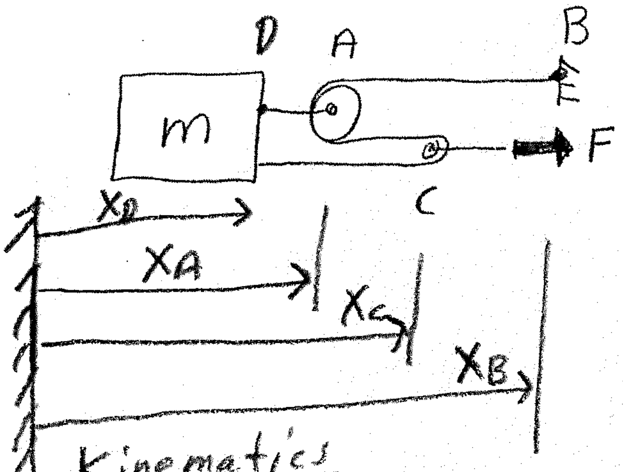
Problem 14: ____ /25

Problem 15: ____ /25

Problem 16: ____ /25

Problem 17: ____ /25

13) Making all the usual assumptions about masses and pulleys, find the acceleration of point C in terms of F and m. Neglect gravity.



Kinematics

L = Rope length = const

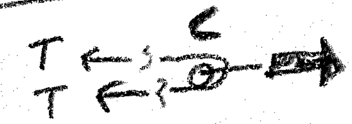
$$= (x_C - x_D) + \pi r_C + (x_C - x_A) + \pi r_A + (x_B - x_A)$$

$$0 = \dot{L} = (\dot{x}_C - \underbrace{\dot{x}_D}_{=\dot{x}_A}) + 0 + (\dot{x}_C - \dot{x}_A) + 0 + (\underbrace{\dot{x}_B}_{=0} - \dot{x}_A)$$

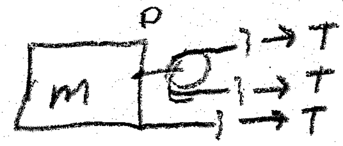
$$\Rightarrow \quad 2\dot{x}_C - 3\dot{x}_A = 0$$

$$\Rightarrow \boxed{\ddot{x}_C = \frac{3}{2}\underbrace{\ddot{x}_A}_{=\ddot{x}_D}} \quad ①$$

FBDs



$\Rightarrow T = F/2$



$\{LMB\}\cdot \hat{\imath}$

$$\underbrace{\sum F}_{3T} = \underbrace{\dot{L}}_{m\ddot{x}_D}$$

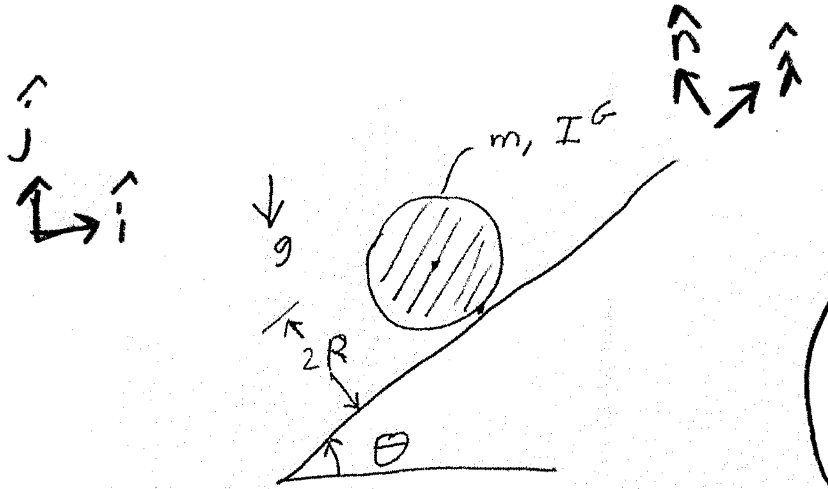
$$\boxed{\ddot{x}_A = 3T/m = 3F/2m} \quad ②$$

① + ② $\Rightarrow$

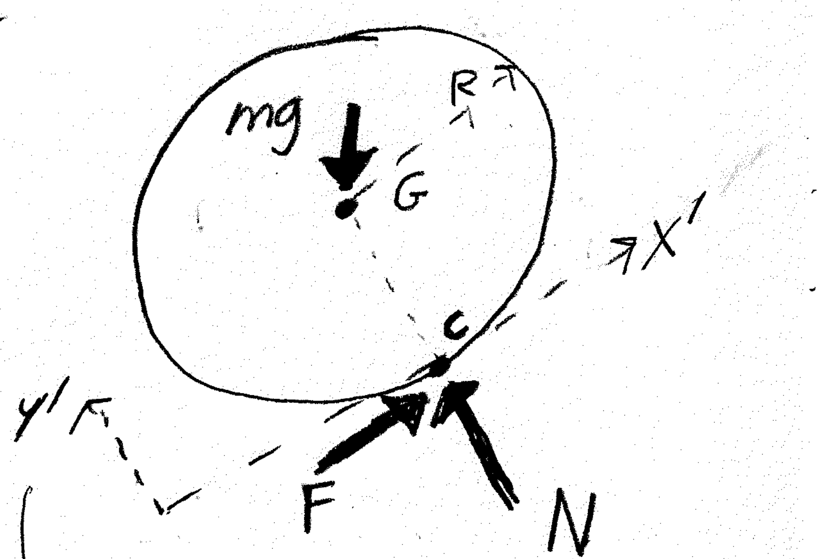
$$\ddot{x}_C = \frac{3}{2}\ddot{x}_A$$

$$\Rightarrow \boxed{\boxed{\ddot{x}_C = \frac{9}{4}\frac{F}{m}}}$$

14) A disk rolls down a ramp without slipping. How big does μ have to be in order to prevent slip? (That is, if μ is too small, slip would not successfully be prevented). Answer in terms of some or all of θ, g, R, I^G and m.



FBD



Kinematics

$$\vec{0} = \vec{V}_c = \dot{x}'\hat{\lambda} + \omega\hat{k}\times\underbrace{\vec{r}_{c/G}}_{-R\hat{n}}$$

$$\vec{0} = \dot{x}'\hat{\lambda} - \omega R(-\hat{\lambda})$$

$$\{\vec{0} = (\dot{x}' + \omega R)\hat{\lambda}\}$$

$$\{\}\cdot\hat{\lambda} \Rightarrow \{\dot{x}' = -\omega R\}$$

$$\frac{d}{dt}\{\} \Rightarrow \boxed{\ddot{x}' = -\dot{\omega}R} \quad (1)$$

LMB

$$\sum\vec{F} = \dot{\vec{L}}$$

$$\{F\hat{\lambda} + N\hat{n} - mg\hat{j} = m\ddot{x}'\hat{\lambda}\}$$

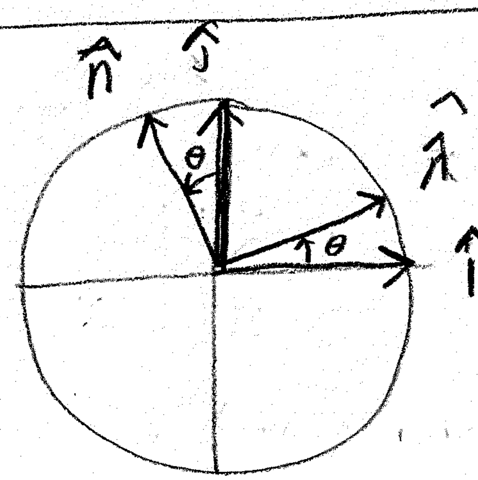
$$\{\}\cdot\hat{n} \Rightarrow N - mg\underbrace{\hat{j}\cdot\hat{n}}_{\cos\theta} = 0$$

$$\boxed{N = mg\cos\theta} \quad (2)$$

$$\text{LMB}\}\cdot\hat{\lambda} \Rightarrow$$

$$F - mg\underbrace{\hat{j}\cdot\hat{\lambda}}_{+\sin\theta} = m\ddot{x}'$$

$$\boxed{F = m[\ddot{x}' + g\sin\theta]} \quad (3)$$



unit circle

AMB$_{/C}$: $\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$

$$\underbrace{\vec{r}_{G/C}}_{R\hat{n}} \times (-mg\hat{j}) = \underbrace{\vec{r}_{G/C}}_{R\hat{n}} \times m\underbrace{\vec{a}_c}_{\ddot{x}'\hat{\lambda}} + I^G\dot{\omega}\hat{k}$$

$$\{-Rmg(-\sin\theta\hat{k}) = Rm\ddot{x}'(-\hat{k}) + I^G\dot{\omega}\hat{k}\}$$

$$\{\}\cdot\hat{k} \Rightarrow \sin\theta\cdot Rmg = -Rm\ddot{x}' + I^G\dot{\omega}$$

$$(1) \Rightarrow \quad \sin\theta\cdot Rmg = R^2m\dot{\omega} + I^G\dot{\omega}$$

$$\Rightarrow \boxed{\dot{\omega} = \frac{Rmg}{I^G+mR^2}\sin\theta} \quad (4)$$

$$\boxed{\ddot{x}' = \frac{-R^2mg}{I^G+mR^2}\sin\theta} \quad (5)$$

$$(3),(5) \Rightarrow \quad F = m\left[\frac{-R^2mg}{I^G+mR^2}\sin\theta + g\sin\theta\right]$$

$$= mg\sin\theta\left[1 - \frac{R^2m}{I^G+mR^2}\right] = mg\sin\theta\frac{I^G}{I^G+mR^2}$$

Sanity checks

A) $\theta = 0 \Rightarrow \mu \geq 0$ ✓

B) $\underbrace{I^G = 0}_{F\to 0 \text{ to cause rotation}} \Rightarrow \mu \geq 0$ ✓

$$\mu \geq \frac{F}{N} = \frac{\cancel{mg}\sin\theta\frac{I^G}{I^G+mR^2}}{\cancel{mg}\cos\theta}$$

$$\boxed{\mu \geq \frac{1}{1+mR^2/I^G}\tan\theta}$$

special cases: $\underbrace{I^G = mR^2}_{\text{hoop}} \Rightarrow \mu \geq \frac{\tan\theta}{2}$

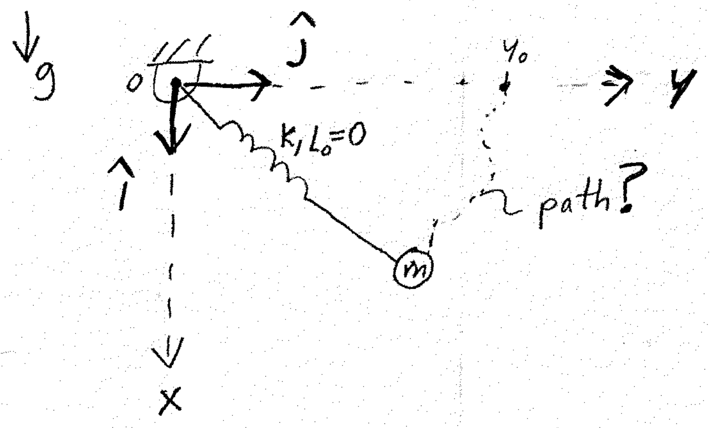
$$\underbrace{I^G = mR^2/2}_{\text{uniform disk}} \Rightarrow \mu = \frac{\tan\theta}{3}$$

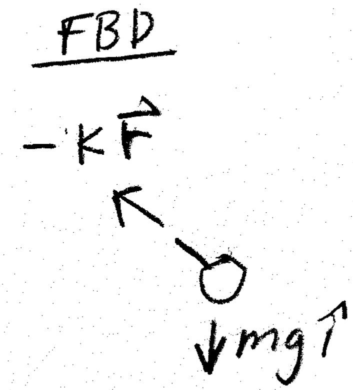
15) A mass m hangs from a spring with constant k and rest length $L_0 = 0$ (the spring is a so-called zero-rest-length spring). The mass is released from rest at the position $\vec{r}_0 = 0\hat{\imath} + y_0\hat{\jmath}$.

a) Find the position of the mass at time t in terms of some or all of k, m, g and y_0.

b) Draw the trajectory (the path that the mass moves on).

c) In words, describe the shape of the trajectory.





LMB

$$\sum \vec{F} = m\vec{a}$$

$$mg\hat{\imath} - k\vec{r} = m\ddot{\vec{r}}$$

$$\{m\ddot{\vec{r}} + k\vec{r} = mg\hat{\imath}\}$$

$\lambda = \sqrt{k/m}$

$$\{\ \}\cdot\hat{\imath} \Rightarrow m\ddot{x} + kx = mg \Rightarrow x = A\cos(\lambda t) + B\sin\lambda t + mg/k$$

$$\{\ \}\cdot\hat{\jmath} = m\ddot{y} + ky = 0 \Rightarrow y = C\cos(\lambda t) + D\sin\lambda t$$

$\dot{x}(0) = 0 \Rightarrow B = 0$

"from rest"

$\dot{y}(0) = 0 \Rightarrow D = 0$

$\Rightarrow x = A\cos\lambda t + mg/k$

$x_0 = 0 \Rightarrow A = -mg/k \Rightarrow x = \frac{mg}{k}(1-\cos\lambda t)$

$y(0) = y_0 \Rightarrow C = y_0$

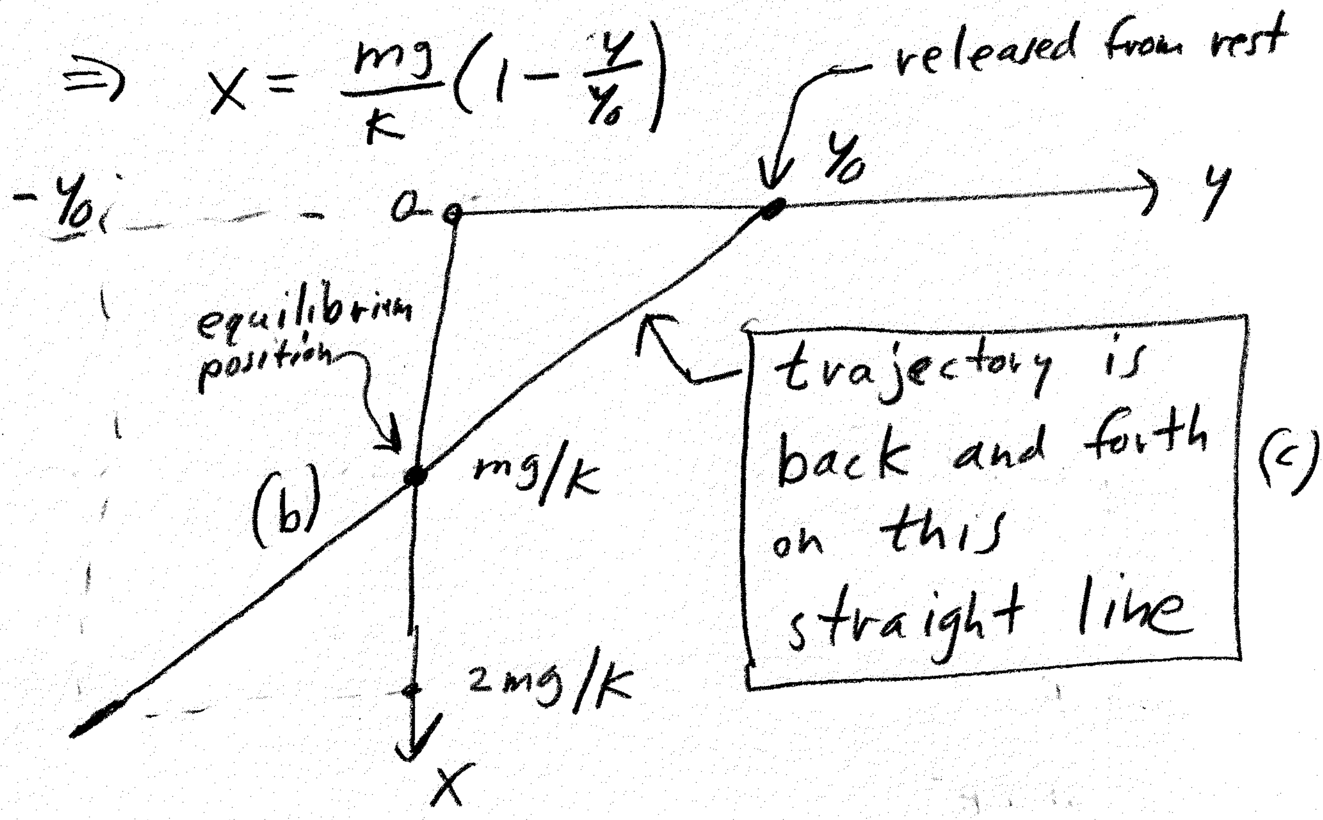
$\Rightarrow$

$y = y_0\cos\lambda t$

$$\Rightarrow \boxed{\vec{r}(t) = \frac{mg}{k}(1-\cos\lambda t)\hat{i} + y_0 \cos\lambda t} \quad (a)$$

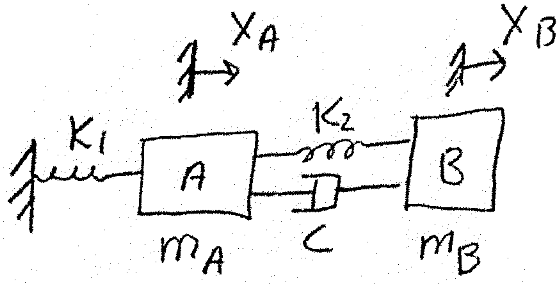
$$\lambda = \sqrt{k/m}$$

$$\Rightarrow \quad x = \frac{mg}{k}\left(1 - \frac{y}{y_0}\right)$$

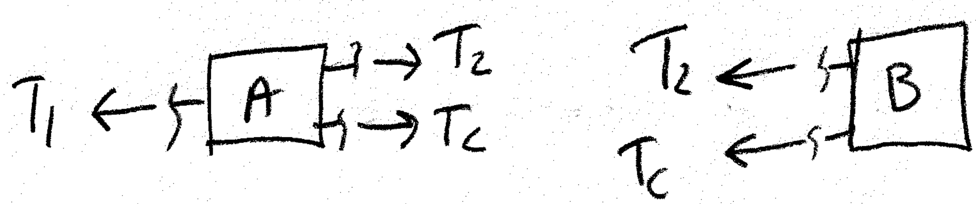


x & y are both simple harmonic motion. In phase with each other, x is offset by $\frac{\text{weight}}{\text{spring constant}}$.

16) Write MATLAB commands to make a plot of $x_B(t)$. Pick any convenient non-zero values (in consistent units) for any variables or constants.



FBDs



LMB

$m\ddot{x}_A = T_2 + T_c - T_1$ $\qquad$ $m\ddot{x}_B = -T_2 - T_c$

Mat. Properties (Constitutive Laws)

$T_2 = (x_B - x_A)k_2$ $\qquad$ $T_1 = k_1 x_A$

$T_c = (\dot{x}_B - \dot{x}_A)c$

$$Z = \begin{bmatrix} x_A \\ x_B \\ v_A \\ v_B \end{bmatrix}$$

Kinematics

$v_A = \dot{x}_A$

$v_B = \dot{x}_B$

MATLAB CODE

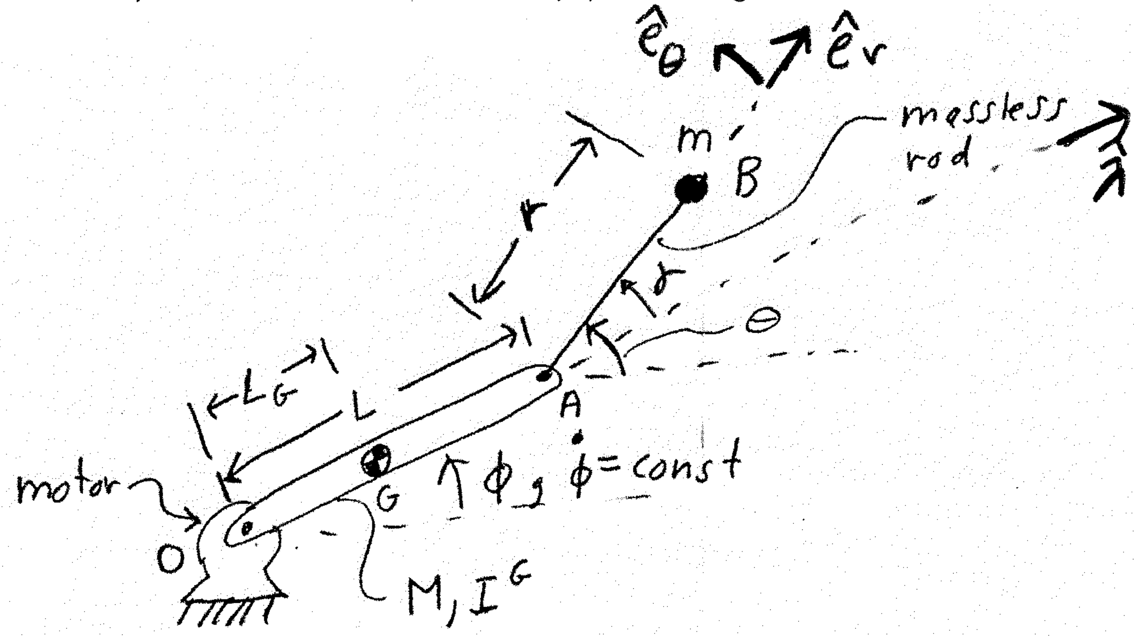
```
function finalexamfun()
XA0 = 1; XB0 = 2; % nonzero things
VA0 = 3; VB0 = 4; %    ''        ''
Z0 = [XA0 XB0 VA0 VB0]';
tspan = [0 10]; % 0≤t≤10
[t zarray] = ode45(@myrhs, tspan, z0)
XB = zarray(:,2); % 2nd column
plot (t,XB); % Skip the labels, its an exam
end

function zdot = myrhs(t,z)
K1=17; K2 =39; C=pi; m=12; % randomish
XA= z(1); XB= z(2); VA= z(3); VB=z(4);
T1 = K1*XA; T2= K2*(XB-XA);
TC = C*(VB-VA);
XADOT=VA; % kinematics
XBDOT=VB; %   ''
VADOT=(1/m)*(T2+TC-T1); % F=ma
VBDOT=(1/m)*(-T2-TC); % F=ma
zdot=[XADOT XBDOT VADOT VBDOT]'
end
```

17) A motor at O turns a rigid rod OA (mass M, moment of inertia I^G) at constant angular rate $\dot{\phi}$. A negligible-mass rod with length r is hinged at A and has mass m at its end. Neglect gravity.

a) Is angular momentum of the system OAB about O constant or not? (Explain your answer.)

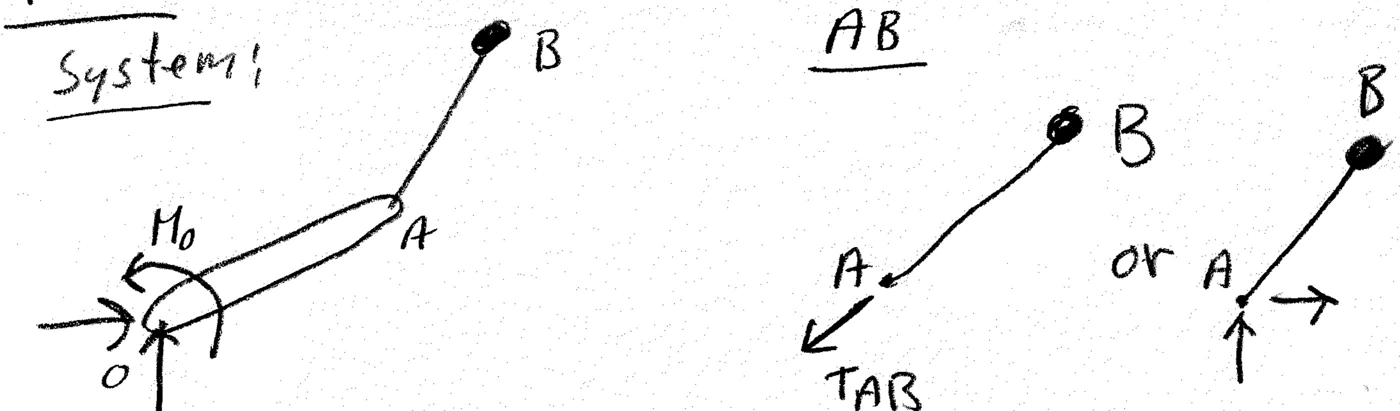
b) Consider the special case that $\phi = 0$ and $\dot{\phi} = 0$ (for all time). Find $\ddot{\theta}$ in terms of as many of these terms are needed: $\theta, \dot{\theta}, L, L_G, r, M, m$ and I^G.

c) Now consider non-zero $\dot{\phi}$. Find $\ddot{\theta}$ in terms of some or all of $\phi, \dot{\phi}, \theta, \dot{\theta}, L, L_G, r, M, m$ and I^G.



FBDs

System:



$$\Sigma \vec{M}_{/0} = \dot{\vec{H}}_{/0}$$

$$M_0 \hat{k} = \dot{\vec{H}}_{/0} \neq \vec{0} \text{ in general}$$

Some $M_0 \neq 0$ needed to keep $\dot{\phi} = \text{const}$

$$\Rightarrow \boxed{\dot{\vec{H}}_{/0} \neq \vec{0}}$$ Ang. Mom not conserved, a

b) $\vec{a}_A = \vec{0}$

$\underline{AMB_{/A}}$ system $AB \Rrightarrow$ $\sum \vec{M}_{/A} = \dot{\vec{H}}_{/A}$

$$\vec{0} = \vec{r}_{B/A} \times m\vec{a}_B$$

$$= m r^2 \ddot{\theta} \hat{k}$$

$\Rightarrow \boxed{\ddot{\theta} = 0}$ B goes in circles at const. rate

c) $\underline{AMB}_{/A}$ system AB; $\sum \vec{M}_{/A} = \dot{\vec{H}}_{/A}$

$$\Rightarrow \quad \vec{0} = \underbrace{\vec{r}_{B/A}}_{r\hat{e}_r} \times \cancel{m}\vec{a}_B$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

$$\vec{a}_A = -L\dot{\phi}^2\hat{\lambda}, \quad \vec{a}_{B/A} = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta$$

$$\Rrightarrow \quad \vec{0} = -rL\dot{\phi}^2 \underbrace{\hat{e}_r \times \hat{\lambda}}_{-\sin(\theta-\phi)\hat{k} = -\sin\gamma\,\hat{k}} - r^2\dot{\theta}^2 \underbrace{\hat{e}_r \times \hat{e}_r}_{\vec{0}} + r^2\ddot{\theta} \underbrace{\hat{e}_r \times \hat{e}_\theta}_{\hat{k}}$$

$$\left\{ \quad = \left[\cancel{r}L\dot{\phi}^2 \sin\theta + \cancel{r}^{\text{for}}\ddot{\theta}\right]\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad \boxed{\ddot{\theta} = \underbrace{-\frac{L}{r}}_{\text{const.}}\dot{\phi}^2 \sin(\theta-\phi)}$$

$\boxed{\ddot{\gamma} = \frac{-L}{r}\dot{\phi}^2 \sin(\gamma)}$ $\rightarrow$ pendulum eqn. w/ g replaced by centripetal accel. of pt. A: $a_A = L\dot{\phi}^2$

Because $\theta - \phi = \gamma$

and $\ddot{\theta} - \underbrace{\ddot{\phi}}_{0 \text{ \{given\}}} = \ddot{\gamma}$

Your TA, Section # and Section time:

Your name:

Cornell TAM/ENGRD 2030

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

Catch-all Makeup Prelim

May 7, 2011

How to get the highest score?

Please do these things:

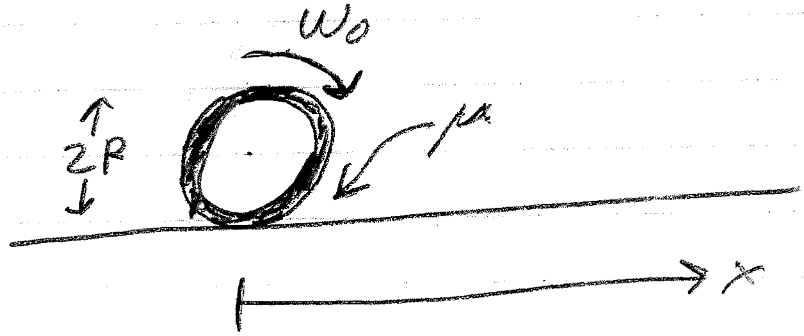
- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60 – 100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** Put your name on each extra sheet, fold it in, and refer to it at the relevant problem.
  Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 10: ____/25

Problem 11: ____/25

Problem 12: ____/25

1. A uniform hoop spins and then is dropped onto a horizontal surface. The hoop spins and slides for a while and eventually ends up in pure rolling.



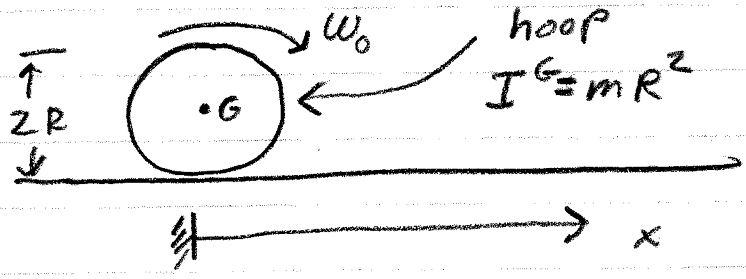
a) When the cylinder eventually rolls what is its velocity (vel. of C.O.M.)

b) What, then, is $\vec{\omega}$?

c) How far does it travel before it enters pure rolling?

Make-up prob. 1 "Solution"

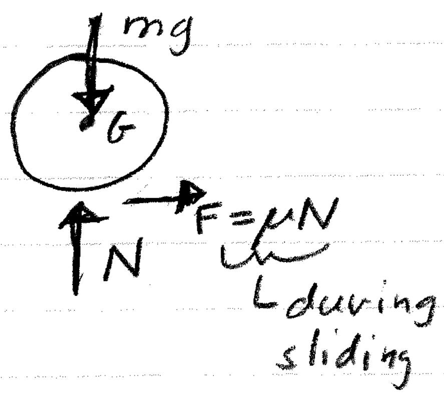
$\hat{j} \uparrow \rightarrow \hat{i}$



initially:
$\vec{\omega} = -\omega_0 \hat{k}$
$\vec{v} = \vec{0}$

Find when & where rolling starts and at what v, ω?

FBD



$\{\sum \vec{F} = m\vec{a}\} \cdot \hat{j} \Rightarrow \boxed{N = mg}$

$\Rightarrow \boxed{F = \mu mg}$

Signs: $\overset{\curvearrowleft}{\omega} = +$, $\vec{v_x} = +$

$AMB_{/G} \Rightarrow \sum \vec{M}_{/G} = \dot{\vec{H}}_{/G} \Rightarrow R\mu mg = I^G \dot{\omega}$ ①

$\{LMB\} \cdot \hat{i} \Rightarrow F = ma_x \Rightarrow \mu mg = m\dot{v}_x$ ②

① $\Rightarrow$ $\omega = -\omega_0 + \dfrac{R\mu mg}{I^G} t = -\omega_0 + \dfrac{\mu g}{R} t$

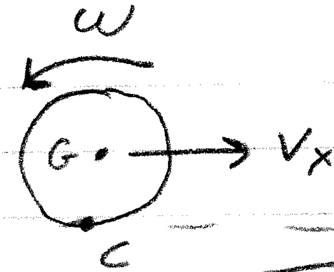
$\Downarrow$

② $\Rightarrow$ $v_x = \mu g t$

$\omega = -\omega_0 + \dfrac{v_x}{R}$ ③

1, (cont'd)

Rolling:



$$V_c = 0 \Rightarrow V_x + \omega R = 0 \Rightarrow \boxed{\omega = -V_x/R} \quad ④$$

Sliding turns to rolling when ③ & ④ agree:

$$\Rightarrow \frac{-V_x}{R} = -\omega_0 + \frac{V_x}{R} \Rightarrow \boxed{V_x = \frac{\omega_0 R}{2}} \quad (a)$$

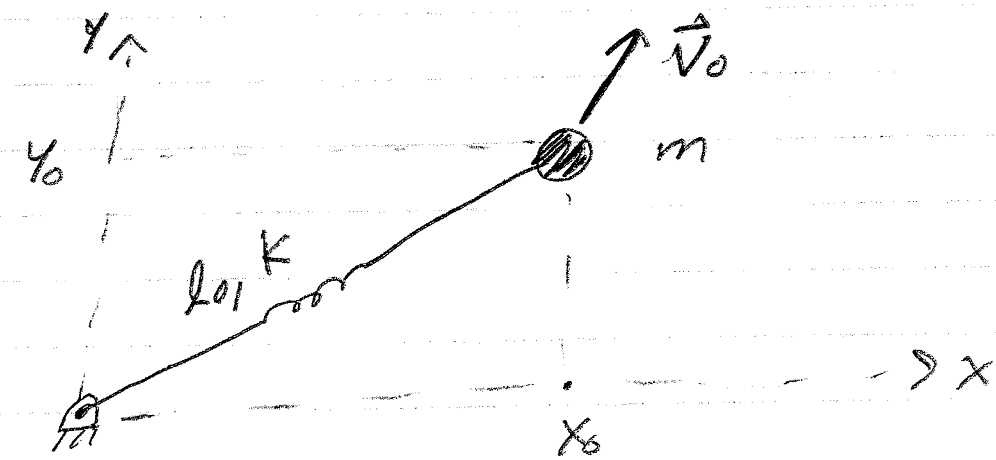
$$\Rightarrow \boxed{\omega = \frac{-\omega_0}{2}} \quad (b)$$

At transition to rolling:

$$V_x = \mu g t = \omega_0 R/2 \Rightarrow t = \frac{\omega_0 R}{2\mu g}$$

$$x = \cancel{x_0}^{\,0} + \mu g t^2/2 = \frac{\mu g}{2}\left(\frac{\omega_0 R}{2\mu g}\right)^2 = \boxed{\frac{\omega_0^2 R^2}{8\mu g}} \quad (c)$$

2) Neglect gravity, 2D.

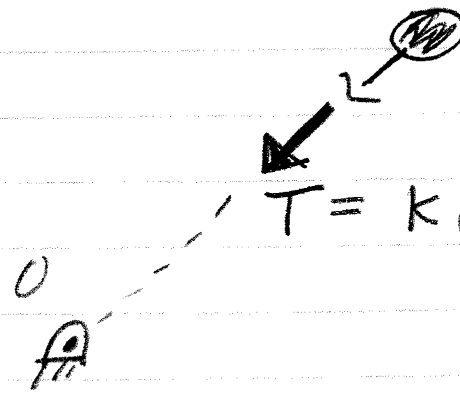


Given $m = 3$ kg
$l_0 = 2$ m
$K = 100$ N/m
$\vec{r}_0 = 5\text{m}\,\hat{i} + 4\text{m}\,\hat{j}$
$\vec{V}_0 = 6\text{ m/s}\,\hat{j}$

Write matlab code to find $x(t = 3s)$.

2) "Solution" $\nearrow \hat{e}_r = \vec{r}/|\vec{r}|$

FBD



$T = k(r - \ell_0)$

LMB: $\Sigma \vec{F} = m\vec{a}$

$$-T\hat{e}_r = m\ddot{\vec{r}}$$

$$T = k(r - \ell_0), \quad r = \sqrt{x^2 + y^2}$$

$$\hat{e}_r = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$$

$$\ddot{\vec{r}} = (\dot{v}_x\hat{i} + \dot{v}_y\hat{j})$$

(Matlab code on next page) $\longrightarrow$

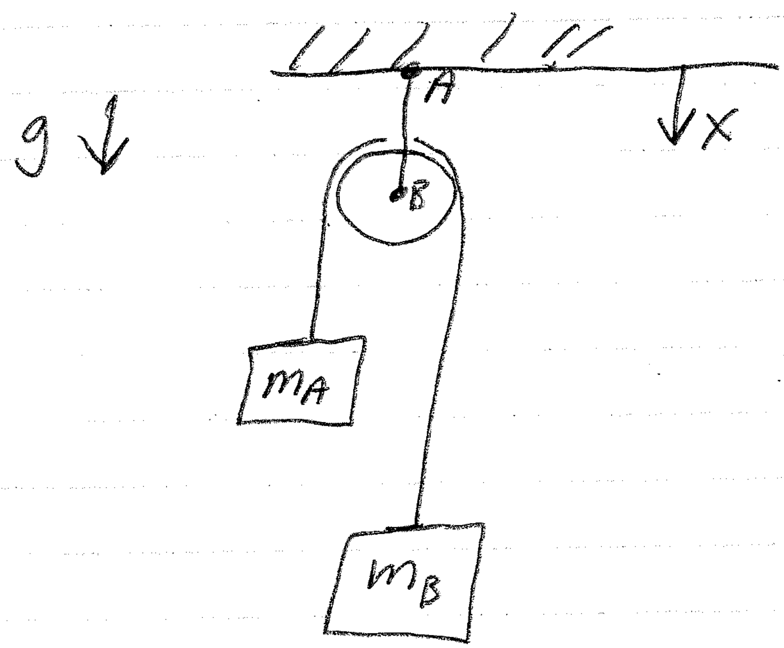
2)(cont'd)

MATLAB CODE

```
function makeupprob2()

r0=[5;4]; v0=[0;6]; z0=[r0;v0];
tspan = [0 3];

[t zarray]=ode45(@myrhs,tspan,z0);

lastx= zarray(:,1); % THE ANSWER

end


function zdot=myrhs(t,z)
m=3; L=2; k=100;

r = z(1:2);    v = z(3:4); magr=norm(r);

rdot= v;                      % 2 ODEs
vdot= -(1/m)*k*(magr-L)*r/magr;
                              % 2 more ODEs

zdot=[rdot;vdot];
end
```

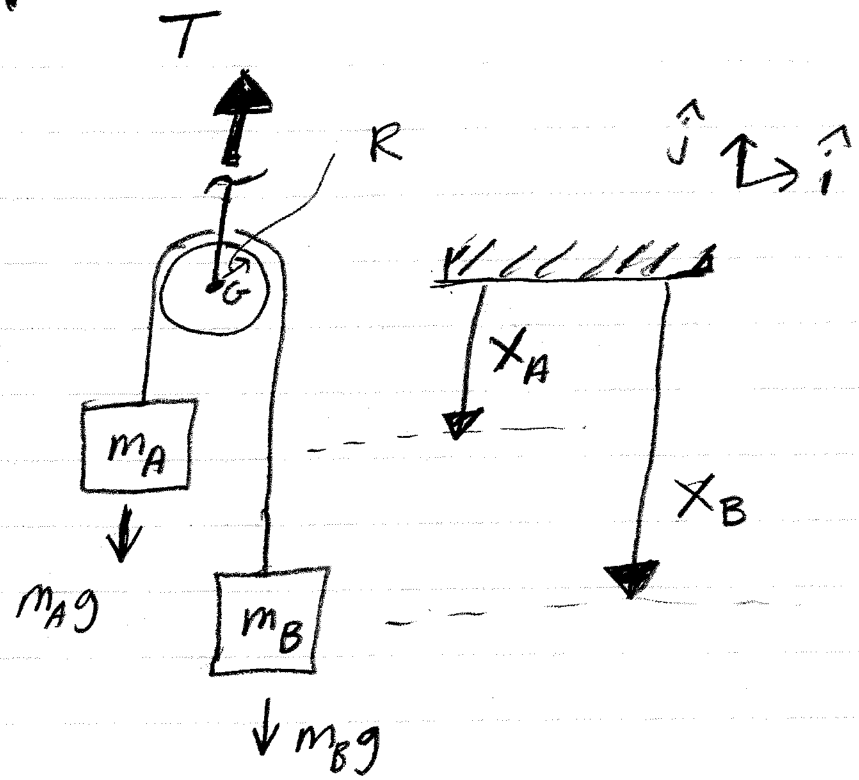
one text file

3) In terms of m_A, m_B & g find the tension in cable AB.



3) "Solution"

FBD



Kinematics:

$$\underbrace{\ddot{x}_A = -\ddot{x}_B}_{\text{string length=const}}$$

AMB$_{/G}$: $\{\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}\}\cdot\hat{k}$

$$gR(m_A - m_B) = R(m_A\ddot{x}_A - m_B\underbrace{\ddot{x}_B}_{=-\ddot{x}_A})$$

$$\Rightarrow \quad \ddot{x}_A = \frac{g\cancel{R}(m_A-m_B)}{\cancel{R}(m_A+m_B)} = g\left(\frac{m_A-m_B}{m_A+m_B}\right)$$

$$\Rightarrow \quad \ddot{x}_B = -g\left(\frac{m_A-m_B}{m_A+m_B}\right)$$

LMB: $\{\sum \vec{F} = \sum m_i \vec{a}_i\}\cdot\hat{j}$

$$T - m_A g - m_B g = -m_A\ddot{x}_A - m_B\ddot{x}_B$$

3) Cont'd

$$T = (m_A + m_B) g - (m_A - m_B) \frac{m_A - m_B}{m_A + m_B} g$$

$$= \frac{(m_A + m_B)^2 - (m_A - m_B)^2}{m_A + m_B} g$$

$$\boxed{T_{AB} = \frac{4 m_A m_B}{m_A + m_B} g}$$

Checks

$$\left.\begin{array}{l} m_A = 0 \Rightarrow T_{AB} = 0 \\ m_B = 0 \Rightarrow T_{AB} = 0 \end{array}\right\} \text{free fall} \checkmark$$

$$m = m_A = m_B \Rightarrow T_{AB} = 2mg \quad \text{statics} \checkmark$$

Your TA, Section # and Section time:

"SOLUTIONS"

Your name:

ANDY RUINA

Cornell TAM/ENGRD 2030

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

Prelim 1

March 1, 2011

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifiable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** Put your name on each extra sheet, fold it in, and refer to it at the relevant problem. Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 1: ______ /25

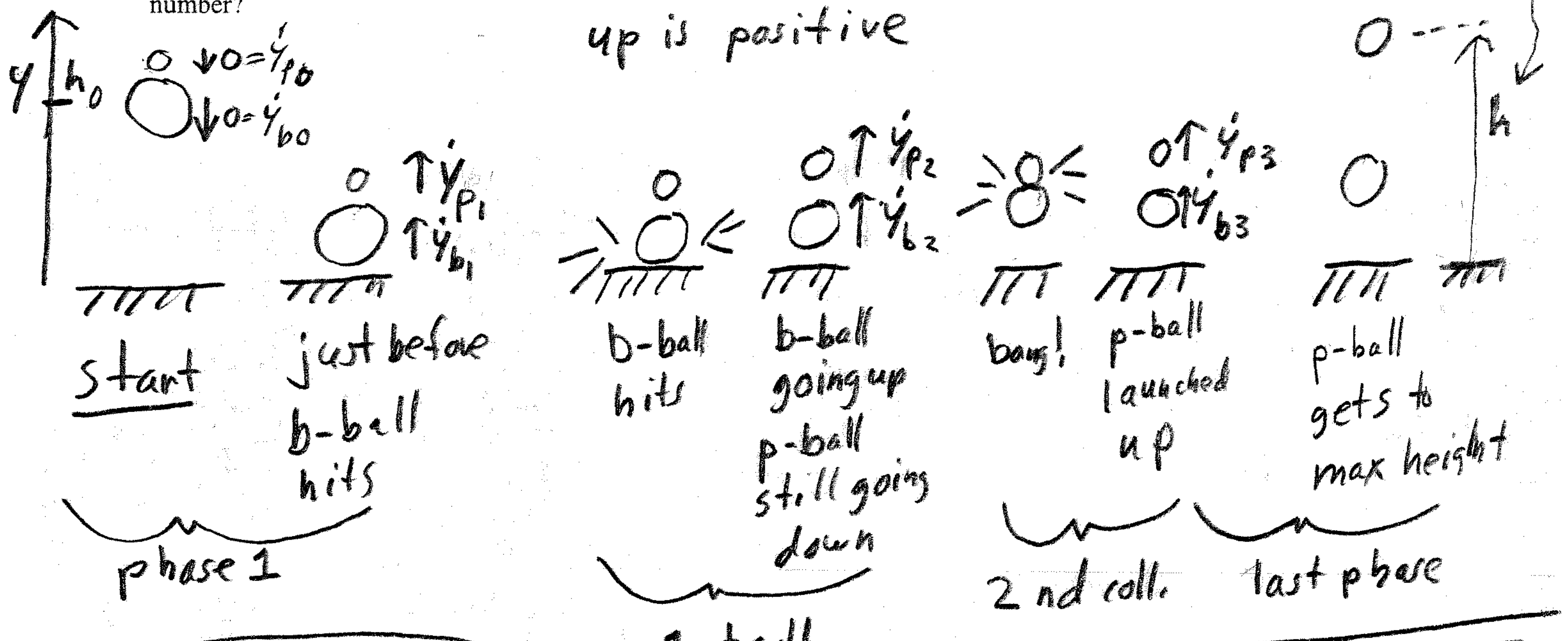
Problem 2: ______ /25

Problem 3: ______ /25

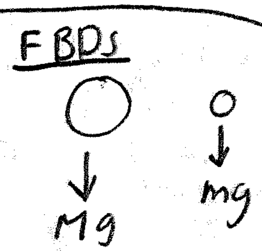
1a) A basketball M is dropped from height h_0 onto a hard surface. Falling just directly above it is a small ping-pong ball m. The Basketball hits the surface, bounces up and immediately hits the ping-pong ball which bounces up off it. Assume $M \gg m$. Assume $e = 1$ for both collisions. Neglect the size of the balls in relation to h ($h \gg$ ball radius). Neglect air friction. How high does the ping-pong ball bounce? of interest

b) Now assume that both collisions have coefficient of restitution e but $e \neq 1$. For what what value of e is the height of the ping-pong ball flight = h_0?

c) The answer to (b) above is a famous number. For one bonus point, do you know anything interesting about that number?



Phase 1



Cons. of En. $\Rightarrow Mgh_0 = \frac{1}{2} M \dot y_{b1}^2 \Rightarrow \dot y_{b1} = -\sqrt{2gh_0}$

Likewise, $\dot y_{p1} = -\sqrt{2gh_0}$ going down

1st Coll $\dot y_{b2} = -e \dot y_{b1}$, $\dot y_{p2} = \dot y_{p1}$

2nd Coll. Cons. of mom. $\Rightarrow \cancel{M} \dot y_{b2} = \cancel{M} \dot y_{b3}$ (because $M \gg m$)

Coll. eqn. $\dot y_{b3} - \dot y_{p3} = -e(\dot y_{b2} - \dot y_{p2})$ (1)

$\dot y_{p2} = \dot y_{p1} = \dot y_{b1}$

$\dot y_{b2} = -e \dot y_{b1}$

$\dot y_{b3} = \dot y_{b2} = -e \dot y_{b1}$

1a) cont'd

① is one eqn for $\dot{y}_{p3}$, solving

$$\dot{y}_{p3} = -e\dot{y}_{b1} + e(-e\dot{y}_{b1} - \dot{y}_{b1})$$
$$\dot{y}_{p3} = -(e^2+2e)\dot{y}_{b1} \quad ②$$

Take $e=1$, $\dot{y}_{p3} = -3\dot{y}_{b1} = -3(-\sqrt{2gh_0}) = 3\sqrt{2gh_0}$

Last phase

Cons. of energy $\cancel{m}gh = \frac{1}{2}\cancel{m}V_{p3}^2$

$$gh = \frac{1}{2}(3\sqrt{2gh_0})^2$$
$$= 9gh_0 \Rightarrow \boxed{h = 9h_0} \text{ (a)}$$

b) $h = h_0 \Rightarrow |\dot{y}_{p3}| = |\dot{y}_{p1}| \xRightarrow{②} e^2+2e = 1$

$$e^2+2e-1=0 \Rightarrow e = \frac{-2\pm\sqrt{8}}{2} \Rightarrow e = -1+\sqrt{2}$$

only sensible value

$\boxed{e \approx 0.414}$ (b)

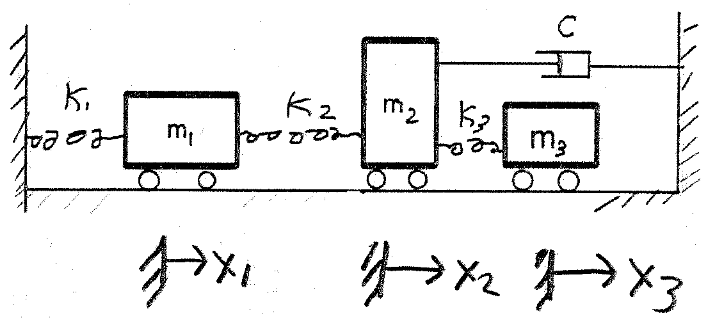
c) Actually, no!

2) As for missing information, please read the front cover.

a) Given the positions and velocities of the three masses find the acceleration of mass 2.

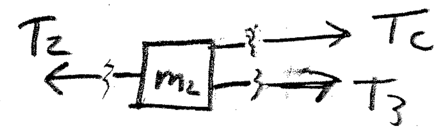
b) Assuming $k_2 = 0$ and $k_3 = 0$ and that mass 2 has an initial speed v, find the position of mass 2 as a function of time.

call it v_0



Assume all springs relaxed at $x_1 = x_2 = x_3 = 0$

FBD of mass 2



$T_2 = k_2(x_2 - x_1)$

$T_3 = k_3(x_3 - x_2)$

$T_c = c\dot{l} = -c\dot{x}_2$

LMB $\qquad \sum F = ma$

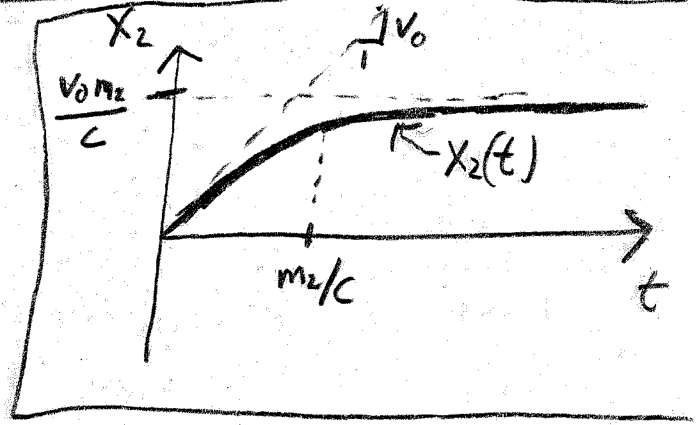
$$-T_2 + T_3 + T_c = m_2 a_2$$

$$\boxed{a_2 = \frac{1}{m_2}\left[-k_2(x_2 - x_1) + k_3(x_3 - x_2) - c\dot{x}_2\right]} \quad (a)$$

b) $k_2 = k_3 = 0 \Rightarrow \quad a_2 = \frac{-c}{m_2}\dot{x}_2$

define $v_2 = \dot{x}_2 \quad \Rightarrow \ddot{x}_2 = \frac{-c}{m_2}\dot{x}_2$

$$\Rightarrow \dot{v}_2 = \frac{-c}{m_2} v_2$$



$$\Rightarrow v_2 = v_0 e^{-(c/m_2)t}$$

$$\Rightarrow x_2(t) = \int_0^t v_2(t')dt' = \int_0^t v_0 e^{-(c/m_2)t'} dt' = \frac{-v_0 m_2}{c} e^{-(c/m_2)t'}\Big|_0^t$$

(assume $x_2(0) = 0$)

$$\Rightarrow \boxed{x_2(t) = \frac{v_0 m_2}{c}\left[1 - e^{-(c/m_2)t}\right]} \quad (b)$$

3) The Matlab text below is in one file. It is known to run without error. It is bad code because it has no comments and does not use suggestive variable names.

a) As accurately as possible show the output from running this file. Please annotate (write comments about and explain) the output with any key features or numbers.

b) Same, but changing `p = 0;` to `p = .1;`.
Do not do any long calculations (e.g., arithmetic with square roots) in in detail. Just show clearly how the solution is different than (a) above.

$Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ } just names for z_1 & z_2

$t_0 = 0$, $t_f \approx 2\pi$ (a bit less)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function prellq3()
 tspan=[0 6.2];      n = 10000;
 z0    = [1 0]';     p = 0;
 [tarray zarray] = eulmeth(tspan,z0,n,p);
 u = zarray(:,1);
 w = zarray(:,2);
 plot(u,w,'r',tarray, u,'b');
 axis('equal')
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

$\Rightarrow \boxed{X_0 = 1,\ Y_0 = 0}$ initial conditions

$\underbrace{Y \text{ vs } X}_{\text{red}}$ $\underbrace{X \text{ vs } t}_{\text{blue}}$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [tmat zmat] = eulmeth(tspan,z0,n,p)
tmat    = linspace(tspan(1),tspan(2),n+1);
h       = tmat(2)-tmat(1);
zmat    = zeros(n+1,length(z0));
zmat(1,:) = z0';
 for i=1:n;
    z            = zmat(i,:)';
    t            = tmat(i);
    znew         = z + h*rhs(t,z,p);
    zmat(i+1,:)  = znew';
 end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Solves ODEs numerically. [Exactly from lecture.]

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function  zdot = rhs(t,z,p)
  z1   = z(1);  X
  z2   = z(2);  Y
  zdot = [ z2;   -z1 - p*z2 ];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

$\Rightarrow \boxed{\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= -X - pY \end{aligned}}$ ODEs $\begin{pmatrix} 1a \\ 1b \end{pmatrix}$

$1a \Rightarrow \ddot{X} = \dot{Y} \overset{1b}{\Longrightarrow} \ddot{X} = -X - p\dot{X} \Rightarrow \boxed{\ddot{X} + p\dot{X} + X = 0}$ (2)

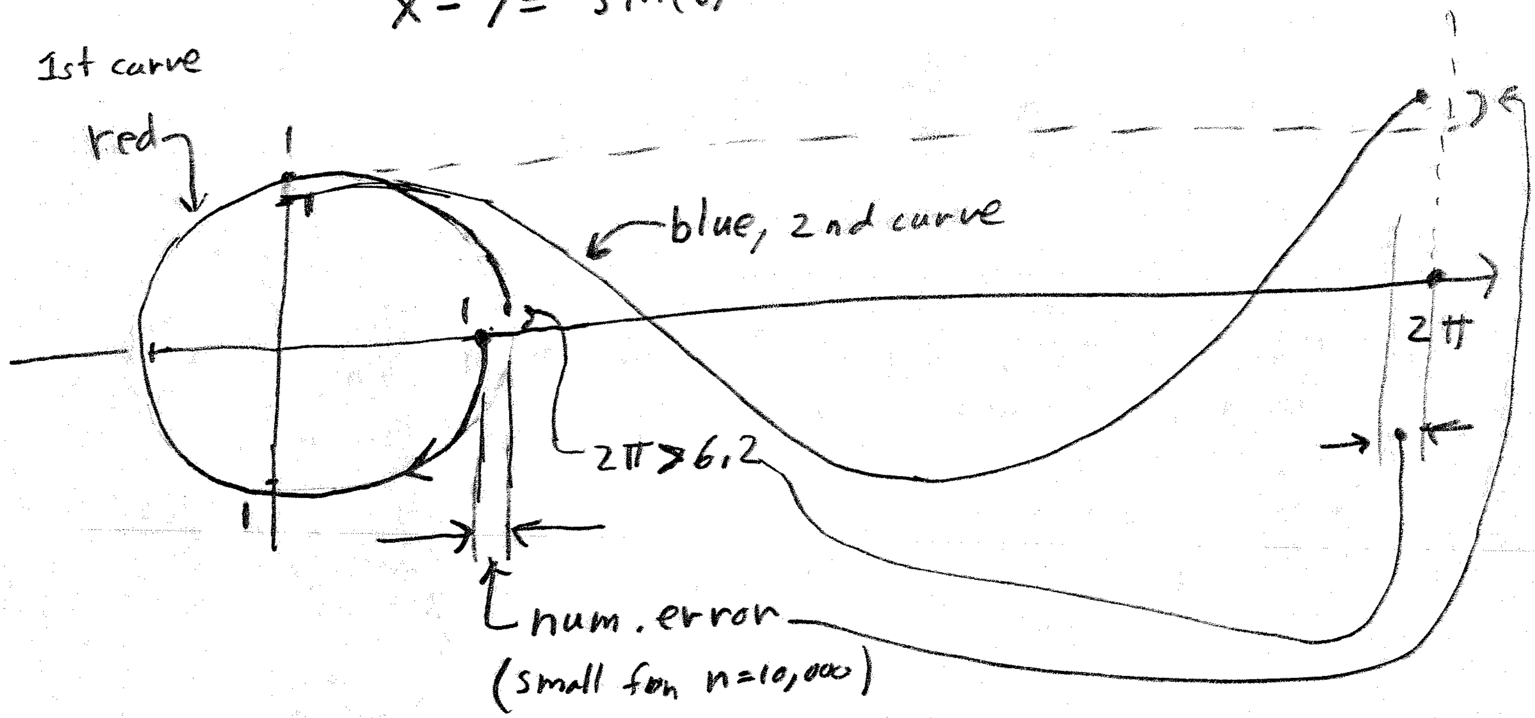
Damped harmonic oscillator!

3a) $p=0 \Rightarrow$ Computer is finding approx. soln. to this problem:

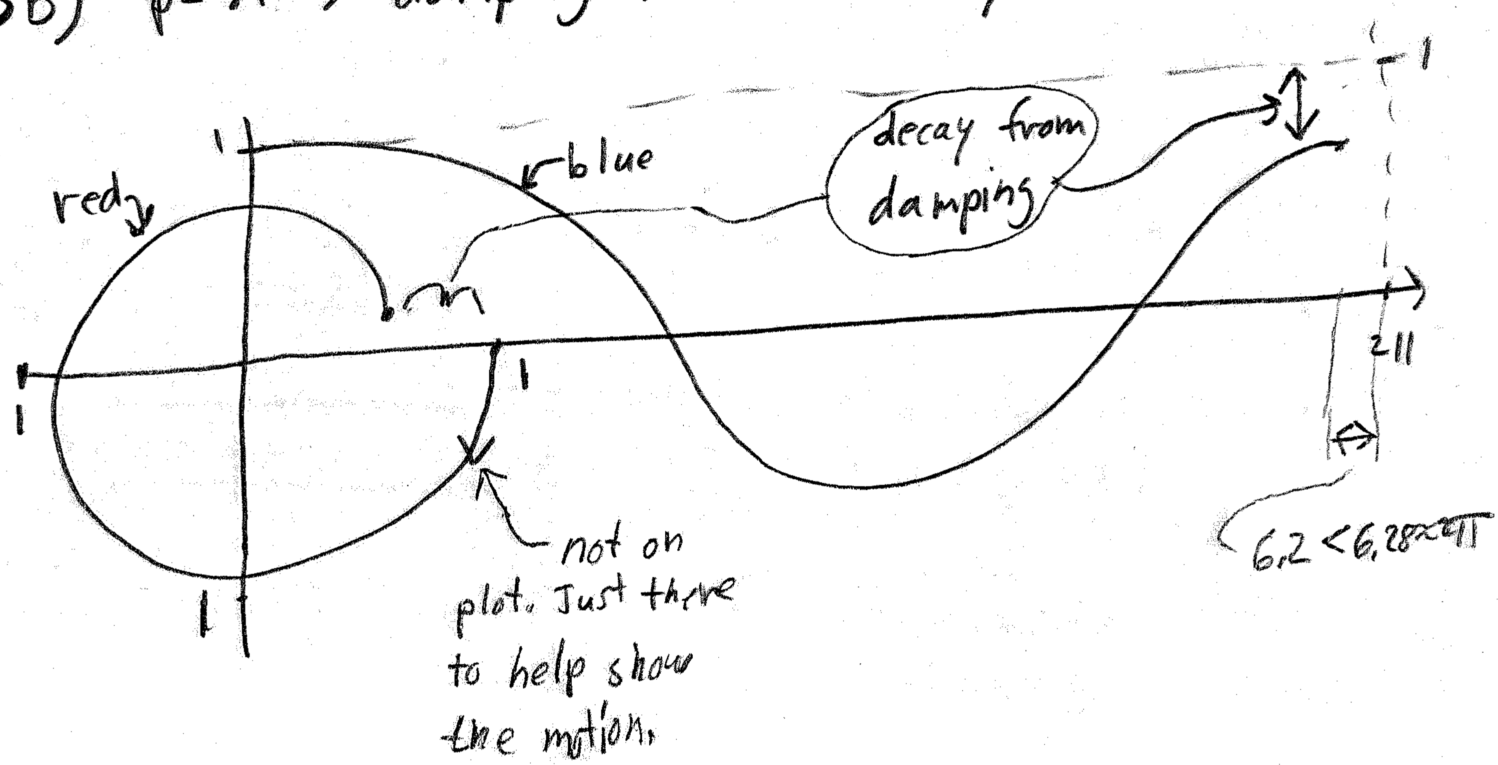
$$\boxed{\begin{array}{ll} \ddot{x}+x=0 & x(0)=1 \\ 0\le t\le 2\pi - \text{a little} & \dot{x}(0)=0 \end{array}}$$

Soln.: $x=\cos(t)$
$\dot{x}=y=-\sin(t)$

$x=u$
$y=w$



3b) $p=.1 \Rightarrow$ damping $\Rightarrow$ motions decay $[\ddot{x}+p\dot{x}+x=0]$



Your TA, Section # and Section time:

"SOLUTIONS"

Your name:

RUINA, ANDY

Cornell TAM/ENGRD 2030

Prelim 2

March 29, 2011

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- ☐ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
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Note the last page is **blank** for your use. Ask for more extra paper if you need it.

Problem 4: ____ /25

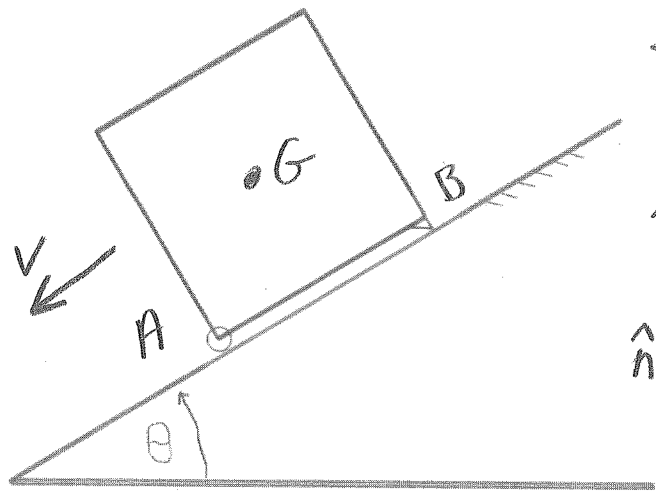
Problem 5: ____ /25

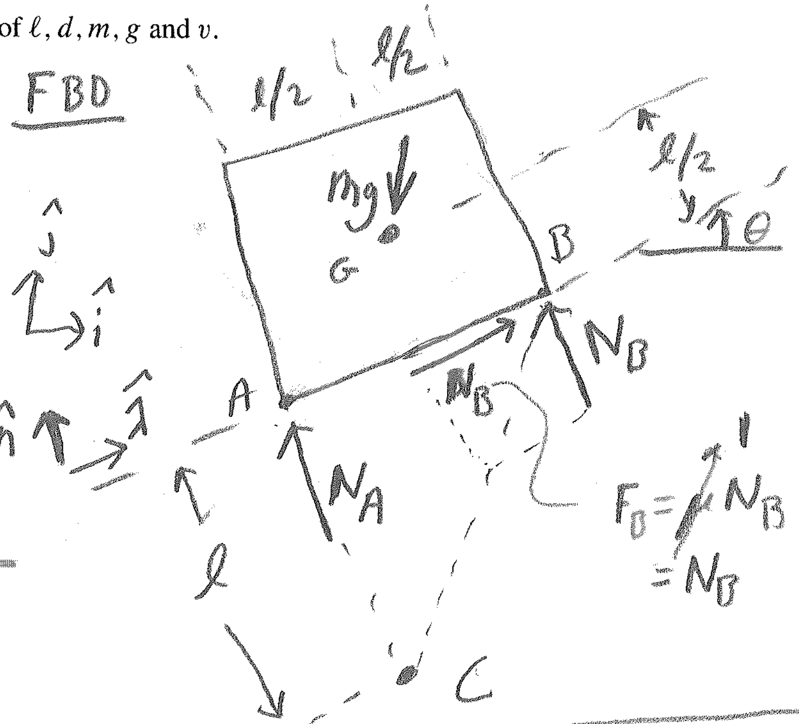
Problem 6: ____ /25

1) At the time of interest a uniform square suitcase with mass m, sides ℓ and thickness d is rolling and sliding down a ramp with speed $v > 0$. The front wheels (downhill) are ideal massless wheels. The other end is sliding on a rubber stub with $\mu = \tan(\phi) = 1$. Do not neglect gravity g.

a) For $\theta = 45°$ find $\dot v$. Answer in terms of some or all of ℓ, d, m, g and v.

b) For what value of θ is $\dot v = 0$?





AMB/C

$$\sum \vec M_{/C} = \dot{\vec H}_{/C}$$

$$\vec r_{G/C} \times (-\cancel{m} g\hat j) = \vec r_{G/C} \times \left(\cancel{m}\,\overbrace{\dot v(-\hat\lambda)}^{\vec a}\right)$$

$$\vec r_{G/C} = \frac{3\ell}{2}\hat n + \frac{\ell}{2}\hat\lambda$$

$$\left(\frac{3\ell}{2}\hat n + \frac{\ell}{2}\hat\lambda\right) \times (-g\hat j) = \left(\frac{3\ell}{2}\hat n + \frac{\ell}{2}\hat\lambda\right) \times (-\dot v\hat\lambda)$$

$\hat n \times \hat j = -\sin\theta\hat k$, $\quad \hat\lambda \times \hat\lambda = \vec 0$, $\quad \hat n \times \hat\lambda = -\hat k$, $\quad \hat\lambda \times \hat j = \cos\theta\hat k$

$$\left\{\left[\frac{3g\cancel{\ell}}{\cancel{2}}\sin\theta - \frac{\cancel{\ell}g}{\cancel{2}}\cos\theta\right]\hat k = \frac{-3\cancel{\ell}}{\cancel{2}}\dot v\hat k\right\}$$

$$\{\ \}\cdot\hat k \Rightarrow \quad \dot v = g\cdot\left[\frac{\cos\theta}{3} - \sin\theta\right] \quad (*)$$

$$\theta = 45° \Rightarrow \quad \dot v = g\frac{\sqrt2}{2}(-2/3) = \boxed{-g\sqrt2/3 = \dot v}\ (a)$$

$$\dot v = 0 \overset{(*)}{?} \Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac13 \Rightarrow \boxed{\theta = \tan^{-1}(1/3)}\ (b)$$

Alt. soln. to b.

$\dot v = 0 \Rightarrow$ statics $\Rightarrow$ G above C

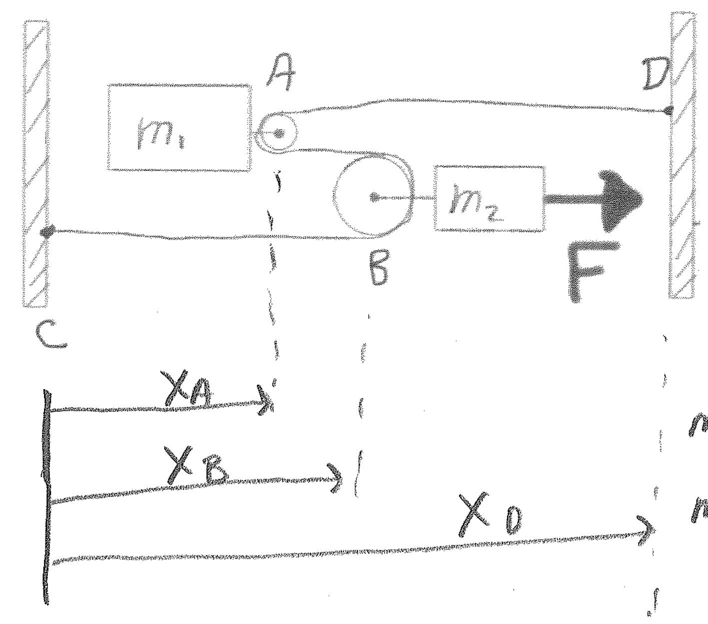
$\Rightarrow \tan\theta = 1/3$

Check (*)

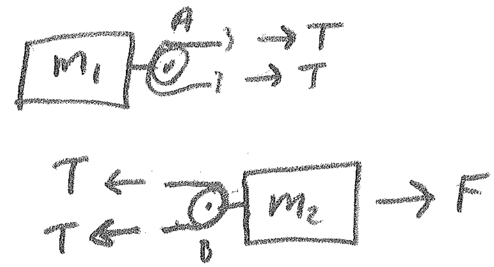
$\theta = 90° \Rightarrow \cos\theta = 0$, $\sin\theta = 1$

$\underbrace{\dot v = -g}_{\text{as expected}}$ ✓

2) In terms of m_1, m_2 and F find a_B, the acceleration of B.



FBDs



LMB

m_2: $\quad F - 2T = a_B m_2 \quad$ (2)

m_1: $\quad 2T = a_A m_1 \quad$ (3)

Kinematics

$$\{ \text{const} = l = (x_D - x_A) + (x_B - x_A) + x_B + \pi r_A + \pi r_B \}$$

$$\frac{d^2}{dt^2}\{\ \} \Rightarrow \quad 0 = (0 - \ddot{x}_A) + (\ddot{x}_B - \ddot{x}_A) + \ddot{x}_B + 0 + 0$$

$$\Rightarrow \underbrace{\ddot{x}_A}_{a_A} = \underbrace{\ddot{x}_B}_{a_B} \quad (1)$$

[both masses have same motion.]

Apply (1) to (3) and sub into (2)

$$\Rightarrow \quad F - a_B m_1 = a_B m_2 \quad \Rightarrow \quad \boxed{a_B = \frac{F}{m_1 + m_2}}$$

to the right

3) A canonball of mass $m = 2\,\text{kg}$ is launched from the top edge of a tall cliff, with a velocity of $\vec{v}_0 = (300\hat{\imath} + 400\hat{\jmath})\,\text{m/s}$. x and y are the distances from the launch to the right and up, respectively. Gravity points down with $g = 10\,\text{m/s}^2$. There is a quadratic drag force on the ball, opposing the motion, with magnitude

$$F = cv^2 \qquad \text{where } c = 0.1\,\text{N}/(\text{m/s})^2.$$

Goal: find the$x$ position of the ball, in meters, after 100 seconds.

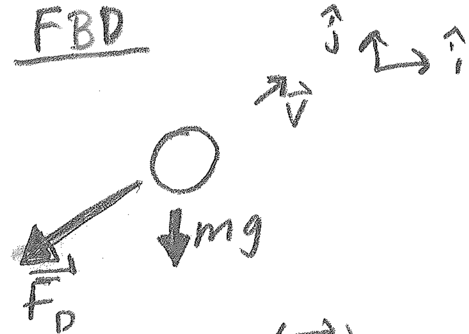
a) Supply the two missing blocks of code (`A` and `B` below), so that running `prel2q3.m` below will provide the desired numerical solution to the equations of motion.

Please use reasonably-named intermediate variables to make your code as readable as possible. No other changes or additions to the code below are needed, but you can make other changes if you like.

b) Extra credit (easy): Estimate the velocity at $t = 100\,\text{s}$. The answer should be a single arithmetic formula involving the numbers above. An exact analytic formula is not possible. But it is possible, making reasonable assumptions based on the nature of the motion, to get an accurate estimate.

b) Extra credit (challenge): Estimate the value of x at $t = 100\,\text{s}$. This estimate need not be as good as that above. Again an arithmetic formula is desired.

FBD




```
function prel2q3()
t_end = 100; x0=0; y0=0; vx0=300; vy0 = 300;
m = 2; c = 0.1;   g = 10;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%      A. FILL IN MISSING LINES HERE     %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[tarray zarray] = rk2(tspan,z0,n,p);
% The next line nicely prints out z(end,1)
disp(['x(100s)='  num2str(z(end,1))  'm'])
end
```

400 array zarray

$\vec{F}_D = -cv^2\left(\frac{\vec{v}}{v}\right)$

unit vector in $\vec{v}$ dir.

```
function [tmat zmat] = rk2(tspan,z0,n,p)
%Midpoint integration. No problems here.
tmat    = linspace(tspan(1),tspan(2),n+1);
h       = tmat(2)-tmat(1);
zmat    = zeros(n+1,length(z0));
zmat(1,:) = z0';
 for i=1:n;
    z       = zmat(i,:)';      t     = tmat(i);
    ztemp   = z +(h/2)*rhs( t,       z,      p);
    znew    = z +   h   *rhs((t+h/2),ztemp,p);
    zmat(i+1,:) = znew';
 end
end
```

LMB

$\sum \vec{F} = m\vec{a}$

$-mg\hat{j} - cv|\vec{v}| = m\dot{\vec{v}}$

$\Rightarrow$ (1) $\boxed{\begin{aligned}\dot{\vec{r}} &= \vec{v}\\ \dot{\vec{v}} &= \frac{1}{m}\vec{F}_D - mg\hat{j}\end{aligned}}$

$\vec{F}_D$: $-cv\vec{v}$ or $-cv^2\left(\frac{\vec{v}}{v}\right)$

Eqs. of motion

```
function   zdot = rhs(t,z,p)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%      B. FILL IN MISSING LINES HERE     %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
end
```

```
%A. MISSING CODE
p.m = m; p.c = c  ;  p.g = g;
tspan = [0 t_end];  n = 10000;
z0    = [x0 y0 vx0 vy0]';
```

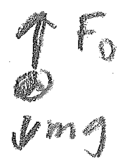
A

```
% B. MISSING CODE
m=p.m; c = p.c;  g = p.g;
r= z(1:2);  vvec = z(3:4); %position and velocity vectors
v = norm(vvec);  uv = vvec/v; % mag of vel, unit vector in dir. of vel.

Fdrag = -c*v^2*uv; % drag force, a 2-comp vector

rdot    = vvec;                         % first two ODEs
vvecdot = (1/m)*(Fdrag -[0 m*g]'); % 2nd   two ODEs (F = ma)

zdot = [rdot; vvecdot];  % All 4 comps of rhs.
```

1

a) Assume terminal velocity

$\Rightarrow \dot{v}=0$



$F_D = mg$

$cv^2 = mg$

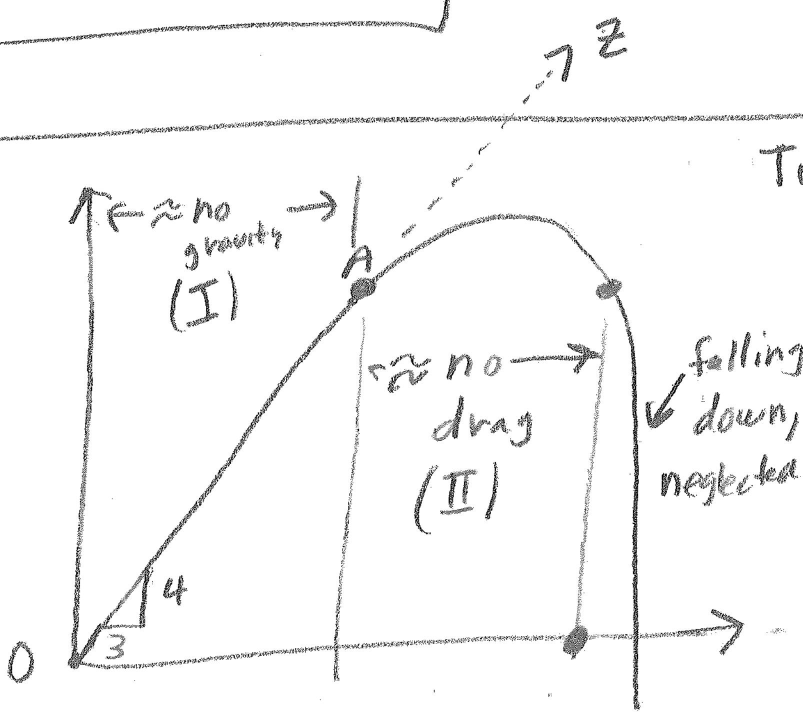
$v = \sqrt{mg/c}$

$v = \sqrt{\dfrac{(2\ kg)(10\ m/s^2)}{0.1\ N/(m/s)^2}}$

$v = \sqrt{200}\ m/s$

a $\boxed{\vec{V} = -\sqrt{200}\ m/s\ \hat{j} = -10\sqrt{2}\ m/s\ \hat{j}}$

b)



Two regimes:

I: slowing (neglect gravity)

II. top of trajectory (neglect drag)

3b) (cont'd)

Regime I:

$$m\frac{dV}{dt} = -cV^2 \Rightarrow \frac{dV}{dz}V = -\frac{c}{m}V^2$$

$$\Rightarrow \frac{dV}{V} = -\frac{c}{m}dx \Rightarrow \ln V_A - \ln V_0 = -\frac{c}{m}z$$

$$\Rightarrow z_I = \frac{m}{c}\ln\frac{V_0}{V_A}$$

use V_A = term vel. for transition

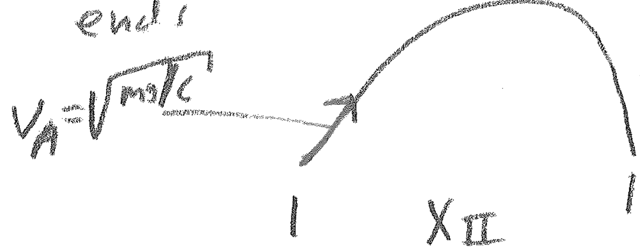
$$\Rightarrow z_I = \frac{m}{c}\ln\left(\frac{V_0}{\sqrt{mg/c}}\right)$$

$$x_I = \frac{3}{5}z_I \quad \text{(project on to z axis)}$$

Regime II

parabolic flight starts where regime I ends

t = time of flight $= 2\frac{4}{5}V_A/g$



$$x_{II} = \overbrace{\frac{3}{5}V_A}^{x\,vel}\cdot t = 2\cdot\frac{3}{5}\cdot\frac{4}{5}V_A^2/g$$

$$\text{Estimate} = \boxed{x_{TOT} = x_I + x_{II} = \frac{3}{5}\frac{m}{c}\ln\left(\frac{\sqrt{mg/c}}{V_0}\right) + \frac{24}{25}\,mg/\cancel{k}g}$$

Using #s from problem $\Rightarrow$

$$x_{TOT} = \left[\frac{3}{5}20\ln\left(\frac{500}{\sqrt{200}}\right) + \frac{24}{25}\frac{20}{1}\right]\text{m} \quad *$$

POSTPLAY MATLAB check:

ODE Soln $\Rightarrow x_{TOT} = 62.4$m

$*$ above $\Rightarrow x_{TOT} = 61.98$m

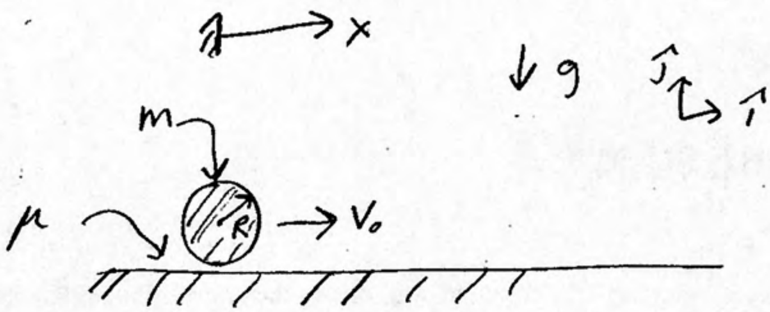
Pretty close! (Better than it deserves)

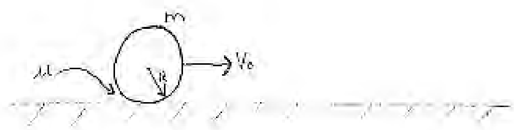
1) A uniform cylinder (mass m, radius R) is initially moving horizontally (velocity of its center of mass is $\vec{v}(0) = v_0\hat{\imath}$, with $v_0 > 0$) and not rotating ($\vec{\omega}_0 = \vec{0}$) when placed on a horizontal flat smooth frictional surface with friction coefficient μ. It slides for a while and then rolls. Answer in terms of some or all of $v_0, m, R, g, \mu, \hat{\imath}$ and $\hat{\jmath}$.

a) When the cylinder eventually rolls what is the velocity of the center of mass?

b) When it eventually rolls what is its angular velocity?

c) How far does it slide before it starts rolling?

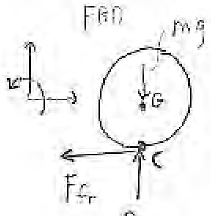




Cylinder is initially moving horizontally at $V = V_0\hat{i}$ without rotation $\vec{\omega}_0 = \vec{0}$ is on a frictional surface and eventually starts rolling after sliding

a) what is the cylinder velocity once it starts rolling

FBD



AMB/c

$$\sum \vec{M}_{/c} = \sum \dot{\vec{H}}_{/c}$$

$$\int_0^t 0 = \int_0^t I^G\dot{\vec{\omega}} + \vec{r}_{G/c} \times m\vec{a}_G \, dt$$

$$0 = I^G(\vec{\omega} - \vec{\omega}_0) + \vec{r}_{G/c} \times m(\vec{V} - \vec{V}(0))$$

$(\vec{a}_G = a\hat{i}$ due to horizontal surface

$\vec{r}_{G/c} = R\hat{j}\ )$

$I^G = \frac{mR^2}{2}$

$\vec{V} = V\hat{i}$

let $\vec{\omega} = (\omega)(-\hat{k})$

$$I^G\cancel{\vec{\omega}_0} + m\vec{r}_{G/c} \times \vec{V}(0) = I^G\vec{\omega} + \vec{r}_{G/c} \times m\vec{V}$$

$$\left\{ mR\hat{j} \times V_0\hat{i} = \frac{-mR^2}{2}\omega\hat{k} + R\hat{j} \times mV\hat{i} \right\} \cdot \hat{k}$$

(1) $\quad -mRV_0 = \frac{-mR^2}{2}\omega - mRV$

from rolling condition, $\vec{\omega} \times R_{C/G} + \vec{V} = 0 \quad \Rightarrow \quad \omega(-\hat{k}) \times R(-\hat{j}) + V\hat{i} = 0$

$$-\omega R\hat{i} + V\hat{i} = 0$$

$$V = \omega R \quad \text{or} \quad \omega = V/R \qquad (2)$$

subs (2) into (1) and simplifying somewhat

$$V_0 = \frac{1}{2}V + V \Rightarrow \boxed{V = \frac{2}{3}V_0}$$

substituting V into (2), $\boxed{\omega = \frac{2}{3}\frac{V_0}{R}}$

c) using energy, translational only

$W = \Delta KE$

$-F_{fr}\, d = \frac{1}{2} m V^2 - \frac{1}{2} m V_o^2$

$d = \dfrac{(\frac{2}{3}V_o)^2 - V_o^2}{-2\mu g}$

$d = \dfrac{\frac{4}{9}V_o^2 - V_o^2}{-2\mu g}$

$\boxed{d = \frac{5}{18}\frac{V_o^2}{\mu g}}$

LMB

$mg(-\hat{j}) + R\hat{j} - F_{fr}(\hat{i}) = m\vec{a}$

$\{\}\cdot\hat{j} \Rightarrow -mg + R = 0$, $R = mg$

from friction, $F_{fr} = \mu R$

alternately;

from $\{\}\cdot\hat{i}$, $a = -\mu g$

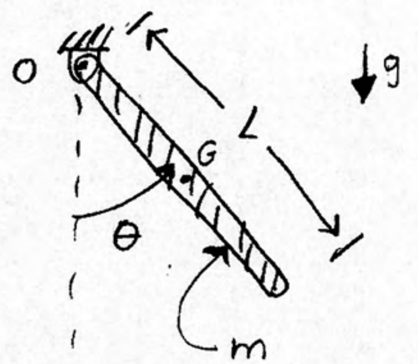
$V = -\mu g t + V_o$

$X = \frac{-\mu g}{2} t^2 + V_o t$

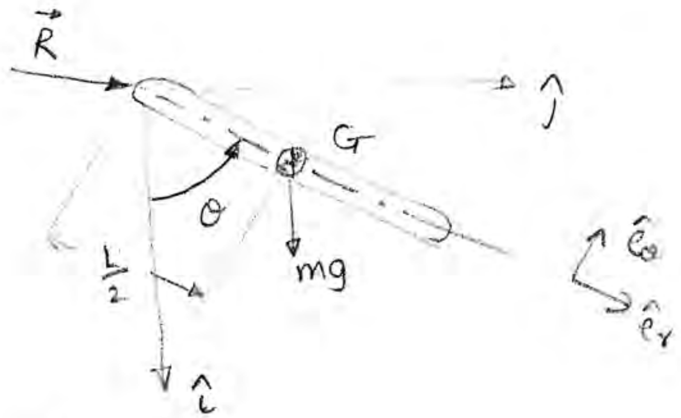
find t for $V = \frac{2}{3}V_o \Rightarrow \frac{2}{3}V_o = -\mu g t + V_o \Rightarrow t = \dfrac{V_o}{3\mu g}$

$X = \frac{-\mu g}{2}\left(\frac{V_o}{3\mu g}\right)^2 + \frac{V_o^2}{3\mu g} = \left(\frac{-1}{18} + \frac{1}{3}\right)\frac{V_o^2}{\mu g} = \boxed{\frac{5}{18}\frac{V_o^2}{\mu g}}$

2) A uniform rigid stick (length L, mass m) hangs from a hinge with negligible friction at one end (point O). Immediately after it is released from rest with initial angle $\theta = \theta_0$ what is the force (a vector) of the hinge on the stick? Answer in terms of some or all of $m, g, L, \theta_0, \hat{\imath}$ and $\hat{\jmath}$. Define $\hat{\imath}$ and $\hat{\jmath}$ any way you like with a clear sketch



② FBD



Find $\vec{R}$.

AMB$_{/O}$ $\quad \dot{\vec{H}}_{/O} = \vec{M}_{/O}$

$$\Rightarrow \quad \vec{r}_{G/O} \times m\vec{a}_G + I_G \ddot{\theta}\,\hat{k} = \vec{r}_{G/O} \times mg\,\hat{\imath}$$

$$\Rightarrow \quad m\frac{L}{2}\,\hat{e}_r \times \left\{ -\frac{L}{2}\dot{\theta}^2\hat{e}_r + \frac{L}{2}\ddot{\theta}\,\hat{e}_\theta \right\} + I_G\ddot{\theta}\,\hat{k} = mg\frac{L}{2}\,\overbrace{\hat{e}_r \times \hat{\imath}}^{-\sin\theta\,\hat{k}}$$

$$\Rightarrow \quad \left(\frac{mL^2}{4} + I_G\right)\ddot{\theta}\,\hat{k} = -mg\frac{L}{2}\sin\theta\,\hat{k}$$

$\{\ \}\cdot\hat{k}$ and put $I_G = \frac{mL^2}{12}$

$$\ddot{\theta} = -\frac{3}{2}\,\frac{g}{L}\sin\theta$$

LMB $\quad \vec{F} = m\vec{a}_G$

$$\vec{R} + mg\,\hat{\imath} = m\left\{ -\frac{L}{2}\cancelto{0}{\dot{\theta}^2}\hat{e}_r + \frac{L}{2}\ddot{\theta}\,\hat{e}_\theta \right\}$$

$$\vec{R} + mg\,\hat{\imath} = \frac{mL}{2}\left\{ -\frac{3}{2}\frac{g}{L}\sin\theta\left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right)\right\}$$

Solving for $\vec{R}$ and putting $\theta = \theta_0$ gives

$$\boxed{\vec{R} = mg\left\{ \left(\frac{3}{4}\sin^2\theta_0 - 1\right)\hat{\imath} - \frac{3}{4}\sin\theta_0\cos\theta_0\,\hat{\jmath} \right\}}$$

3) A two-dimensional object $\mathcal{B}$ moves in the plane. At the instant of interest its center of mass has position $\vec{r}_G = \vec{r}_{G/O}$, velocity $\vec{v}$. and counter-clockwise angular velocity $\omega \neq 0$.

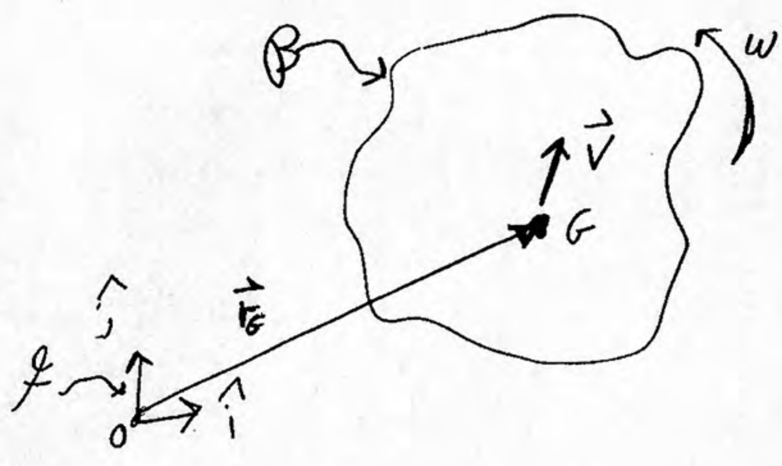
> *Interesting fact:*
> *So long as $\omega \neq 0$ a point C, called the 'instantaneous center of rotation' (COR), always exists such that*
>
> - *point C is instantaneously stationary: $\vec{v}_C = \vec{0}$, and*
> - *the velocities of all points D on the object are calculated by treating the object as rotating about C: $\vec{v}_D = \vec{\omega} \times \vec{r}_{D/C}$.*
>
> *Point C is not necessarily literally on the object, but rather is somewhere on an infinite rigid extension of the object (that is, C is on a large imagined rigid piece of graph paper glued to the object).*

a) Find $\vec{r}_C = \vec{r}_{C/O}$ in terms of some or all of $\vec{r}_G, \vec{v}, \omega, \hat{\imath}, \hat{\jmath}$ and $\hat{k}$. That is, write a formula that answers the question: $\vec{r}_C =$? If you happen to have memorized this formula, you must show how to obtain it.

b) For the special case that

$$\vec{r}_G = 2\,\mathrm{m}\hat{\imath},$$
$$\vec{v} = 3\,\mathrm{m/s}\hat{\imath} + 4\,\mathrm{m/s}\hat{\jmath} \qquad \text{and}$$
$$\omega = 1\,\mathrm{s}^{-1}$$

find x_C and y_C. A neat sketch may help your work and may help you better communicate your understanding.



(3) Using the definition of instantaneous center

a)

$$\vec{V}_G = \vec{\omega} \times \vec{r}_{G/C}$$

$$\vec{V} = \vec{\omega} \times (\vec{r}_G - \vec{r}_C)$$

Cross both sides with $\vec{\omega}$

$$\vec{\omega} \times \vec{V} = \vec{\omega} \times \{ \vec{\omega} \times (\vec{r}_G - \vec{r}_C) \}$$

$$= -|\vec{\omega}|^2 (\vec{r}_G - \vec{r}_C)$$

Solving for $\vec{r}_C$ gives

$$\boxed{\vec{r}_C = \vec{r}_G + \frac{\vec{\omega} \times \vec{V}}{|\vec{\omega}|^2}}$$

b) Put $\vec{r}_G = 2\hat{i}$ $\quad \vec{\omega} = 1\hat{k}$ $\quad \vec{V} = 3\hat{i} + 4\hat{j}$

$$\vec{r}_C = 2\hat{i} + \frac{\hat{k} \times \{3\hat{i} + 4\hat{j}\}}{1^2}$$

$$= 2\hat{i} + 3\hat{j} - 4\hat{i}$$

$$\boxed{\vec{r}_C = -2\,\mathrm{m}\,\hat{i} + 3\,\mathrm{m}\,\hat{j}}$$

Your TA, Section # and Section time:

"SOLUTIONS"

Your name:

ANDY RUINA

Cornell TAM/ENGRD 2030

Final exam

May 10, 2013

No calculators, books or notes allowed.

5 Problems, 150 minutes (+no extra time: University policy $\Rightarrow$ *budget your time!*)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 13: ____/25

Problem 14: ____/25

Problem 15: ____/25

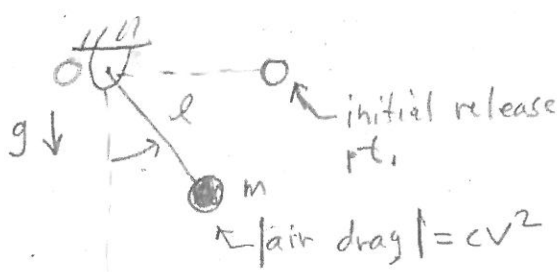
Problem 16: ____/25

Problem 17: ____/25

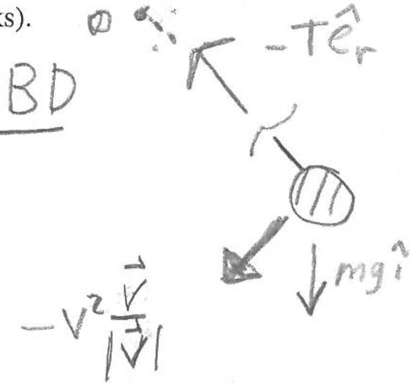
13) 2D, with gravity g. A mass m is attached to the end of a negligible-mass rigid rod with a length ℓ. The other end of the rod is attached to a hinge with negligible friction. The mass is slowed by air friction which resists motion with a force with magnitude $|F| = cv^2$ where $v = |\vec{v}|$ is the speed of the mass. The pendulum is released from rest at time $t = 0$ with the rod horizontal and to the right of the hinge. Assume any non-zero positive values that please you for all parameters (*e.g.*, g, m, ℓ, c, and t_1). Do *not* attempt an analytic solution.

a) Using ODE23 or ODE45 write all the Matlab commands needed to find the speed of the mass at time t_1. Some partial credit if you never learned ODE23 or ODE45 and can write your own ODE solver.

b) The output to the command window should be 'At time ____ the speed is ____.' (with numbers, calculated by the computer, instead of blanks).



FBD



Drag Force

$$\vec{F}_D = -v^2\frac{\vec{v}}{|\vec{v}|}c$$

$$\vec{F}_D = -c(\ell\dot{\theta})^2\frac{\ell\dot{\theta}\hat{e}_\theta}{+\sqrt{(\ell\dot{\theta})^2}}$$

$$\boxed{\vec{F}_D = -c\ell^2\dot{\theta}|\dot{\theta}|\hat{e}_\theta}$$

AMB$_{/0}$:

$$\sum \vec{M}_{/0} = \dot{\vec{H}}_{/0}$$

$$\Rightarrow \ell\hat{e}_r\times(-mg\hat{j} - \ell^2\dot{\theta}|\dot{\theta}|\hat{e}_\theta) = \ell\hat{e}_r\times[-\ell\dot{\theta}^2\hat{e}_r + \ell\ddot{\theta}\hat{e}_\theta]m$$

$$\Rightarrow -\ell mg\sin\theta\hat{k} - c\ell^3\dot{\theta}|\dot{\theta}|\hat{k} = \ell^2 m\ddot{\theta}\hat{k}$$

$$\{\}\cdot\hat{k} \Rightarrow \boxed{\ddot{\theta} = \underbrace{-\frac{g}{\ell}\sin\theta - \frac{c\ell}{m}\dot{\theta}|\dot{\theta}|}_{*}}$$

$$\boxed{|\vec{v}| = |\dot{\theta}|\ell}$$

$$\omega \equiv \dot{\theta} \Rightarrow \boxed{\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= *\end{aligned}}$$

```
function   givemeahighgrade( )
p.L=1;  p.c=1; p.m=1; p.g=1;  t1=10;
tspan= [0  t1];
theta0 = pi/2;  omega0 = 0;
z0= [ theta0   omega 0]';

[t zarray]= ode45(@myrhs, tspan, z0,[], p);

vend= abs(p.L * zarray(end, 2));

disp(['At time ' num2str(t1) ...
     ' the speed is ' num2str(vend)])

end


function  zdot= myrhs(t, z, p)
theta= z(1);  omega= z(2)

thetadot= omega;

omegadot= -p.c *(p.L/p.m) * omega * abs(omega) ...
               - (p.g/p.L) sin(theta);
zdot = [ thetadot ; omegadot];
end
```

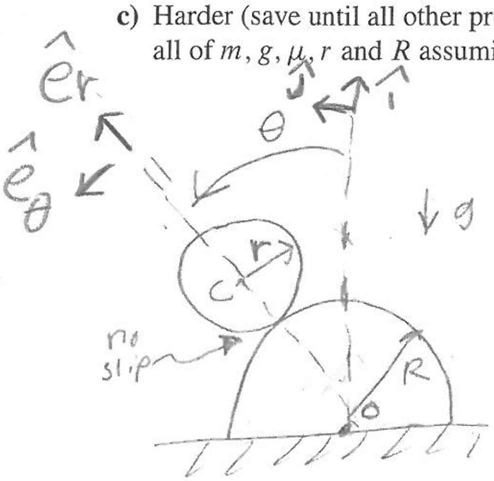
the heart of the matter

14) 2D, with gravity g. A solid uniform disk with radius r and mass m rolls on the top of a rigid unmoving hollow pipe with radius R. Line OC, between the center of the pipe and the center of the disk makes an angle of θ CCW (counterclockwise) from straight up. Assume θ and $\dot{\theta}$ are small enough, and μ big enough, so there is no separation or slip.

a) Find the equations of motion (That is, find $\ddot{\theta}$ in terms of some or all of $\theta, \dot{\theta}, m, g, r, \mu$ and R).

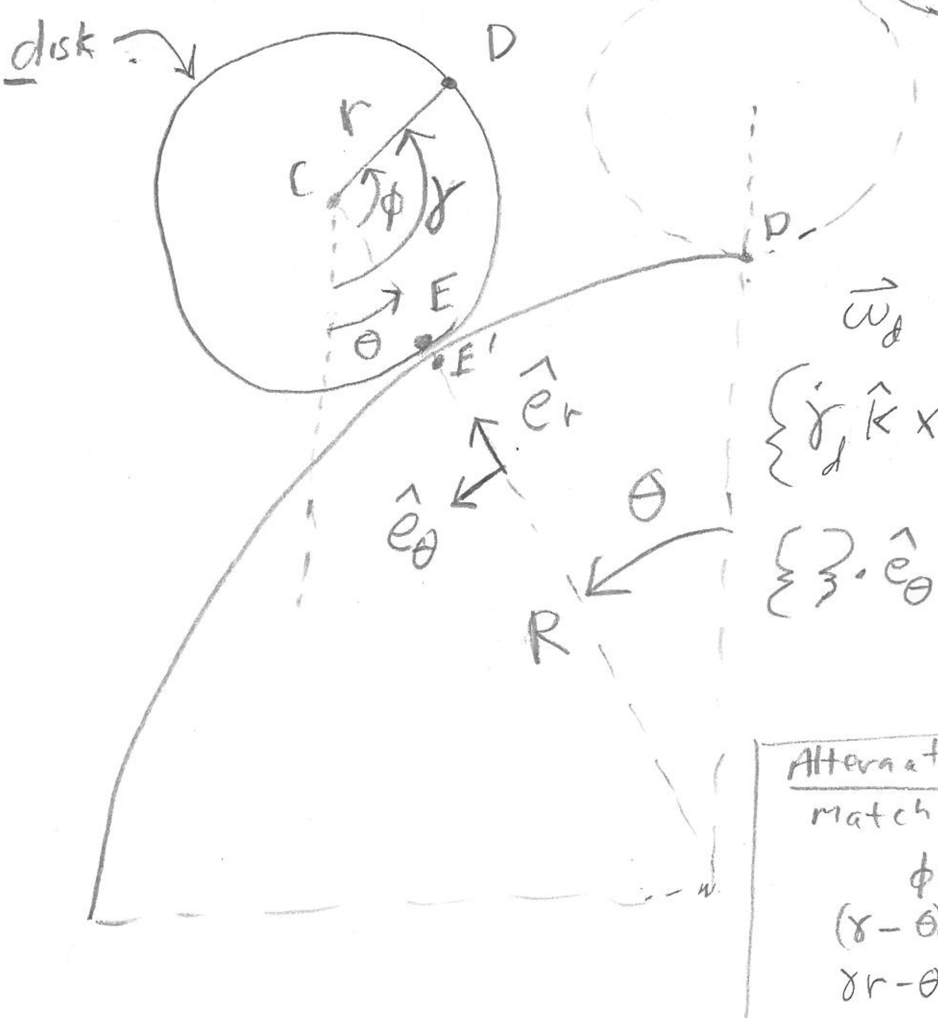
b) Find a function f so that the equation $0 = f(\theta, \dot{\theta}, m, g, \mu, r, R)$ describes the condition when the wheel would first lose contact.

c) Harder (save until all other problems are done). Find the angle θ when contact is first lost in terms of some or all of m, g, μ, r and R assuming that the rolling starts from rest at $\theta_0 = 0^+$ and all rolling is without slip.



FBD

Kinematics



$$\boxed{\vec{\omega}_d = \dot{\gamma}\hat{k}}$$

$$\vec{V}_E = \vec{V}_{E'}$$

$$\vec{\omega}_d \times \vec{r}_{E/C} + \vec{V}_C = \vec{0}$$

$$\left\{ \dot{\gamma}_d \hat{k} \times (-r\hat{e}_r) + (R+r)\dot{\theta}\hat{e}_\theta = \vec{0} \right\}$$

$$\{\} \cdot \hat{e}_\theta \Rightarrow -\dot{\gamma} r + (R+r)\dot{\theta} = 0$$

$$\Rightarrow \boxed{\dot{\gamma} = \frac{R+r}{r}\dot{\theta}} \; *$$

Alternative Kinematics

match arc lengths:

$$\phi r = \theta R$$

$$(\gamma - \theta) r = \theta R$$

$$\gamma r - \theta r = \theta R$$

$$\gamma r = (r+R)\theta$$

$$\boxed{\gamma = \frac{R+r}{r}\theta} \; *$$

(again)

(14 cont'd)

AMB$_{/E}$:
$$\sum \vec{M}_{/E} = \dot{\vec{H}}_{/E}$$

$$\underbrace{\vec{r}_{c/E}}_{r\hat{e}_r} \times (-mg\hat{\imath}) = \vec{r}_{c/E} \times m\vec{a}_c + I\dot{\omega}_d \hat{k}$$

$\dot{\omega}_d = \frac{R+r}{r}\ddot{\theta}$ from *

$$\vec{a}_c = -\dot{\theta}^2(R+r)\hat{e}_r + \ddot{\theta}(R+r)\hat{e}_\theta$$

$$\left\{ mgr\sin\theta\,\hat{k} = \left[mr(R+r)\ddot{\theta} + I\left(\frac{R+r}{r}\right)\ddot{\theta}\right]\hat{k} \right\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow \quad \ddot{\theta} = \frac{mgr\sin\theta}{mr(R+r) + I\frac{R+r}{r}}$$

$I = mr^2/2$ uniform disk

$$= \frac{g\sin\theta}{(R+r) + (R+r)/2}$$

$$\boxed{\ddot{\theta} = \frac{2g}{3(R+r)}\sin\theta} \quad \text{a}$$

LMB $\left\{\sum \vec{F} = \dot{\vec{L}}\right\}\cdot\hat{e}_r$

$$\Rightarrow N = -(R+r)\dot{\theta}^2 m + mg\cos\theta$$

Liftoff $\Rightarrow$ $N=0$

$$\Rightarrow \text{(b)} \quad \boxed{\cos\theta = \frac{(R+r)\dot{\theta}^2}{g}}$$

C) Cons. of Energy

$$E_0 = E_1$$

$$(R+r)\cancel{m}g = (R+r)\cancel{m}g\cos\theta + m(R+r)^2\dot{\theta}^2/2 + I\left(\frac{R+r}{r}\dot{\theta}\right)^2/2$$

$I = \frac{1}{2}\cancel{m}r^2$

$$\Rightarrow (R+r)g(1-\cos\theta) = \left[\frac{(R+r)^2}{2} + \frac{(R+r)^2}{4}\right]\dot{\theta}^2$$

$$\Rightarrow \dot{\theta}^2 = \frac{4}{3}\frac{(1-\cos\theta)g}{R+r} \quad (**)$$

apply ** to b $\Rightarrow$

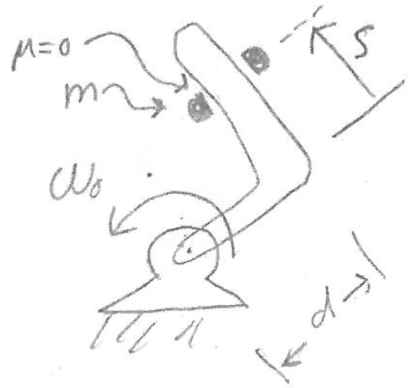
$$\cos\theta = \frac{4}{3}(1-\cos\theta)$$

$$\Rightarrow \frac{7}{3}\cos\theta = \frac{4}{3} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{4}{7}\right)} \quad \text{c}$$

15) 2D, no gravity. A bead with mass m slides on an L-shaped (right angle at bend) frictionless rigid rod which is turned by a motor at constant angular velocity $\vec{\omega}_0 = \omega_0\hat{k}$. The bead only moves on the straight part of the bar marked by s. If s passes through zero in your solution, only consider until $s = 0$.

a) Find the equations of motion of the bead. That is, find $\ddot{s}$ in terms of some or all of $\omega_0, s, \dot{s}, m$ and d.

b) Given that $s(0) = s_0 > 0$ and $\dot{s}(0) = 0$, find s in terms of some or all of t, ω_0, m, s_0 and d. (No Matlab).



FBD

Kinematics:
$$\vec{a}_p = \vec{a}_{o'} + \vec{a}_{p/o'}$$
$$= -\omega_0^2 d\hat{e}_r + (\ddot{s} - s\omega_0^2)\hat{e}_\theta + (0 + 2\dot{s}\omega_0)(\hat{e}_r)$$

LMB:
$$\sum \vec{F} = m\vec{a}$$
$$\{N\hat{e}_r = m[\quad]\}$$
$$\{\}\cdot\hat{e}_\theta \Rightarrow \quad 0 = \ddot{s} - s\omega_0^2 \Rightarrow \boxed{\boxed{\ddot{s} = s\omega_0^2}} \text{ (a)}$$

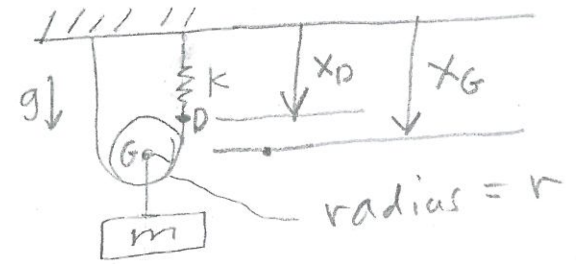
ODE soln is: $s = A\cosh(\omega_0 t) + B\sinh(\omega_0 t)$

$$\dot{s}(0) = 0, \quad s(0) = s_0 \Rightarrow \boxed{\boxed{s = s_0\cosh(\omega_0 t)}} \text{ (b)}$$

16) 1D with gravity g. A mass m hangs from an ideal round negligible-mass frictionless pulley, an inextinsible strings, and spring k as shown. Give all answers in terms of some or all of m, g and k. As for all problems, clearly define any other variables you may use in your solution.

a) At equilibrium how much lower is the pulley than when there is no mass (but the string and pulley are not slack)?

b) What is the frequency of small oscillation (so small that the strings do not go slack)? You can find ω or f, as you please.



String length $= L_S$

Spring rest length $= L_0$

Spring stretched length $= X_D$

Spring stretch $= X_D - L_0 = \Delta l$

Before stretch: $2X_{G0} + \pi r = L_S + L_0$ (1)

After stretch: $2X_G + \pi r = L_S + X_D$ (2)

Subtract (2) − (1) ⟹ $\boxed{2\Delta X_G = \underbrace{X_D - L_0}_{\Delta l}}$

FBD: $T \uparrow \uparrow T = k(\Delta l)$; $\downarrow mg$

LMB: $\sum \vec{F} = m\vec{a}$

$$mg - 2T = m\ddot{X}_G$$

$$T = k\Delta l = 2k\,\Delta X_G$$

$$\ddot{X}_G = g - \frac{4k(\Delta X_G)}{m} \quad (1)$$

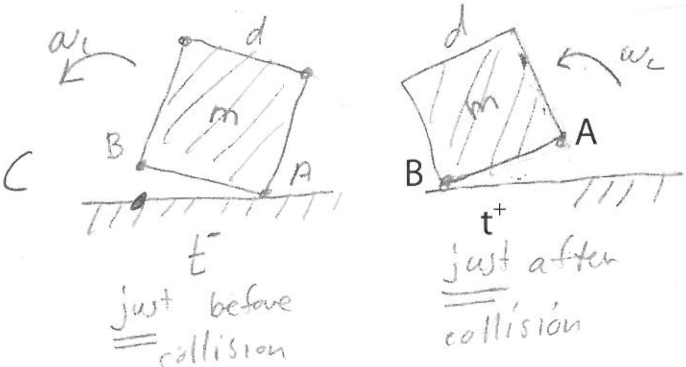
equilib stretch ⟹ $\ddot{X}_G = 0$ ⟹ $\boxed{\Delta X_G = \frac{mg}{4k}}$ (a)

(1) ⟹ $m\ddot{X}_G + 4kX_G = 4kX_{G0}$

$$\Rightarrow (X_G - X_{G0}) = A\sin\sqrt{\tfrac{4k}{m}}\,t + B\sin\sqrt{4k/m}\,t$$

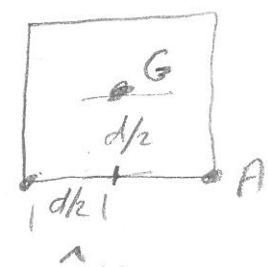
$$\Rightarrow \boxed{\omega = 2\sqrt{\frac{k}{m}}, \quad f = \frac{1}{\pi}\sqrt{\frac{k}{m}}}$$ b

17) 2D, with gravity g. A uniform cube with mass m and side d rocks on edge A and tips until it has a collision with edge B. Then edge A breaks free, and then the cube rocks about edge/hinge B. *Just* before the collision, at $t = t^-$, the angular velocity of the cube is known to be $\omega_1 \hat{k}$. Just before and after the collision the tip angles are negligibly small. What is the angular velocity $\omega_2 \hat{k}$ *just* after the collision at $t = t^+$? Answer in terms of some or all of m, g, d and ω_1.



collisional FBD

geometry



$\hat{j}$ $\hat{i}$

$|\vec{r}_{G/C}| = \frac{\sqrt{2}}{2} d$

AMB$/_C$ $\qquad \vec{H}_C^- = \vec{H}_C^+$

$$\underbrace{\vec{r}_{G/C} \times m\vec{v}^-}_{=\vec{0}} + I\omega^-\hat{k} = \vec{r}_{G/C} \times m\vec{v}^+ + I\omega^+\hat{k}$$

$\vec{v}^- = \vec{\omega}^- \times \vec{r}_{G/A}$ $\qquad \vec{v}^+ = \vec{\omega}^+ \times \vec{r}_{G/C}$

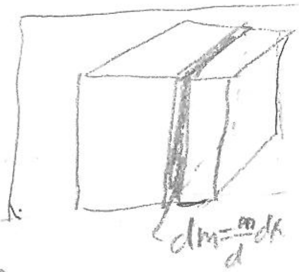
$= \vec{0}$ because $\vec{v}^-$ is $\parallel$ to $\vec{r}_{G/C}$

$$\Rightarrow \left\{ I\omega^-\hat{k} = |\vec{r}_{G/C}|^2 m\omega^+\hat{k} + I\omega^+\hat{k} \right\}$$

$$\{\}\cdot\hat{k} \Rightarrow \quad \omega^+ = \frac{I}{I + md^2/2}\,\omega^-$$

$$\omega^+ = \frac{(1/6)\,md^2}{(1/6)md^2 + 1/2\,md^2}\,\omega^-$$

$$\boxed{\omega^+ = \frac{1}{4}\omega^-}$$



for cube

$I^G = \frac{1}{6} md^2$

Check: $\int r^2 dm = \int x^2 + y^2\, dm$

$= 2\int x^2 dm \qquad dm$

$= 2\int_{-d/2}^{d/2} x^2 \left(\frac{m}{d}\right) dx$

$= 2 \left.\frac{x^3}{3}\right|_{-d/2}^{d/2} \frac{m}{d}$

$= md^2/6$

Your TA, Section # and Section time:

Your name:

Cornell
TAM/ENGRD 2030

Makeup Prelim

May 4, 2013

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

How to get the highest score?

Please do these things:

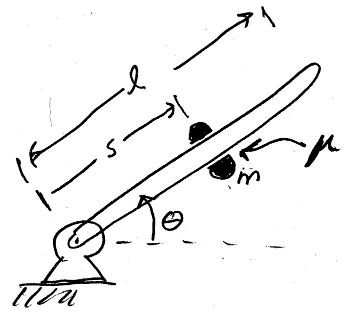
- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- ◻ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** Ask for more extra paper if you need it. Put your name on each extra sheet.

Problem 10: ____/25

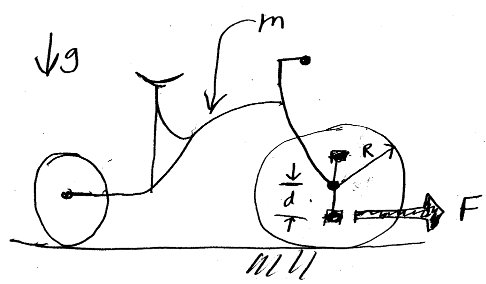
Problem 11: ____/25

Problem 12: ____/25

10) 2D. No gravity. A bead m slides with friction coefficient μ on a rigid straight rod with length ℓ that is rotated by a motor. At the instant of interest the angle of the rod is θ, the rotation rate is $\dot{\theta}$ and the angular acceleration is $\ddot{\theta}$. The bead is a distance s from the motor axle and has rate of sliding $\dot{s} > 0$. In terms of some or all of $\mu, \theta\dot{\theta}, \ddot{\theta}, \ell, s$ and $\dot{s}$, find $\ddot{s}$.



11) 2D. A tricycle has weight mg and wheels with negligible mass. The steering is locked straight forwards. Assume the friction μ is big enough so that the wheels roll without slip. The front wheel has radius R and the front crank has length $d < R$. A forwards force $F > 0$ is applied to the bottom pedal from a person standing at the side. In terms of some or all of m, g, R, d, F, μ and g, which direction does the tricycle accelerate (right or left) and with what acceleration?



12) Write all of the Matlab commands to solve the following problem using ODE23 or ODE45. The result should be printed by Matlab in the command window.

The equation of a damped simple pendulum is $\ddot{\theta} = -\frac{g}{\ell}\sin\theta - c\dot{\theta}$.

Find the angle θ at $t = t_f$.

Use any non-zero values you like for g, ℓ, c and t_f and for the initial conditions.

Your TA, Section # and Section time:

"Solutions"

Your name:

ANDY RUINA

Cornell TAM/ENGRD 2030

No calculators, books or notes allowed. This version slightly improves the given version.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

Prelim 1

February 26, 2013

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation.**
- A+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 1: ____ /25

Problem 2: ____ /25

Problem 3: ____ /25

1) A mass m bounces back and forth between two springs each with constant k. The space between the springs is L. Neglect the width of the mass. Neglect the mass of the springs and their supports. In this motion energy is conserved. For some initial condition, assume that the peak speed of the mass in a cycle of oscillation is v_p. Answer the questions below in terms of some or all of m, k, L and v_p.

a) What is the total energy? Assume the potential energy is zero when the springs are unstretched.

$E_T = E_K + \cancel{E_P}$

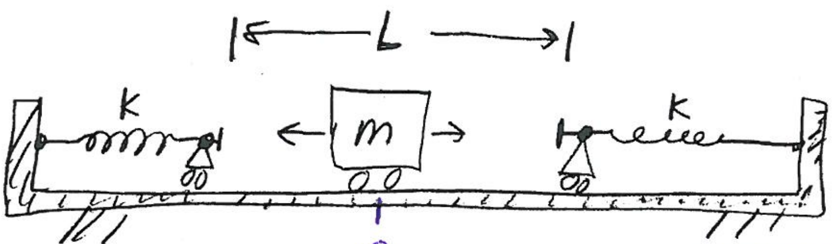
$\boxed{E_T = \frac{1}{2} m V_p^2}$ (a)

b) In a cycle of oscillation what is the maximum deflection of the right spring?

c) What is the period T of oscillation?

d) Plot the cyclic frequency f (defined as $f = 1/T$) *vs* peak speed v_p. Clearly mark and label any key slopes, intercepts, intersections or asymptotes on this plot.

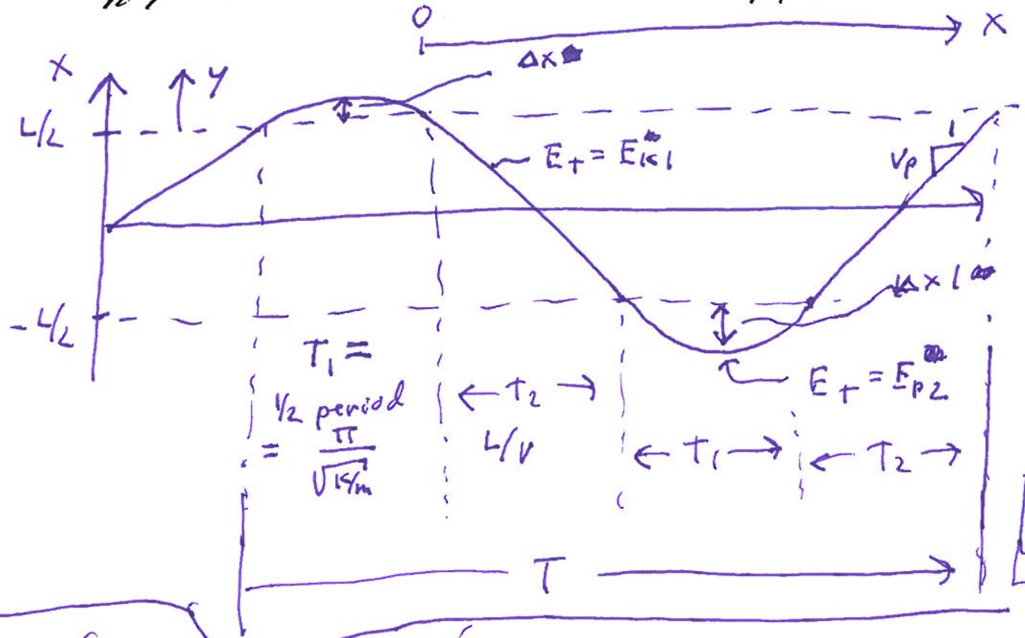
e) EXTRA CREDIT. Hard. *Only think about this if you have nothing more to add to anything else on the exam.* Assume that the ends of the springs also each have mass $m = m/3$. Assume that all collisions are with restitution coefficient $e = 0$. Assume there is no other friction. Can you find a situation where oscillations persist with no decay? This means finding the right combination of v_p, k, m and L, as well as the right initial conditions. Hint: How, with $e = 0$, can there be collisions with no energy loss?



$E_{K1} = E_{P2}$

$\frac{1}{2} m V_p^2 = \frac{1}{2} k (\Delta x)^2$

$\boxed{|\Delta x| = \sqrt{m/k}\, V_p}$ (b)



during contact

$y = A\sin(\omega t) + B\sin(\omega t)$

$\omega = \sqrt{k/m}$

$2\omega T_1 = 2\pi$

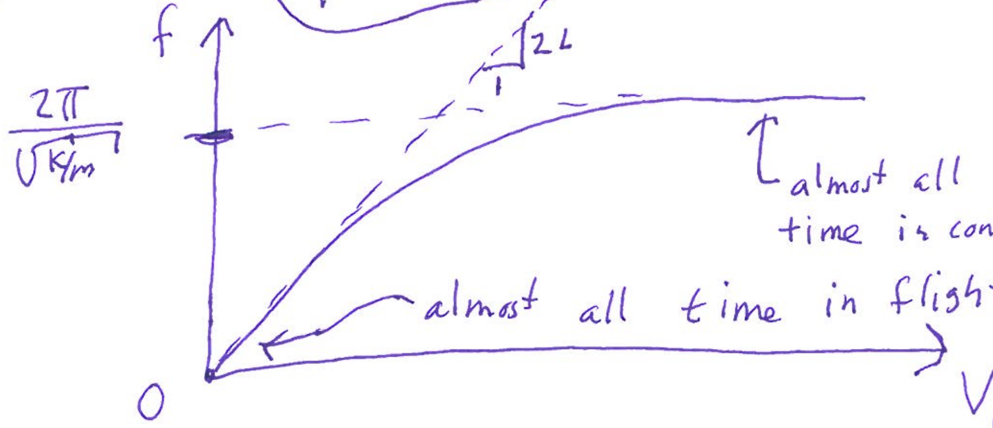
$T_1 = \frac{\pi}{\sqrt{k/m}}$

$T = 2(T_1 + T_2)$ (c)

$\boxed{T = \frac{2L}{V_p} + \frac{2\pi}{\sqrt{k/m}}}$

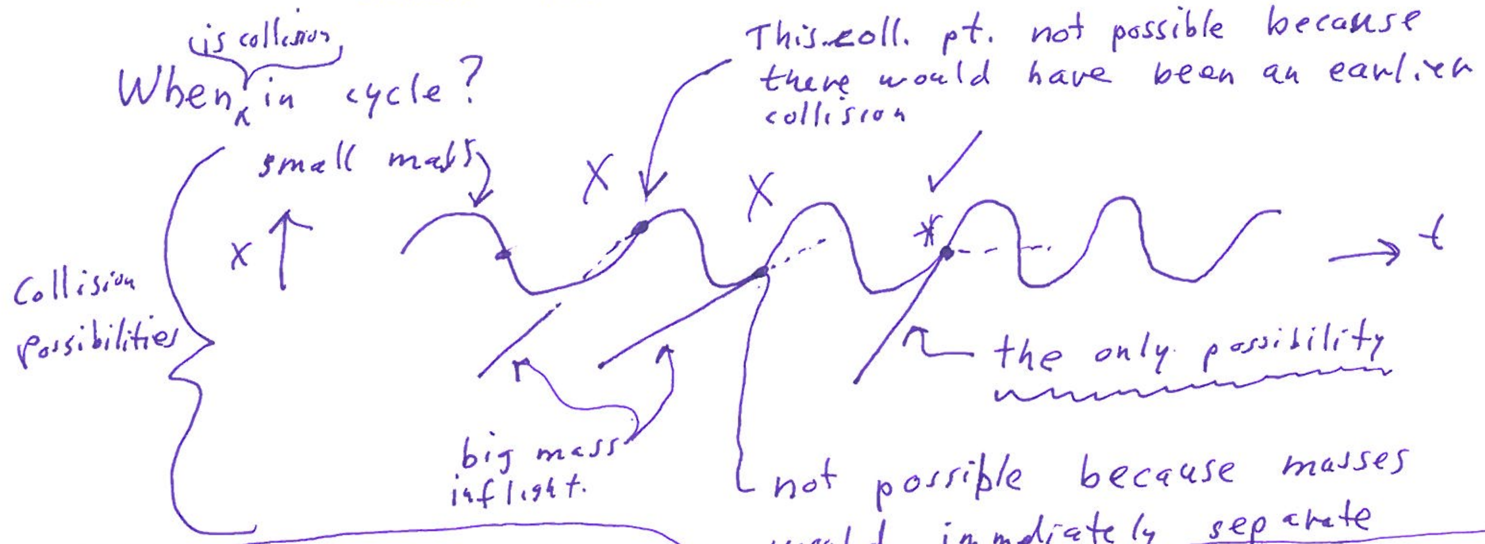
$f = 1/T$

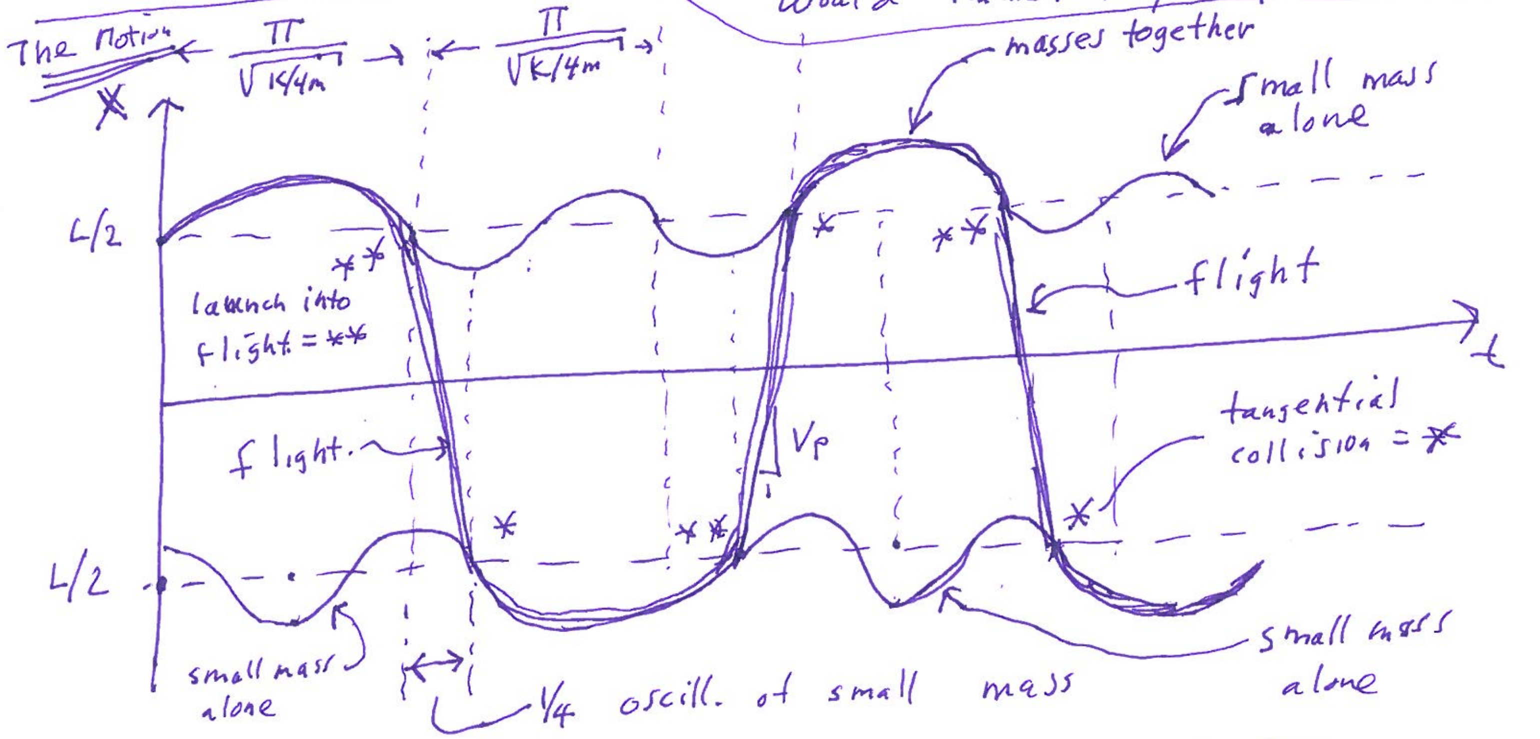
(d) $\boxed{f = \frac{1}{2L/V_p + 2\pi/\sqrt{k/m}}}$



e) No loss ⇒ rel. vel. of collision = 0.
⇒ velocities of masses match
⇒ flight vel. of big mass matches osc. vel of small mass. } ✱

When (is collision) in cycle?





When masses all in contact $M_{tot} = 3m + m = 4m$

⇒ oscillations are ~~twice~~ half as fast.

One full cycle = 2 half oscillations of 2 masses together + 2 flights

$$= \cancel{4}\left[\frac{2\pi}{\sqrt{K/4m}} + \frac{\pi}{\sqrt{K/m}}\right] = \boxed{\frac{5\pi}{\sqrt{K/m}} = T}$$ Period of one full oscillation.

flight time

$$\frac{L}{V_p} = \frac{1}{4}\text{ oscill. of small mass} = \frac{1}{4}\frac{2\pi}{\sqrt{K/m}} \Rightarrow \boxed{V_p = \frac{2L\sqrt{K/m}}{\pi}}$$ Flight speed.

2) The year is 2018 and Dennis Tito's space ship (mass = m) is flying around Mars (mass = $M \gg m$)[1]. The forces acting on the ship include

- a gravity force from mars due to the universal law of gravitation with constant G;
- a drag force with magnitude cv^2, due to motion through the thin martian atmosphere;
- a thrust force F_0, from the ion generator, along the motion.

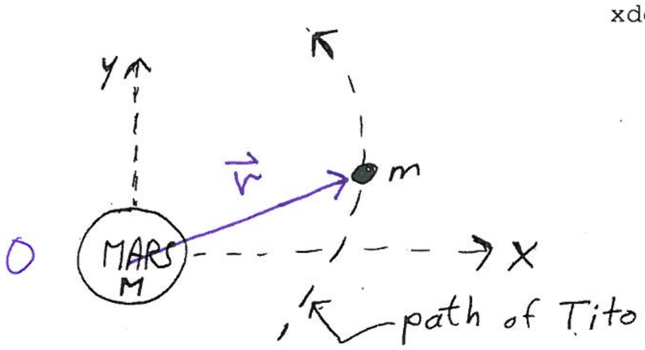
a) (5 points for your signature) *I have read all the directions on the front cover.* ________________________.

(sign above)

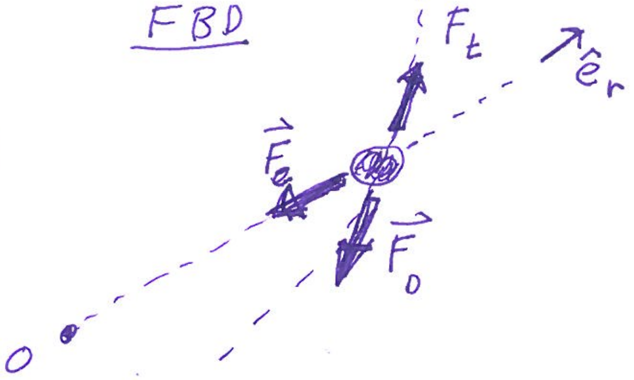
b) (20 points) In terms of some or all of $m, M, G, c, F_0, t, x, y, \dot{x}$ and $\dot{y}$ given at some time t, write Matlab commands to find $\ddot{x}$ at that time ~~(No need to solve the ODEs by hand or numerically.)~~ That is, assume non-zero values have already been assigned for the given variables, and you write Matlab commands the last of which should start:

```
xddot =
```

$\hat{\jmath} \uparrow \rightarrow \hat{\imath}$



FBD



$$\vec{F}_G = -\frac{GMm}{r^2}\hat{e}_r \qquad \vec{F}_D = -cV^2\frac{\vec{V}}{|V|} = -c|\vec{V}|\vec{V}$$

$$\vec{F}_t = F_0\frac{\vec{V}}{|\vec{V}|} \qquad \vec{F}_{TOT} = \vec{F}_g + \vec{F}_t + \vec{F}_D$$

LMB! $\quad \vec{F}_{TOT} = m\vec{a} \Rightarrow \vec{a} = \vec{F}/m \Rightarrow a_x = \ddot{x} = [\vec{F}/m]\cdot\hat{\imath}$

```
r = [x y];
V = [xdot ydot];
er = r/norm(r);
FG = -G*M*m*er/r^2;
Ft = F0*er;
FD = -c*V*norm(V);
```

$a = (F_G + F_t + F_D)/m$

$$\boxed{\text{Xddot} = a(1)}$$

[1]No kidding. Dennis Tito is a real person and this, according to the New York Times this past weekend, is his plan. The ship is to be designed by Space X, many of whose top engineers, so the rumor goes, learned dynamics in TAM 2030 at Cornell. And by 2018, one of them could be you.

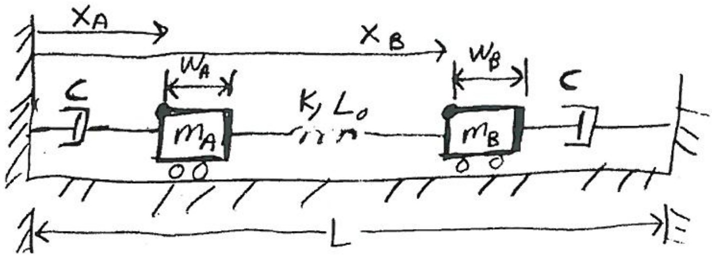
3) Two masses are connected to two identical dashpots and a spring, as shown. The positions of the left edges of the masses x_A and x_B, as well as all other variables, are defined as shown.

a) Draw free body diagrams for both masses, and write the equations of linear momentum balance ($F = ma$) for both masses clearly enough so that a differential equations expert would know what to solve.

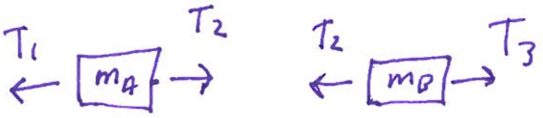
b) That expert wrote some code to generate an approximate numerical solutions, but left some parts blank. Wherever there is a ____ , you fill in the missing code. The meanings of variables are implicitly defined by the physical problem and by other parts of the code.

c) Based on your understanding of both Matlab and mechanics, draw the plot Matlab would make, as accurately as you can.

Symmetric Prob. has symmetric Soln., Like one mass oscillating w/ spring w/ stiffness 2K.



FBDs



$T_1 = c\dot{x}_A$ $\quad$ $T_2 = K(x_B - x_A - L_0 - w)$ $\quad$ $T_3 = -c\dot{x}_B$

LMB

A) $\sum F = m_A \ddot{x}_A$

$\ddot{x}_A = (T_2 - T_1)/m_A$

above

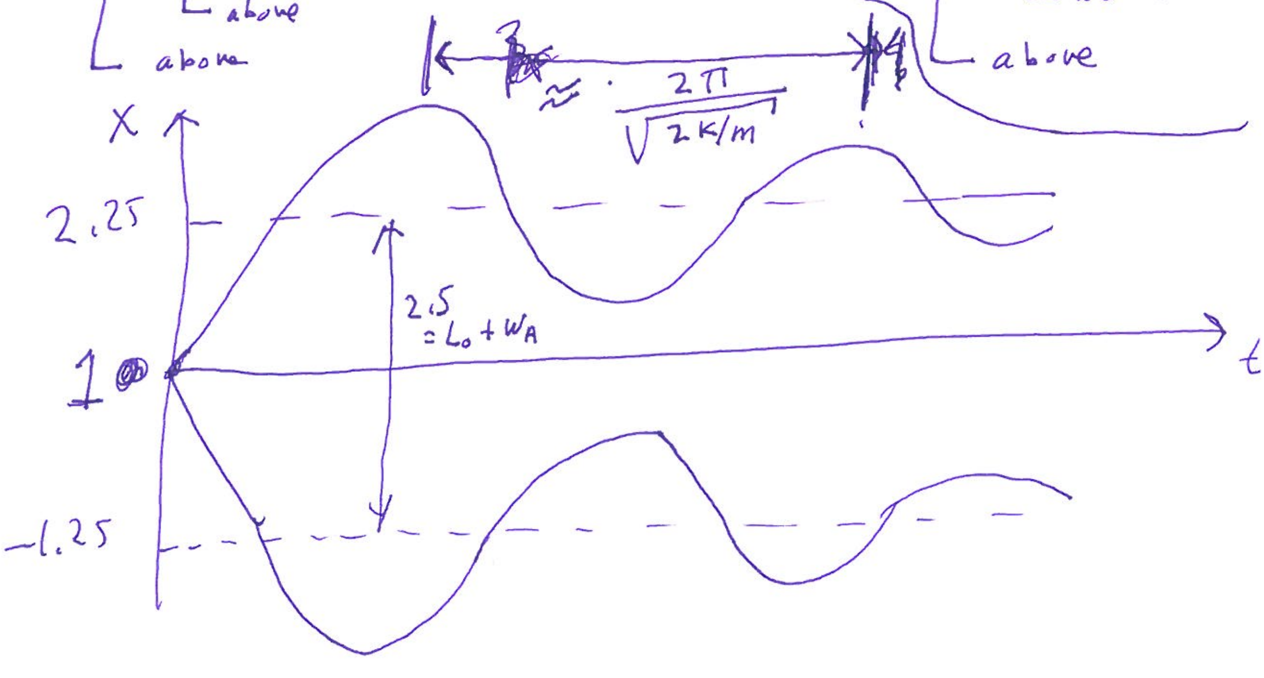
above

B) $\sum F = m_B \ddot{x}_B$

$\ddot{x}_B = (T_3 - T_2)/m_B$

above

above



masses on top of each other but computer doesn't care

This code is poorly commented. It is an exam problem, not an example to emulate.

```
function    Prel1Q1
  tarray =   0:.01:20 ;   n = length(tarray);
  z0     = [1 1 0 0]';   %in the original exam this was [1 1 0 0]';
  p.mA = 1;    p.mB = 1;    p.k  = 1;    p.c = .5;
  p.wA = .5;   p.wB = 2;    p.L0 = 2;    p.L = 10;

  [tarray zarray] = mysolver(tarray, z0, p);

  x1array = zarray(:,1);x2array = zarray(:,2);
  plot(tarray, x1array,  'b--',tarray, x2array,  'r-' )
end
```

Spring relaxed when $x_B - x_A = 2.5$

initial stretch is $\Delta L = -2.5$

```
function   [tarray zarray] = mysolver(tarray, z0, p)
  n          = length(tarray);
  zarray   = zeros(n,length(z0));
  z          = z0;                 zarray(1,:) = z;

  for i = 2:n
    t     = tarray(i-1);   h = tarray(i) - tarray(i-1);

    zdot = rhs(t,z,p);

    z    = z + zdot*h;
    zarray(i,:) = z';
  end
end
```

```
function   zdot=rhs(t,z,p)

  x=z(1:2);  v=z(3:4);
  L1dot     = v(1);

  L2        = x(2) - x(1) - p.wA;        % length of spring

  L3dot     = -v(2);

  T1  =  p.c  * L1dot;

  T2  =  p.k * (L2 - p.L0);
  T3  =  p.c * (L3dot);

  xdot   = v;

  v1dot = (T2 - T1)/p.mA;

  v2dot = (T3 - T2)/p.mB;
  vdot    = [v1dot; v2dot] ;

  zdot   = [xdot; vdot];
end
```

Your TA, Section # and Section time:

"Solutions"

Your name:

Matthew Kelly

Cornell TAM/ENGRD 2030

Prelim 2

March 26, 2013

No calculators, books or notes allowed. This version slightly improves the given version.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- ☐ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct. You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`". Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
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  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 4: ____ /25

Problem 5: ____ /25

Problem 6: ____ /25

1) Pulleys. One dimensional mechanics. Draw three pulley systems. Each one has only one mass m and only one applied force F. For each system you can use any number of ideal massless pulleys and any number of pieces of inextensible massless string. Neglect gravity.

You can label any number of points on one drawing. On your drawings find and label a point

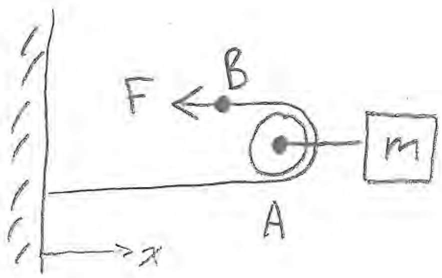
a) A with acceleration whose magnitude is $2F/m$.

b) B with acceleration whose magnitude is $4F/m$.

c) C with acceleration whose magnitude is $F/(2m)$.

d) D with acceleration whose magnitude is $F/(4m)$.

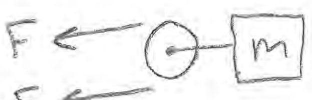
e) E with acceleration whose magnitude is $9F/m$.



$$L = x_A + (x_A - x_B)$$

$$2\ddot{x}_A = \ddot{x}_B$$

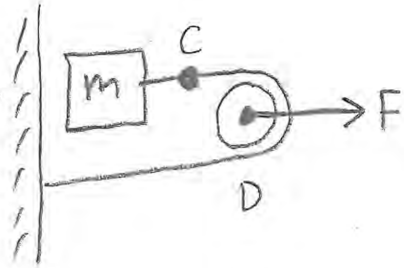
FBD:



LMB: $\Sigma F = ma$

$$-2F = m\ddot{x}_a$$

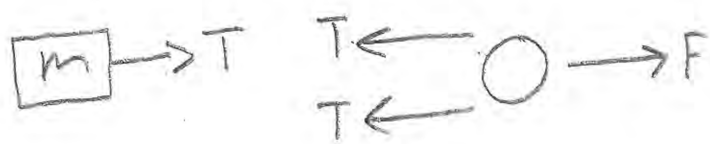
$$\boxed{\ddot{x}_a = \frac{-2F}{m} \qquad \ddot{x}_b = 2\ddot{x}_a = \frac{-4F}{m}}$$



$$L = x_D + (x_D - x_C)$$

$$2\ddot{x}_D = \ddot{x}_C$$

FBD



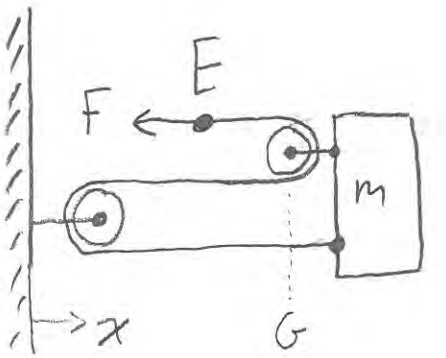
LMB: $\Sigma F = ma$

massless pulley

$$T = m\ddot{x}_c \qquad -2T + F = 0$$

$$\ddot{x}_c = \frac{T}{m} \qquad T = \frac{-F}{2}$$

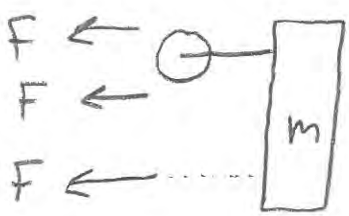
$$\boxed{\ddot{x}_c = \frac{-F}{2m} \qquad \ddot{x}_D = \frac{1}{2}\ddot{x}_c = \frac{-F}{4m}}$$



$$L = 2x_G + (x_G - x_E)$$

$$3\ddot{x}_G = \ddot{x}_E$$

FBD



LMB

$$-3F = m\ddot{x}_G$$

$$\ddot{x}_G = \frac{-3F}{m}$$

$$\ddot{x}_E = 3\ddot{x}_G = \frac{-9F}{m}$$

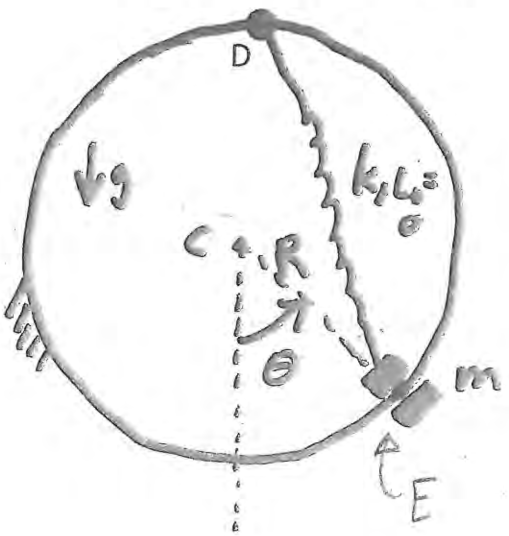
$$\boxed{\ddot{x}_E = \frac{-9F}{m}}$$

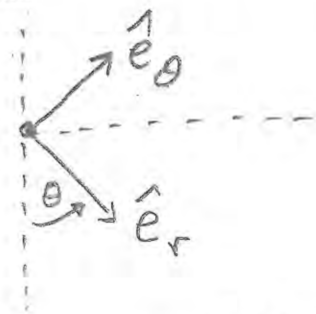
2) A small collar m slides on a rigid stationary hoop with radius R. There is gravity g but no friction. A spring with constant k and rest length $L_0 = 0$ pulls on the mass. One end of the spring is at the fixed point D directly above C. Answer in terms of some or all of m, R, g, θ and $\dot{\theta}$ find

a) Find $\ddot{\theta}$

a) Find the net force (a vector) on the block from the hoop (Please read the 6th line in the directions on the front cover: "Clearly define . . . ").

a) Are there any special values of k (in terms of m, g and R) for which you can find the general exact solution to the equations of motion? If so, name the k and give the solution. This problem part depends on correct solution of (a)). No partial credit for Matlab on this problem.



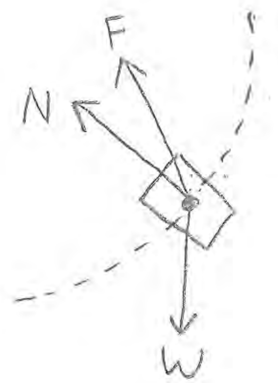


$\hat{e}_r = \sin\theta\,\hat{i} - \cos\theta\,\hat{j}$

$\hat{e}_\theta = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Note: not the conventional def. in terms of $\hat{i}$ and $\hat{j}$

FBD



$\vec{W} = -mg\hat{j}$

$\vec{N} = -N\hat{e}_r$

Assumes $L_0 = 0$

$\vec{F} = K(\vec{r}_{D/E}) = K(R\hat{j} - R\hat{e}_r)$

LMB $\quad \Sigma \vec{F} = \dot{\vec{L}}$

$\vec{W} + \vec{N} + \vec{F} = m(R\ddot{\hat{e}}_r)$

$\{(-mg\hat{j}) + (-N\hat{e}_r) + (KR\hat{j} - KR\hat{e}_r) = mR(\ddot{\theta}\hat{e}_\theta - \dot{\theta}^2\hat{e}_r)\}$

a) $\{\}\cdot\hat{e}_\theta$: $-mg(\hat{j}\cdot\hat{e}_\theta) + 0 + KR(\hat{j}\cdot\hat{e}_\theta) = mR\ddot{\theta}$

$$-mg\sin\theta + KR\sin\theta = mR\ddot{\theta}$$

$$\boxed{\ddot{\theta} = \left(\frac{K}{m} - \frac{g}{R}\right)\sin\theta} \quad (a)$$

b) $\{\}\cdot\hat{e}_r$: $-mg(\hat{j}\cdot\hat{e}_r) - N + \left(KR(\hat{j}\cdot\hat{e}_r) - KR\right) = -mR\dot{\theta}^2$

$$mg\cos\theta - N - KR\cos\theta - KR = -mR\dot{\theta}^2$$

$$\boxed{\begin{array}{l} N = mg\cos\theta - KR(\cos\theta + 1) + mR\dot{\theta}^2 \\ \vec{N} = -N\hat{e}_r \end{array}} \quad (b)$$

c) Sol'n when $\left(\frac{K}{m} - \frac{g}{R}\right) = 0$

$$(c)\ \boxed{K = \frac{gm}{R}} \implies \ddot{\theta} = 0$$

$$(c)\ \boxed{\theta(t) = \omega_0 t + \theta_0} \quad \omega_0 \equiv \dot{\theta}(0)\ ;\ \theta_0 \equiv \theta(0)$$

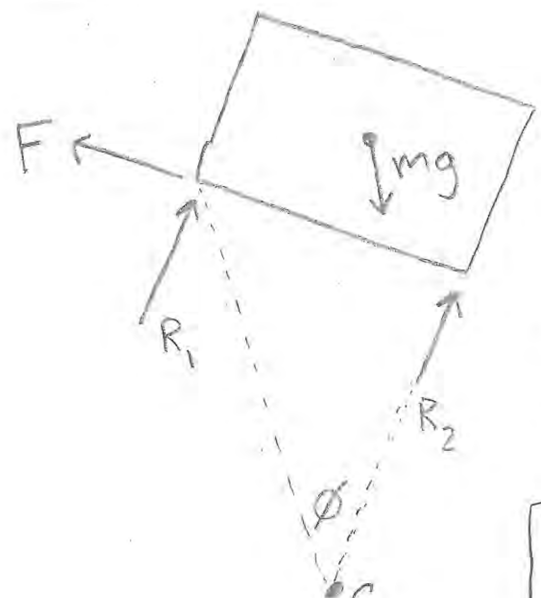
3) A uniformly dense suitcase with mass m slides down (with $v > 0$) a straight ramp with slope θ with $0 < \theta < \pi/2$. The suitcase has height $2h$ and width $2w$. The front (downhill) end is supported by well lubricated and negligible-mass wheels at B. The uphill end drags with friction coefficient $\mu > 0$ at A.

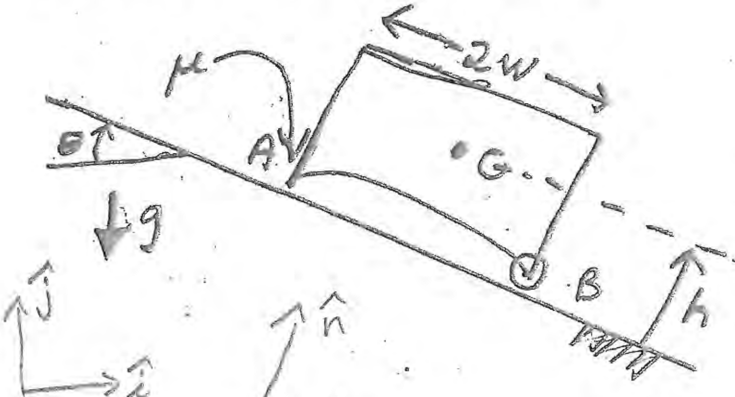
a) What is the acceleration of the suitcase? Answer in terms of some or all of m, g, h, w, θ, v and μ.

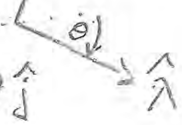
b) For what values of the parameters is the solution not applicable because the suitcase would tip over forwards? Answer in terms of some or all of m, g, h, w, θ, v and μ. [Hint: some people may find the answer surprising]

c) Given the other parameters, for some slopes θ the suitcase is slowing and for some slopes it is speeding. What is the minimum θ for which it is assured that the suitcase will speed up as it goes along no matter how big is the friction μ? Answer in terms of some or all of m, g, h and w. (It is possible to answer this without use of the answer to (a) above. No partial credit for correct algebra based on an incorrect answer to (a) above.)

FBD







$\hat{n} = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\hat{\lambda} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$

Aside: $F = \mu R_1$

$\tan\phi = \dfrac{\mu R_1}{R_1}$

$\tan\phi = \mu$

$\tan\phi = \dfrac{2w}{L} = \mu \longrightarrow L = \dfrac{2w}{\mu}$

$\text{AMB}_{/C}$ $\quad \sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$

$\vec{r}_{G/C} \times (-mg\hat{j}) = \vec{r}_{G/C} \times (ma\hat{\lambda})$

$\vec{r}_{G/C} = \left(\dfrac{2w}{\mu} + h\right)\hat{n} + (-w)\hat{\lambda}$

$$\left(\frac{2w}{\mu}+h\right)(-mg)(\hat{n}\times\hat{j})+(mgw)(\hat{\lambda}\times\hat{j})$$

$$=\left(\frac{2w}{\mu}+h\right)(ma)(\hat{n}\times\hat{\lambda})+(-maw)(\hat{\lambda}\times\hat{\lambda})$$

$$\left(\begin{array}{ll} \hat{n}\times\hat{j}=\sin\theta & \hat{\lambda}\times\hat{j}=\cos\theta \\ \hat{n}\times\hat{\lambda}=-1 & \hat{\lambda}\times\hat{\lambda}=0 \end{array}\right)$$

$$-mg\left(\frac{2w}{\mu}+h\right)\sin\theta+mgw\overset{\cos\theta}{}=-ma\left(\frac{2w}{\mu}+h\right)+0$$

$$\boxed{a=g\left(\sin\theta-\frac{w\mu\cos\theta}{2w+\mu h}\right)} \quad \boxed{\vec{a}_G=a\hat{\lambda}} \text{ (a)}$$

CHECKS

1. Units: accel = accel ✓
2. $g=0 \Rightarrow a=0$
3. $\mu=0 \Rightarrow a=g\sin\theta$
4. $h=0,\ \theta=0 \Rightarrow a=\mu/2$
5. $w\to 0 \Rightarrow a=g\sin\theta$

LMB $\quad \Sigma\vec{F}=\dot{\vec{L}}$

$$\left\{(-\mu R_1\hat{\lambda})+(R_1\hat{n})+(R_2\hat{n})+(-mg\hat{j})=(ma\hat{\lambda})\right\}$$

$$\{\}\cdot\hat{\lambda}:\ -\mu R_1+0+0-mg(-\sin\theta)=ma$$

$$R_1=\frac{m}{\mu}(g\sin\theta-a)$$ *Now use Sol'n

$$R_1=\frac{m}{\mu}\left(\frac{wg\mu\cos\theta}{2w+\mu h}\right)=\frac{mgw\cos\theta}{2w+\mu h}$$

Assume $\theta<\pi/2 \Rightarrow \cos\theta>0$

Tip Forwards $\longleftrightarrow (R_1<0)$

NEEDS TO BE TRUE

For all h,μ,θ, ect...

$$\frac{mgw}{2w+\mu h}>0 \quad \therefore \quad R_1>0 \quad \therefore \quad \boxed{\text{Cannot Tip Forward}}$$

Always True

Slowing Down $\longleftrightarrow (a<0)$

Speeding Up $\longleftrightarrow (a>0)$

$$a = g\left(\sin\theta - \frac{w\mu\cos\theta}{2w+\mu h}\right)$$

$$a(\theta) \equiv 0 \longrightarrow \sin\theta - \frac{w\mu\cos\theta}{2w+\mu h} = 0$$

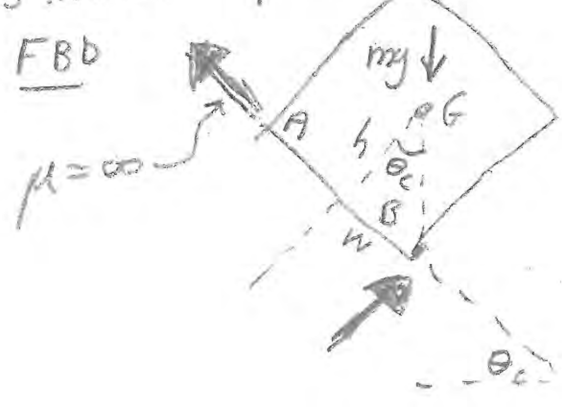
$$\boxed{\begin{gathered}\theta_{crit} = \tan^{-1}\left(\frac{w\mu}{2w+\mu h}\right) \\ \text{if } \theta > \theta_{crit} \text{ then } a>0\end{gathered}}$$

For all $\mu \Rightarrow$ worst case is $\mu \to \infty$

$$\Rightarrow \boxed{\boxed{\underset{\text{for } a>0 \text{ guarantee}}{\theta > \tan^{-1}\left(\frac{w}{h}\right)}}} \text{(c)}$$

Alt. derivation of (c) w/out using (a). Worst case is $\mu = \infty$ $\Rightarrow R_1 = 0$. Critical case is $a=0 \Rightarrow$ statics. Solve statics problem:

FBD



$\sum \vec{M}_{/B} = \vec{0} \Rightarrow G$ directly above B

$\Rightarrow \tan\theta_c = w/h$

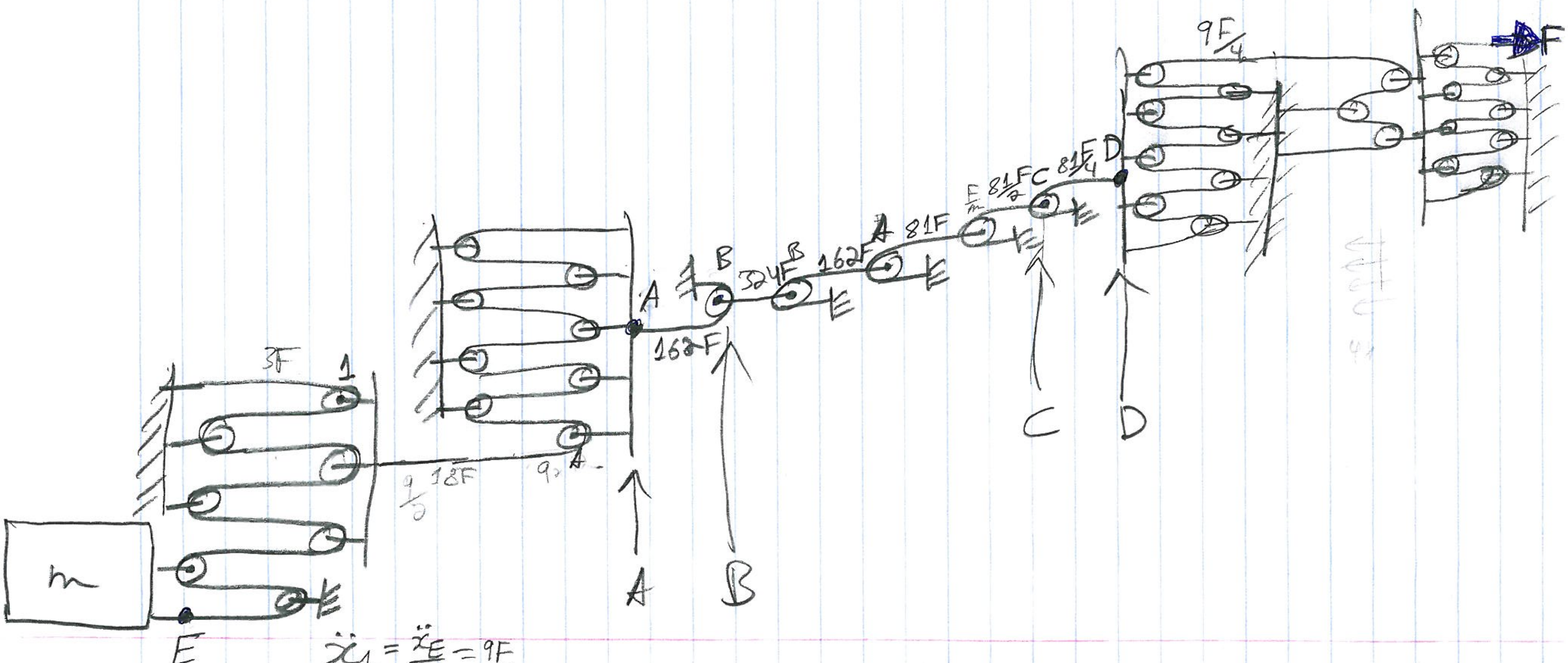
$$\Rightarrow \boxed{\begin{gathered}\theta_c = \tan^{-1}(w/h) \\ \text{accel} > 0 \text{ if } \theta > \theta_c\end{gathered}}$$

Soln. for
Prob. 1,
Prelim 2
Spring 2013

Dynamics extra credit

39 Pulleys

A, B, C, D, E all on one pulley system.



$$\ddot{x}_1 = \frac{\ddot{x}_E}{2} = \frac{9F}{2m}$$

$9\ddot{x}_A = \ddot{x}_1 \rightarrow \ddot{x}_A = \frac{F}{2m}$

$2\ddot{x}_B = \ddot{x}_A \rightarrow \ddot{x}_B = \frac{F}{4m}$

Your TA, Section # and Section time:

"Solutions"

Your name:

RUINA

Cornell
TAM/ENGRD 2030

Prelim 3

April 16, 2013

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- □ TIDILY REDUCE and box in your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct. You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`". Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 7: ____ /25

Problem 8: ____ /25

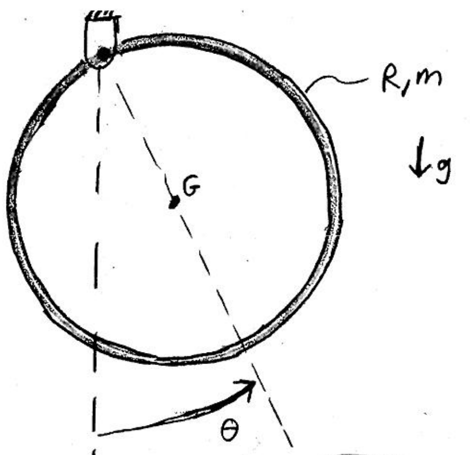
Problem 9: ____ /25

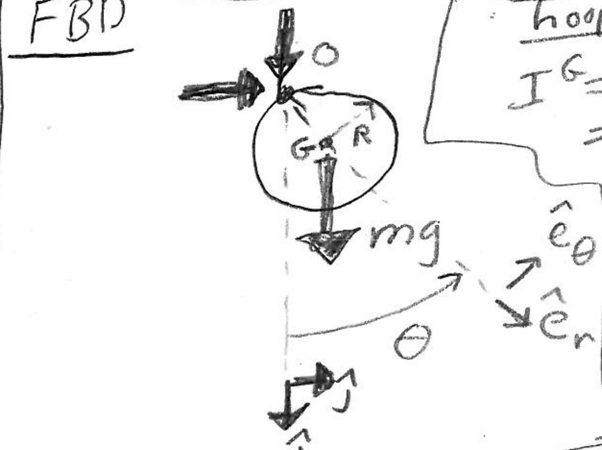
1) A circular hoop swings as a pendulum from a hinge at a point on its edge. Answer in terms of some or all of m, R and g.

a) What is the period of small oscillation near to hanging straight down?

b) If it was launched from the straight down position, what is the minimum launch speed needed by the center of mass in order to swing the hoop over the top?

c) In terms of some or all of m, R and g, what is the length ℓ of a simple pendulum (ie, point mass m and string with the same gravity g) that has the same period of small oscillation as does this hoop?





hoop

$I^G = \int r^2 dm$
$= mR^2$

AMB$_{/O}$: $\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$

$\{-mg\sin\theta R\hat{k} = R\hat{e}_r \times m\underbrace{\vec{a}_G}_{R\ddot{\theta}\hat{e}_\theta - R\dot{\theta}^2\hat{e}_r} + I^G\ddot{\theta}\hat{k}\}$

$\{\}\cdot\hat{k} \Rightarrow -mgR\sin\theta = (I^G + R^2m)\ddot{\theta}$

$-\cancel{m}g\cancel{R}\sin\theta = 2\cancel{m}R^{\cancel{2}}\ddot{\theta}$

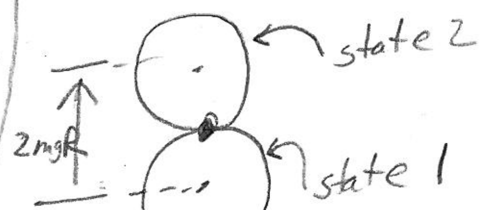
$\theta \ll 1 \Rightarrow$ $\ddot{\theta} = \frac{-g}{2R}\theta$ (*) (ODE)

$\theta =$, e.g. $\theta = \theta_0 \cos\sqrt{\frac{g}{2R}}\,t$ (one soln)

one period $= t^*$: $\sqrt{\frac{g}{2R}}\,t^* = 2\pi$

(a) $\boxed{t^* = 2\pi\sqrt{2R/g}}$

Energy



$E_{k_1} + E_{p_1} = E_{k_2} + E_{p_2}$

$\Delta E_k = -\Delta E_p$

$-\frac{mv_1^2}{2} - \frac{I^G \omega_1^2}{2} = -2mgR$, $\omega_1 = v_1/R$

$-\frac{v_1^2}{2}\left(m + \frac{I^G}{R^2}\right) = -2mgR$

$\frac{\cancel{m}v_1^2}{\cancel{2}}(\cancel{2}\cancel{m}) = 2\cancel{m}gR$

$v_1^2 = 2gR$

$\boxed{v_1 = +\sqrt{2gR}}$ (b)

Simple pendulum:

$\ddot{\theta}_p = \frac{-g}{L}\theta_p$ as * above

to have same soln, $\frac{g}{L} = \frac{g}{2R}$

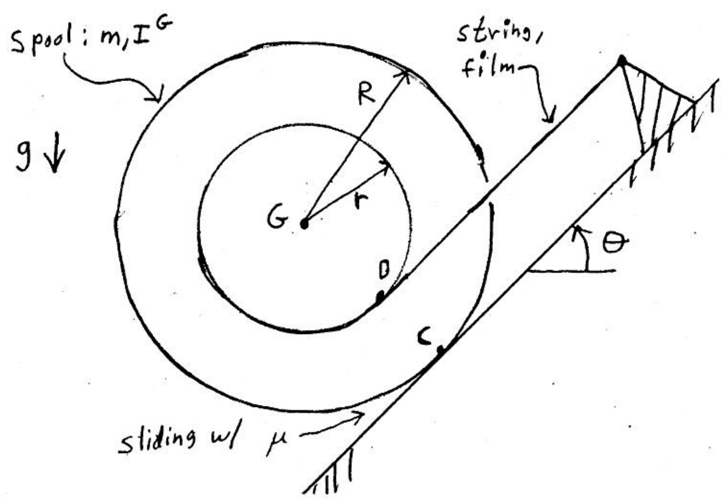
$\Rightarrow \boxed{L = 2R}$ (c)

2) A spool, like the movie spool in lecture, is progressing down a slope. The inextensible film is held firmly at one end and unwinds from the spool. The friction between the spool and the ground is low enough so that the spool slides on the surface. You are given the spool outer radius R, the film radius r, the spool inertia about its COM I^G, the spool mass m, the slope θ, the friction coefficient μ, the gravity constant g and the present speed v_G of the spool down the slope.

Hint: it might help you to picture the motion by first imagining that there is no friction. Then put in the friction.

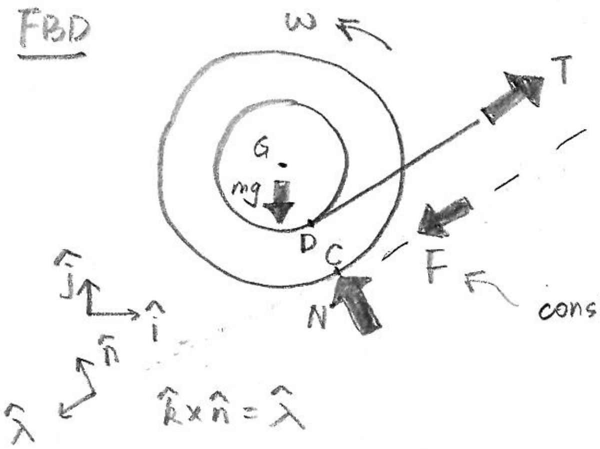
a) Find the normal component of the force of the ground on the spool.

b) Find any one of these quantities: The acceleration a_G of the spool center down the slope, the angular acceleration α of the spool, or the tension in the film/string. If you find more than one quantity, *clearly label the one you want graded.*



2) Solution:

FBD



consistant with $V_G > 0$ & $\underbrace{V_c < 0}$ ← bottom of spool is sliding up hill

$\hat{k} \times \hat{n} = \hat{\lambda}$

Geometry

$\vec{V}_G = V_G \hat{\lambda}$, $\vec{a}_G = a_G \hat{\lambda}$

$\vec{0} = \vec{V}_D = \vec{V}_G + \vec{V}_{D/G} = \vec{V}_G + \omega \hat{k} \times (-r\hat{n})$

$= (V_G - \omega r)\hat{\lambda}$

$\{ \vec{0} = (V_G - \omega r)\hat{\lambda} \}$

$\{ \} \cdot \hat{\lambda} \Rightarrow \boxed{V_G = \omega r}$

$\boxed{a_G = \dot{\omega} r}$ (*)

$\vec{V}_c = \vec{V}_D + \vec{V}_{c/D}$

$= \vec{0} + \omega \hat{k} \times [(R-r)(-\hat{n})]$

$\vec{V}_c = -\omega(R-r)\hat{\lambda}$

└ $\omega = V_G / r$

if $V_G > 0 \Rightarrow V_c < 0$

⇒ slides uphill

⇒ friction points down

LMB $\cdot \hat{n} \Rightarrow N - mg\cos\theta = 0$

$\Rightarrow \boxed{N = mg\cos\theta}$ (a)

LMB $\cdot \hat{\lambda} \Rightarrow F - T + mg\sin\theta = m a_G$ (**)

└ $F = \mu N = \mu mg\cos\theta$

AMB/D: $\sum \vec{M}_{/D} = \dot{\vec{H}}_{/D}$

$\{ [mgr\sin\theta - (R-r)F]\hat{k} = \vec{r}_{G/D} \times m\vec{a}_G + I^G \dot{\omega} \hat{k}$

$= (mr a_G + I^G \dot{\omega})\hat{k}$

$= (mr a_G + \frac{I^G}{r} a_G)\hat{k} \}$

$\{ \} \cdot \hat{k} \Rightarrow mgr\sin\theta - (R-r)F = (mr + \frac{I^G}{r}) a_G$

$$\Rightarrow \boxed{a_G = g \frac{\sin\theta - \mu\cos\theta(R-r)/r}{I^G/mr^2 + 1}} \quad (b)$$

$$* \Rightarrow \boxed{\dot{\omega} = \frac{a_G}{r} = \frac{g}{r} \frac{\sin\theta - \mu\cos\theta(R-r)/r}{I^G/mr^2 + 1}} \quad (b)$$

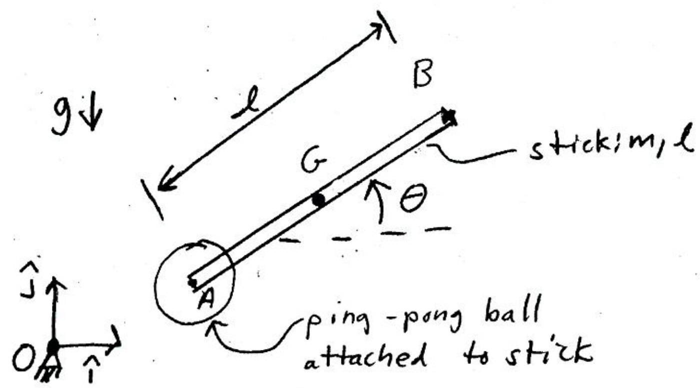
$** \Rightarrow T = F + mg\sin\theta - m a_G$

$$\boxed{T = mg\left[\mu\cos\theta + \sin\theta - \frac{\sin\theta - \mu\cos\theta(R-r)/r}{I^G/mr^2 + 1}\right]} \quad (b)$$

3) Equations of motion for a simple model of an arrow. 2D. A uniform rigid stick (length ℓ, mass m) flies through the air. As it flies, the line segment AB makes an angle $\theta(t)$ with the positive horizontal x axis. Attached to one end, labeled A, is a negligible-mass ping-pong ball that has air friction. The friction is modeled as linear: the ping-pong ball drag-force resists motion of point A with magnitude $F_D = cv_A$. The air friction on the rest of the stick is negligible. The center of mass G is at the center of the stick. You are given $\theta, \dot{\theta}, \vec{r}_G, \vec{v}_G, m, \ell, c, d$ and g.

a) Find $\ddot{\theta}$ and $\vec{a}_G$ (a vector expression for $\vec{a}_G$ without explicit components is fine, but scalar components are also fine). You are given $\theta, \dot{\theta}, \vec{r}_G, \vec{v}_G, m, \ell, c$ and g.

b) Harder. If, instead of a ping-pong ball at A there were feathers. And, more like a real arrow, these feathers had no resistance to the motion of A in the AB direction. But the feathers resisted motion perpendicular to AB with a force $F_L^{\perp} = dv^{\perp}$, where $v^{\perp}$ is the component of $\vec{v}_A$ orthogonal to the line AB[1]. You are given $\theta, \dot{\theta}, \vec{r}_G, \vec{v}_G, m, \ell, d$ and g. Find $\ddot{\theta}$.



[1]Aside: A more realistic model of an arrow would have that 'lift' force depend quadratically on the forward speed. More realistic still would be to have a lift and drag model for the feathers, this would be a model more like that used for an airplane wing.

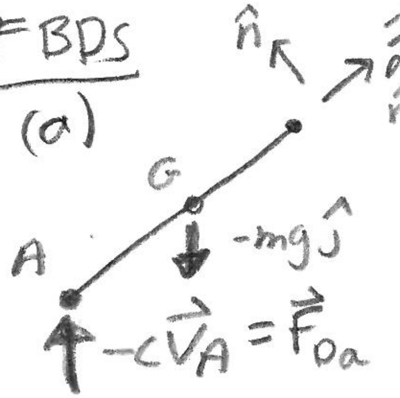
3) **FBDs**

$\hat{\jmath}$, $\hat{\imath}$

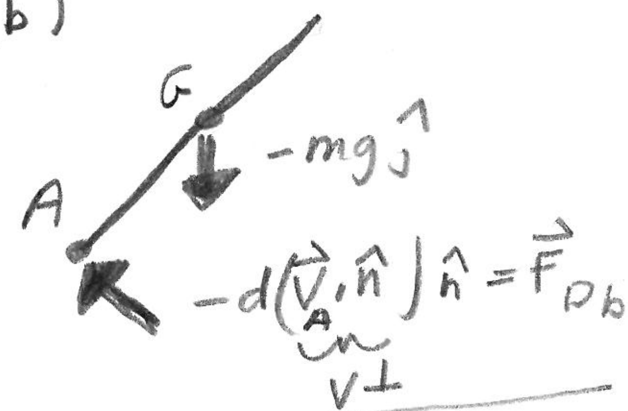
(a)

$\hat{\lambda} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$\hat{n} = \hat{k} \times \hat{\lambda}$



(b)



Kinematics

$$\vec{V}_A = \underbrace{\vec{V}_G}_{\text{given}} + \underbrace{\vec{\omega}}_{\dot{\theta}\hat{k}} \times \overbrace{\vec{r}_{A/G}}^{-\hat{\lambda}\,\ell/2}$$

$$I^G = \int r^2 dm = \int_{-\ell/2}^{\ell/2} s^2 \underbrace{\rho}_{m/\ell}\, ds$$

$$I^G = m\ell^2/12$$

LMB!

$$\sum \vec{F} = m\vec{a}_G$$

$$-mg\hat{\jmath} - c\vec{V}_A = m\vec{a}_G$$

$$\boxed{\vec{a}_G = \frac{-c}{m}\vec{V}_A - g\hat{\jmath}}$$

(b) **AMB**$_{/G}$

$$\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$$

$$\vec{r}_{A/G} \times \vec{F}_{D_b} = I^G \ddot{\theta}\hat{k}$$

$$\boxed{\ddot{\theta} = \frac{\vec{r}_{A/G} \times \vec{F}_{D_b} \cdot \hat{k}}{I^G}} \quad (b)$$

where all quantities are calculated above in terms of given quantities

AMB$_{/G}$ $\quad \sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$

$$\vec{r}_{A/G} \times \vec{F}_{D_a} = \underline{I}^G \ddot{\theta}\hat{k}$$

see above

$$\boxed{\ddot{\theta} = \frac{\vec{r}_{A/G} \times \vec{F}_{D_a}}{I^G} \cdot \hat{k}}$$

see above

see above

Your TA, Section # and Section time:

Your name:

Cornell
ME 2030, Dynamics

Final Exam

May 15, 2014

No calculators, books or notes allowed.

5 Problems, 150 minutes (+ no overtime, Cornell rules)

How to get the highest score?

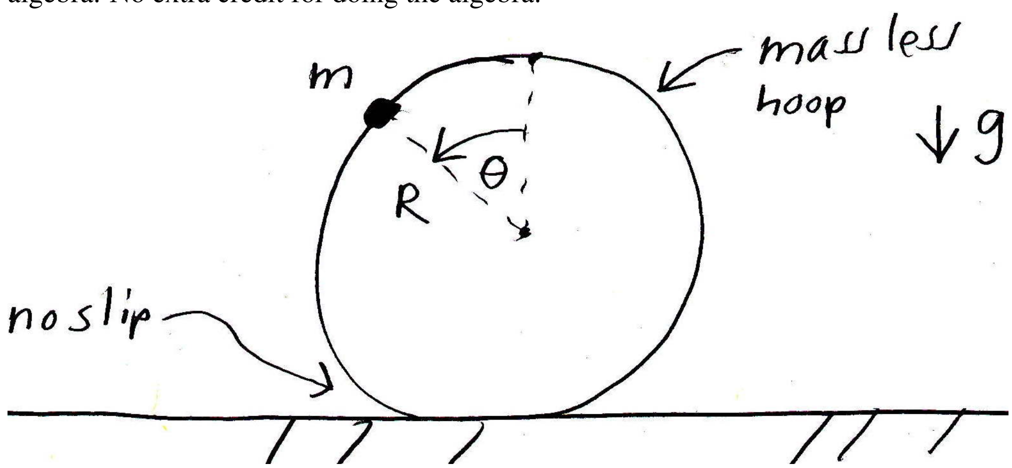
Please do these things:

- : Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- : Use correct **vector notation**.
- A+ : Be (I) neat, (II) clear and (III) well organized.
- □ : TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifiable algebraic expressions).
- >> : Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- : Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}} \ldots$) and signs ($\pm$) with sketches, equations or words.
- $\rightarrow|$: **Justify** your results so a grader can distinguish an informed answer from a guess.
- : If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- $\approx$: Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 13:	/25
Problem 14:	/25
Problem 15:	/25
Problem 16:	/25
Problem 17:	/25

13) 2D. A round rigid hoop with radius R has negligible mass. A point mass m is glued to the hoop. The hoop rolls without slip on a horizontal surface. Given $\theta, \dot{\theta}, m, g$ and R find $\ddot{\theta}$.

Full credit for reducing the problem to a totally clear math problem (one that a skilled person could solve without understanding the problem or looking at any pictures). That is, you should set up the problem correctly while making clear that you could do all of the algebra required (e.g., show all of the planned substitutions and show the results of any needed dot and cross products between base vectors). That is, there is no point in doing any lengthy algebra. No extra credit for doing the algebra.



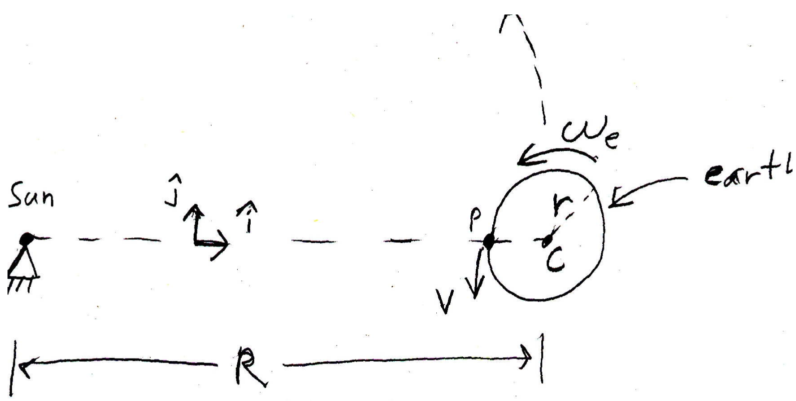
14) 2D. The earth goes around the sun. A plane goes around the earth.

- The earth goes around the sun in a counter-clockwise circular orbit with radius R once per unit time T (that is, if you were to substitute numbers, which you should *not* do, $T = 1$ year).
- The angular velocity of the earth is $\omega_e = \omega_{e/\mathcal{F}}$ relative to the fixed distance Newtonian-reference-frame stars (That is, $\omega_e > 2\pi/\text{day}$ by a little bit).
- The earth radius is r.
- A plane is going counterclockwise around the earth with constant speed v_p relative to the earth (v_p is the so-called "ground speed" of the plane).
- At the time of interest everything is lined up like in the picture below (with the plane at P on the line from the sun to the center of the earth).
- Answer all questions in terms of some or all of $R, r, T, \omega_e, v_p, \hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$.

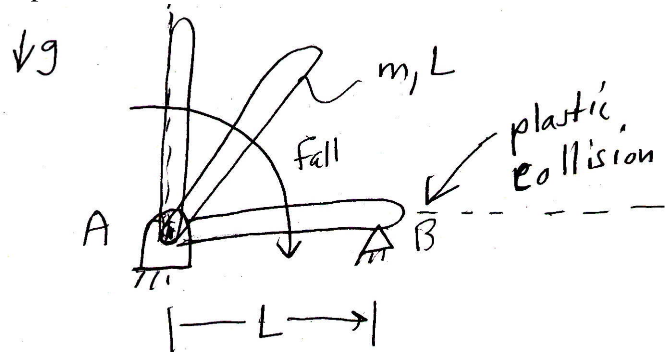
a) What is the position of the plane $\vec{\boldsymbol{r}}_p$?

b) What is the velocity of the plane $\vec{\boldsymbol{v}}_p$?

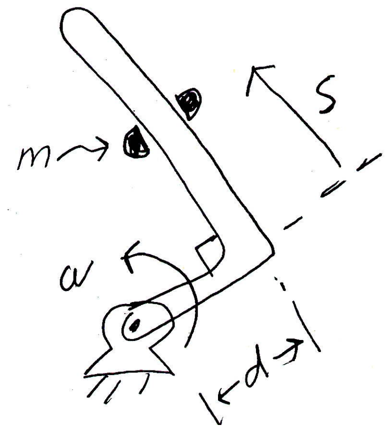
c) What is the acceleration of the plane $\vec{\boldsymbol{a}}_p$?



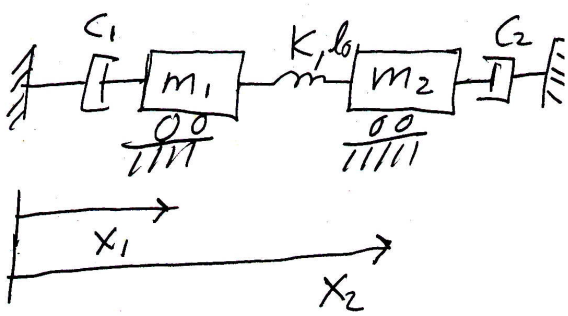
15) 2D. A uniform rigid stick with length L and mass m is balanced upright. It is hinged at A. It is then given a tiny push (big enough to cause a fall, but small enough so that the kinetic energy just after the push is negligible) and eventually falls to the right. It has a plastic collision at B, assume the impulse at B is vertical. What is the impulse at B? Answer in terms of some or all of m, ℓ and g.



16) 2D. No gravity. A rigid L-shaped rod rotates at constant angular speed ω. A point mass m slides on the rod with no friction. Given that $s(0) = 0$ and $\dot{s}(0) = v_0$ find $s(t)$. Your answer can contain some or all of t, v_0, m, d and ω.



17) 1D. No gravity. Spring-mass-dashpot system. The width of the two masses is negligible. At $t = 0$ the masses have given positions x_{10}, x_{20} and speeds to the right v_{10}, v_{20}. Assume any non-zero values, in consistent units, for the parameters $x_{10}, x_{20}, v_{10}, v_{20}, k, \ell_0, c1, c_2$ and t_{end}. Using ODE45, write Matlab code to plot the position of mass one for the interval $0 \le t \le t_{end}$.



Your TA, Section # and Section time: "SOLUTIONS"

Your name: ANDY RUINA

Cornell
TAM/ENGRD 2030

Prelim 1

February 27, 2014

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

How to get the highest score?

Please do these things:

- $\nwarrow\!\!\bullet\!\!\nearrow$: Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- $\vec{\bullet}$: Use correct **vector notation**.
- A+ : Be (I) neat, (II) clear and (III) well organized.
- □ : TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> : Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- ⮥ : Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| : **Justify** your results so a grader can distinguish an informed answer from a guess.
- ➧ : If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ : Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- ◫ **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 1: ____ /25

Problem 2: ____ /25

Problem 3: ____ /25

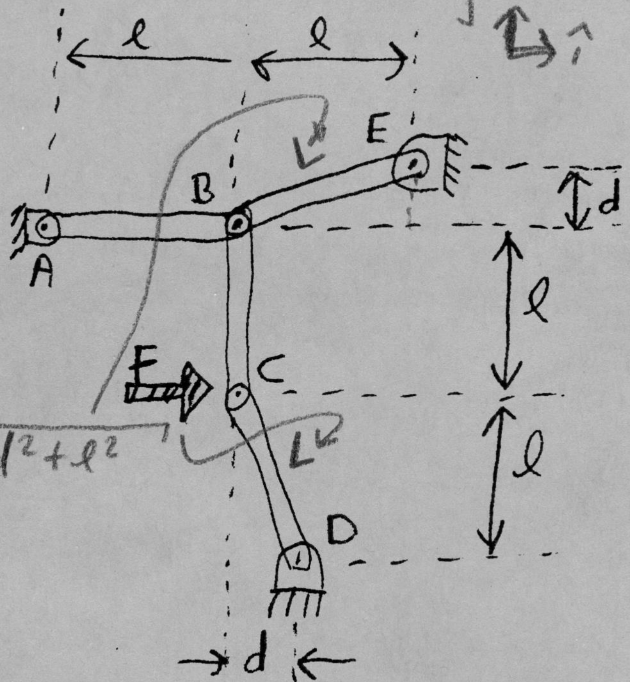
Hint: No part of this test depends on, or will even probably be helped by, extensive calculations.

1) Statics. Consider the small structure below. Assume rigid bars and neglect gravity. [Hint: this problem is prone to sign confusion, so take care.]

a) Read the cover page. Write"I have read the cover page and I understand it." I wrote it!

b) Find T_{AB}, the *'bar force'* in bar AB, in terms of F, d and ℓ .

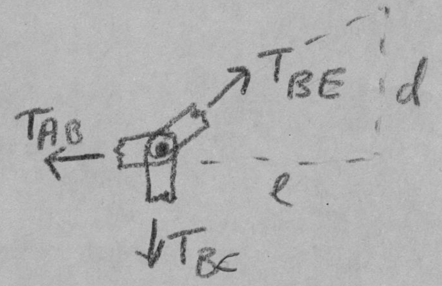
c) If $F = 1\,\mathrm{N}$, $d = 1\,\mathrm{cm}$ and $\ell = 10\,\mathrm{cm}$ what is T_{AB}? If you don't trust your formula for T_{AB} from part (b), you can get full credit here if your answer is about right and justified with some reasonable words.



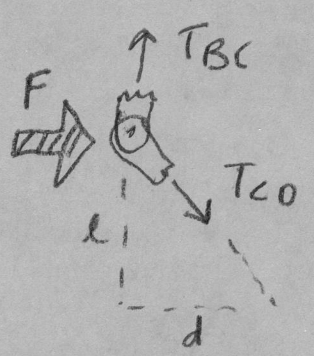
$L = \sqrt{d^2+\ell^2}$

FBDs

joint B;



joint C:



joint C: $\sum \vec{F}_i = \vec{0} \Rightarrow \left\{ F\hat{\imath} + T_{BC}\hat{\jmath} + T_{CD}\left(\frac{d}{L}\hat{\imath} - \frac{\ell}{L}\hat{\jmath}\right) = \vec{0} \right\}$

$\{\}\cdot\hat{\imath} \Rightarrow \quad T_{CD} = \frac{-FL}{d}$

$\{\}\cdot\hat{\jmath} \Rightarrow \quad T_{BC} = \frac{\ell}{L} T_{CD} = \frac{\ell}{L}\frac{-L}{d}F \Rightarrow \boxed{T_{BC} = \frac{-\ell}{d}F}$

joint B: Exactly the same geometry as joint C, rotated 90° & with $-T_{BC}$ playing the role of F.

$$\Rightarrow \quad T_{AB} = -\frac{\ell}{d}(-T_{BC}) = \frac{\ell}{d}\cdot\frac{-\ell}{d}F$$

$$\boxed{T_{AB} = \frac{-\ell^2}{d^2}F} \quad \text{(b)}$$

Why? Each toggle mechanism multiplies force by about 10.

$\ell = 10\,\mathrm{cm}$
$d = 1\,\mathrm{cm}$ $\Rightarrow \boxed{T_{AB} = -100\,\mathrm{N}}$ (c)
$F = 1\,\mathrm{N}$

bar force = 100N compression

2) What's the problem? That is the problem. The computer code on the next page is poorly documented. But once the missing line of code is filled in (at the "% *!*") it runs without error.

a) Describe below *as precisely as you can*, with pictures, words and equations, a mechanics question for which this code gives the answer. Then set up the governing equations.

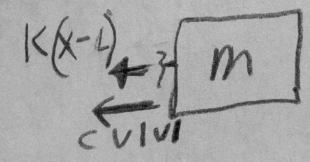
b) Fill in the missing line of code.

c) As precisely as you can, draw the plot that the code at the right makes (assuming you have correctly filled in the missing line).

From code! ① ⇒ $\dot{x} = v$

② ⇒ $\dot{v} = (-k(x-L) - c\,v|v|)/m$

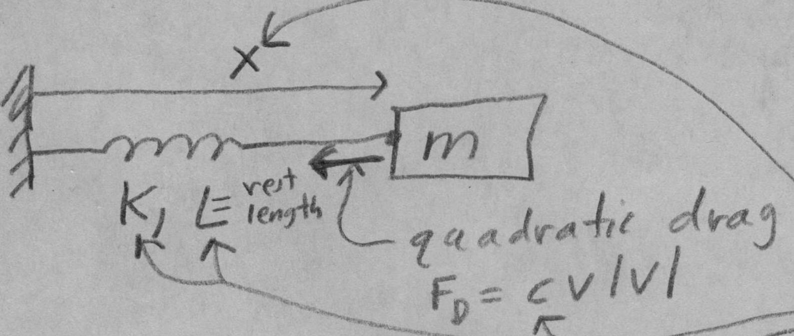
FBD



⇒ $ma = -k(x-L) - \underbrace{c\,v|v|}_{\text{quadratic drag}}$

L rest length

1D



$m = 1$
$K = 1$
$L = 2$
$c = 0.1$

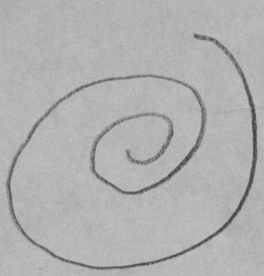
assume consistent units

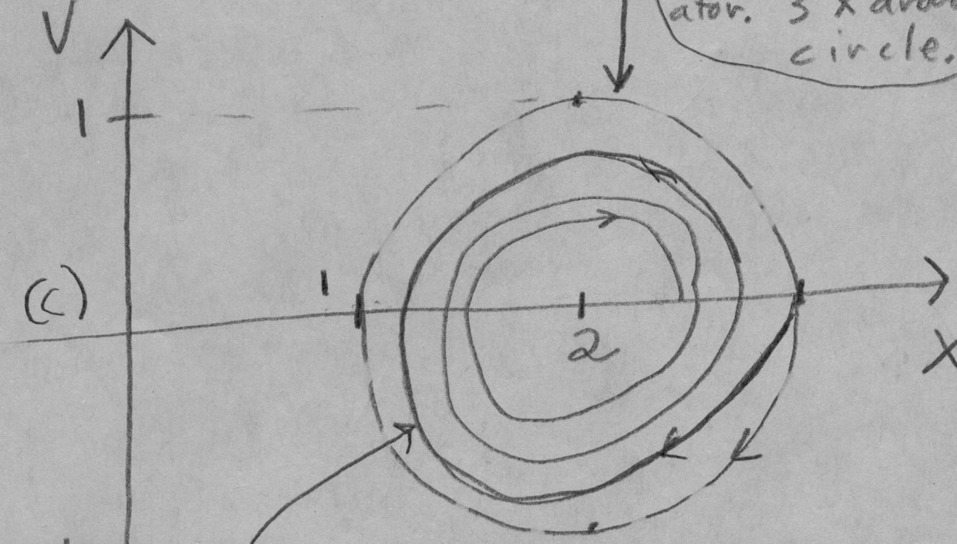
ICs: $x_0 = 3$, $V_0 = 0$

Plot V vs X for $0 \le t \le 6\pi$

If no damping this would be harmonic oscillator. 3 X around circle.

If linear damping ⇒ exponential spiral





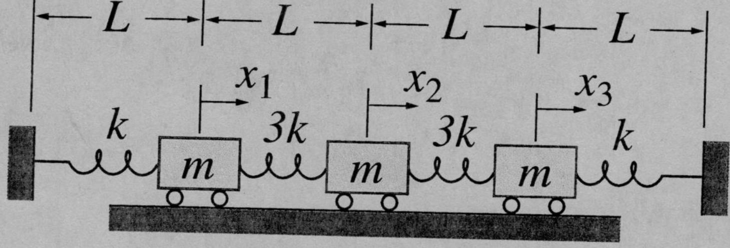
About 3x around circle.
As radius → 0 damping → 0
(because of "v^2") ⇒ more like circles than exponential spiral

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function prel1()
%Poorly documented code
p.M    = 1;     p.K = 1; p.L = 2;  p.c = .1;
tspan  = [0 6*pi];
n      = 1000;
x0     = 3; v0= 0;
z0     = [x0 v0]';

[tarray zarray] = odesolver(tspan,z0,n,p);
x = zarray(:,1); v=zarray(:,2);
plot(x,v);
axis('equal') % makes x and y axis on the same scale
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [tarray zarray] = odesolver(tspan,z0,n,p)
tarray  = linspace(tspan(1),tspan(2),n+1);
zarray = zeros(n+1,2); zarray(1,:) = z0';
   for i=1:n;
      h               = tarray(i+1) - tarray(i);
      zdottemp        = rhs(tarray(i),zarray(i,:),p)';
      zarraytemp      = zarray(i,:) + (h/2)*zdottemp;
      zdot            = rhs(tarray(i),zarraytemp,p)';
(b)    Zarray(i+1,:) = Zarray(i,:) + h*zdot;                %*!*

   end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zdot = rhs(t,z,p)
   x = z(1); v = z(2);
 1 xdot = v;
 2 vdot = (-p.K*(x-p.L) - p.c*v*abs(v))/p.M;   % abs(v) is |v|
   zdot =[xdot; vdot];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

3) For the three-mass system shown assume that at the equilibrium position all of the springs are relaxed and that $x_1 = x_2 = x_3 = 0$.

a) Write the governing equations in matrix form (Use the laws of mechanics to find these).

b) As precisely as you can describe one normal mode of this system (for example, find the frequency of vibration for this mode). [Please do *not* use eigen-values and eigen-vectors to find the mode and its frequency unless you know no other way.]



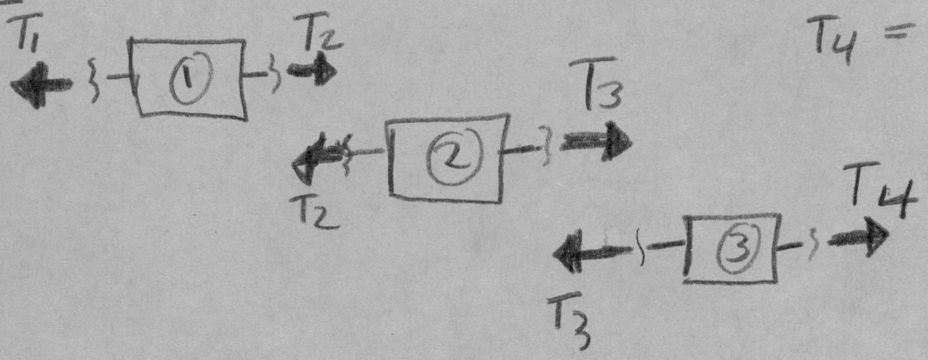
$T_1 = k x_1$

$T_2 = 3k(x_2 - x_1)$

$T_3 = 3k(x_3 - x_2)$

$T_4 = -k x_3$

FBD:



LMB:

1: $m\ddot{x}_1 = T_2 - T_1 = 3k(x_2 - x_1) - k x_1$

2: $m\ddot{x}_2 = T_3 - T_2 = 3k(x_3 - x_2) - 3k(x_2 - x_1)$

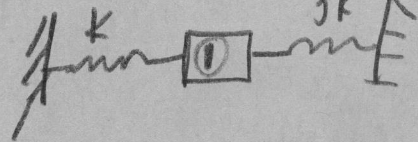
3: $m\ddot{x}_3 = T_4 - T_3 = -k x_3 - 3k(x_3 - x_2)$

$$\Rightarrow \quad (a)\ \boxed{\underbrace{m\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{\underline{M}}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{bmatrix}+\underbrace{k\begin{bmatrix}4&-3&0\\-3&6&-3\\0&-3&4\end{bmatrix}}_{\underline{K}}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}}$$

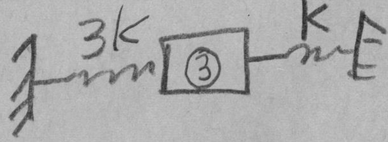
By symmetry, one normal mode is:

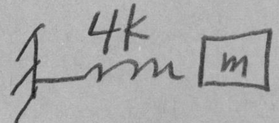
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_0 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \sin(\omega t)$$

Middle mass is still, so the side masses see this



&



$\Rightarrow$ 

$\Rightarrow m\ddot{x} + 4k\,x = 0$

$\Rightarrow \quad x = x_0 \sin \omega t$

$\quad \omega = \sqrt{4k/m} = 2\sqrt{\frac{k}{m}}$

(b) $\boxed{\text{Mode shape} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{frequency } \omega = 2\sqrt{\frac{k}{m}}}$

Your TA, Section # and Section time:

SOLUTIONS

Your name:

ANDY RUINA

Cornell TAM/ENGRD 2030

Prelim 2

March 20, 2014

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

How to get the highest score?

Please do these things:

- ↖•↗ : Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- $\vec{\bullet}$: Use correct **vector notation**.
- A+ : Be (I) neat, (II) clear and (III) well organized.
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Problem 4: ______ /25

Problem 5: ______ /25

Problem 6: ______ /25

4) Consider ideal massless pulleys and massless inextensible strings. Each pulley system as one and only one mass m and one and only one force F. Points A, B, C and D can be any point on any string or at the center of a pulley that you like. Show enough reasoning so it is clear that you could justify your result in detail if needed. You can count the accelerations as being positive in any direction you like.

a) Read the cover page. Initial this statement. "I have read the cover page and I understand it." ALR

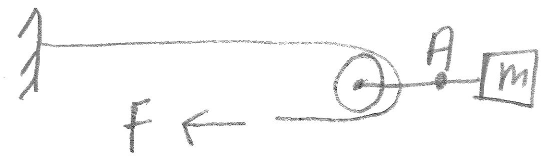
b) Design a pulley system so that $a_A = 2F/m$.

c) Design a pulley system so that $a_B = F/(4m)$.

d) Design a pulley system using 2 or fewer pulleys so that $a_C = 9F/m$.

e) Design a pulley system using 6 or fewer pulleys so that $a_D = 1024F/m$.

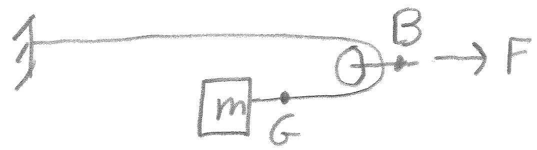
a)



tension at A = 2F

$\Rightarrow \boxed{a_A = 2F/m}$ (a)

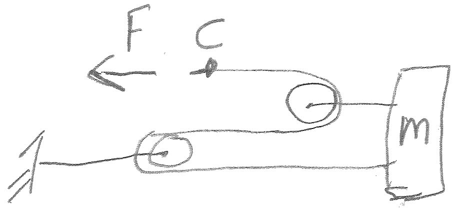
b)



tension at G = F/2

$a_B = a_G/2 \Rightarrow \boxed{a_B = F/(4m)}$ (b)

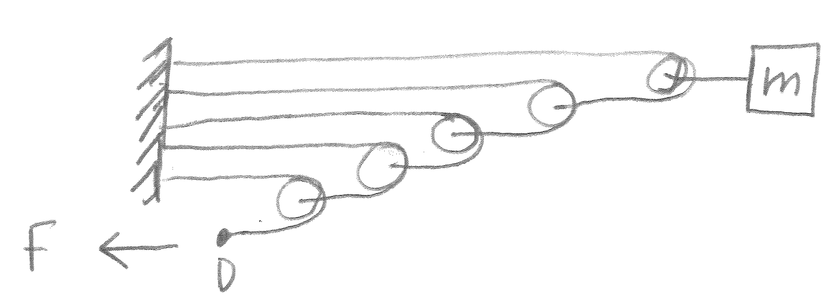
c)



Force on C = 3F

$a_c = 3a_m \Rightarrow \boxed{a_c = 9F/m}$ (c)

d)



Force on mass =

$2^5 \cdot F$

$a_D = 2^5 \cdot a_m$

$a_D = 2^{10}\ F/m$

$\boxed{a_D = 1024F/m}$ (d)

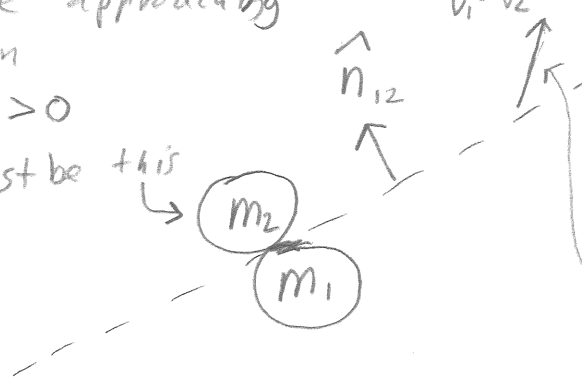
5) 2D. Two disks m_1 and m_2 collide with negligible friction. Their velocities before the collision are known, as is the coefficient of restitution is e. The line of common tangency at the instant of collision makes a 30° angle with the $+x$ axis (that is, the x axis rotated 30° counter clockwise is the common tangent line). Assume consistent units.

Complete the Matlab code at right to find the system kinetic energy after the collision.

For balls to be approaching before collision

$(\vec{V}_1^b - \vec{V}_2^b)\cdot\hat{n}_{12} > 0$

$\Rightarrow$ ball 2 must be this one



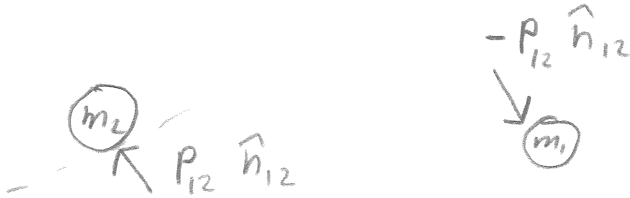
$30° = \theta$

$\vec{V}_1^b = 2\hat{\imath} + 9\hat{\jmath}$

$\vec{V}_2^b = 1\hat{\imath} - 3\hat{\jmath}$

$\vec{V}_1^b - \vec{V}_2^b = \hat{\imath} + 12\hat{\jmath}$

FBDs



$\hat{n}_{12} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

LMB$_1$: $\quad m_1\vec{V}_1^a = m_1\vec{V}_1^b - P_{12}\hat{n}_{12} \qquad (1)$

LMB$_2$: $\quad m_2\vec{V}_2^a = m_2\vec{V}_2^b + P_{12}\hat{n}_{12} \qquad (2)$

Restitution: $\quad (\vec{V}_2^a - \vec{V}_1^a)\cdot\hat{n}_{12} = -e(\vec{V}_2^b - \vec{V}_1^b)\cdot\hat{n}_{12} \qquad (3)$

1, 2, 3 $\Rightarrow$

$$\begin{array}{c} (1)\\ \\ (2)\\ \\ (3)\end{array}\underbrace{\begin{bmatrix} m_1 & 0 & 0 & 0 & +n_{12x}\\ 0 & m_1 & 0 & 0 & +n_{12y}\\ 0 & 0 & m_2 & 0 & -n_{12x}\\ 0 & 0 & 0 & m_2 & -n_{12y}\\ -n_{12x} & -n_{12y} & n_{12x} & n_{12y} & 0\end{bmatrix}}_{\text{Known} \rightarrow A}\underbrace{\begin{bmatrix} V_{1x}^a\\ V_{1y}^a\\ V_{2x}^a\\ V_{2y}^a\\ P_{12}\end{bmatrix}}_{z \leftarrow \text{find}} = \underbrace{\begin{bmatrix} m_1 V_{1x}\\ m_1 V_{1y}\\ m_2 V_{2x}\\ m_2 V_{2y}\\ -e(\vec{V}_2^b - \vec{V}_1^b)\cdot\hat{n}_{12}\end{bmatrix}}_{b \leftarrow \text{Known}}$$

$$\boxed{E_k = \frac{1}{2}m_1\vec{V}_1^a\cdot\vec{V}_1^a + \frac{1}{2}m_2\vec{V}_2^a\cdot\vec{V}_2^a}$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function prel2()
m1 = 1  ;    v1b =   [2   9];
m2 = 2  ;    v2b =   [1  -3];
e  = 0.5;
θ = π/6;
n     =  [ -sin(θ)   cosθ];     % = n12
```

$$b = [\underbrace{\underbrace{m_1 * V1b}_{1\times 2} \quad \underbrace{m_2 * V_2^b}_{1\times 2} \quad \underbrace{-e * (V_2^b - V_1^b) * n_{12}'}_{1\times 1}}_{5\times 1}]';$$

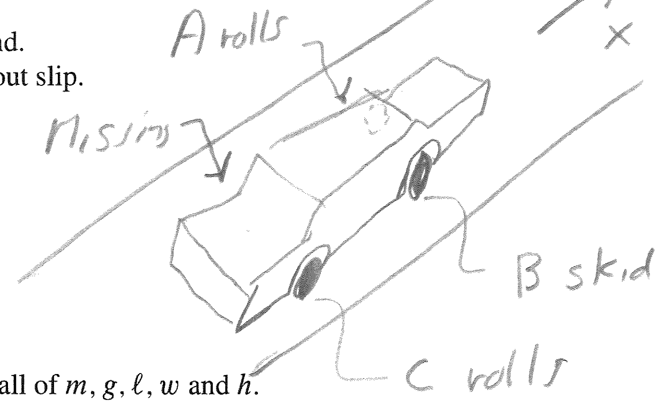
```
A = [ m1    0    0    0    n(1) ;
      0     m1   0    0    n(2) ;
      0     0    m2   0   -n(1) ;
      0     0    0    m2  -n(2) ;
      [-n        n        0]          ];

Z = A\b; % The great backslash!

V1a = z(1:2);
V2a = z(3:4);
```

$$\boxed{E_K = \frac{1}{2}(m_1 * V_{1a} * V_{1a}' + m_2 V_2^a * V_2^{a\prime})}$$

```
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

6) A car with mass m travels without tipping or turning on a straight path on a level road in the $+x$ direction ($+y$ is to the left and $+z$ is up).
* The back left wheel is missing so only three wheels touch the ground.
* The front left wheel A and back right wheel C both roll freely without slip.
* The front right wheel B is jammed and slides with friction $\mu = 1$.

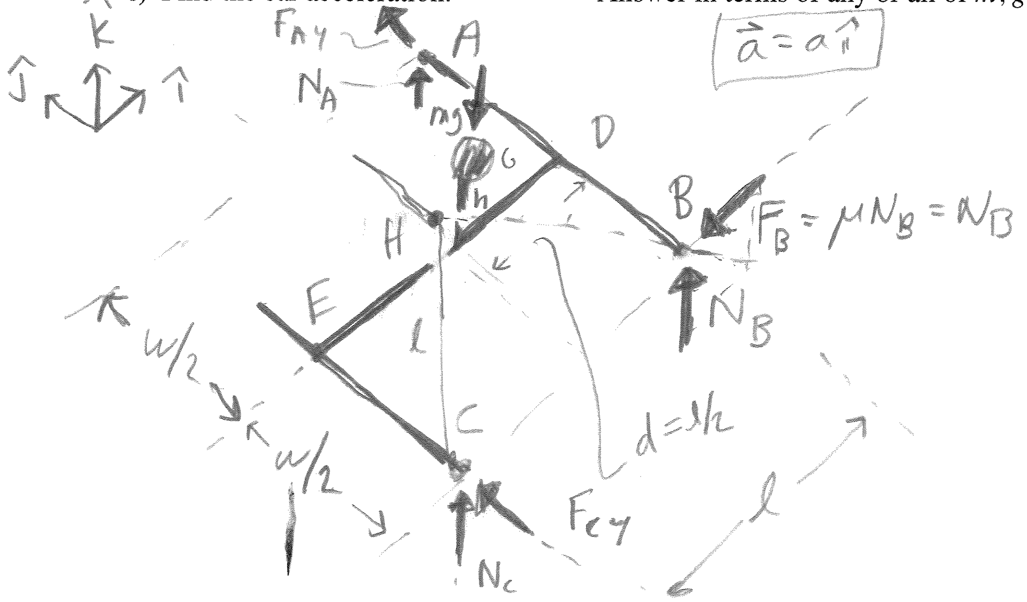
$\ell =$ the distance between the front and rear wheels,
$w =$ the width of the car,
$h =$ the height of G, the car's center of mass (assume $h < \ell/2$),
$d = \ell/2$ the distance of G back from the front axle.



a) Draw a clear sketch. Draw a clear free body diagram.

b) Find any unknown force component. Answer in terms of any or all of m, g, ℓ, w and h.

c) Find the car acceleration. Answer in terms of any or all of m, g, ℓ, w and h.



LMB

$$\left\{\sum \vec{F} = m\vec{a}\right\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow$$

$$N_A + N_B + N_c - mg = 0 \quad (1)$$

AMB$_{ED}$:

$$\left\{\sum \vec{M}_E = \dot{\vec{H}}_{/E}\right\}\cdot\hat{\imath}$$

$$-(N_c + N_B)\frac{w}{2} + N_A\frac{w}{2} = \underbrace{(\vec{r}_{G/E} \times ma\hat{\imath})\cdot\hat{\imath}}_{0} \quad (2)$$

$$\Rightarrow \quad N_A = N_B + N_c$$

$$(2)\ \&\ (1) \Rightarrow \quad \boxed{N_A = mg/2}\ (a)$$

Use pt. H: * directly above C, a height ℓ

* on line of action of $\vec{F}_B = N_B\hat{k} - N_B\hat{\imath}$

Consider axis Hy: through H, parallel to $\hat{j}$.

N_C intersects axis $\Rightarrow$ no moment

F_{Cy} $\parallel$ to axis $\Rightarrow$ " "

$\vec{F}_B$ intersects axis $\Rightarrow$ " " $\left[\vec{F}_B = N_B\hat{k} - N_B\hat{i}\right]$

F_{Ay} $\parallel$ to axis $\Rightarrow$ " "

$$\sum M_{Hy} = \dot{H}_{Hy}$$

$$mg(\underbrace{\ell-d}_{=\ell/2}) - \underbrace{N_A}_{=mg/2}\ell = \overbrace{-(\ell-h)ma}^{\left[\vec{r}_{G/H}\times(ma\hat{i})\right]\cdot\hat{j}}$$

$$0 = -(\ell-h)\cancel{m}a$$

$$\boxed{a = 0} \quad (b)$$

Why? w/ $d = \ell/2$ car is supported by wheels A & C

$\Rightarrow$ No normal force at B

Note: If $h > \ell/2$ then there could be another solution where pt. B locks (like frictional self-locking in statics) & car would flip. But this is beyond where we are in this class, hence the assumption $h < \ell/2$

Your TA, Section # and Section time:

Your name:

Cornell TAM/ENGRD 2030

Prelim 3

April 21, 2014

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes overtime)

How to get the highest score?

Please do these things:

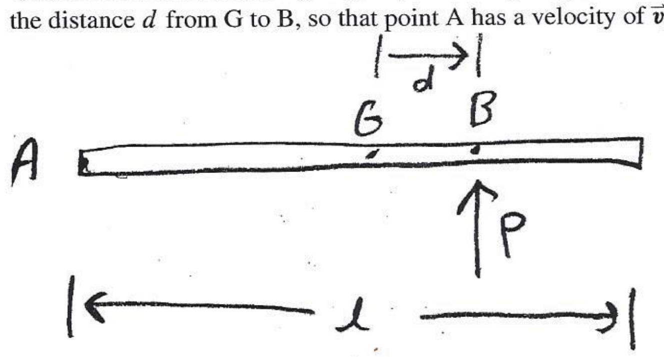
- : Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- $\vec{\bullet}$: Use correct **vector notation.**
- A+ : Be (I) neat, (II) clear and (III) well organized.
- □ : TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifiable algebraic expressions).
- >> : Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$T_7 = 18$" instead of, say, "`T(7) = 18`".
  Small syntax errors will have small penalties.
- : Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| : **Justify** your results so a grader can distinguish an informed answer from a guess.
- : If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ : Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 7: ____/25

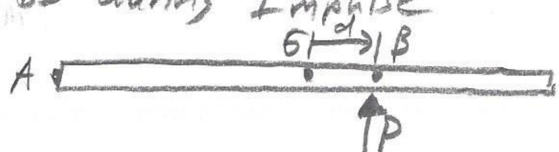
Problem 8: ____/25

Problem 9: ____/25

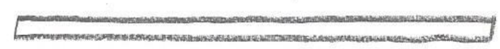
7) A uniform stick (mass m, length ℓ) is floating in space at rest. It is then hit with an impulsive force at B. Find the distance d from G to B, so that point A has a velocity of $\vec{v}_A = \vec{0}$ immediately after the impulse.



FBD during Impulse



FBD after Impulse



Kinematics

$$\vec{0} = \vec{V}_A^+ = \vec{V}_G^+ + \vec{\omega}^+ \times \vec{r}_{A/G} \quad (1)$$

LMB

$$\Sigma \int F\,dt = \int m\,\vec{a}_G\,dt$$
$$\vec{P} = m\Delta\vec{V}_G \Rightarrow \Delta\vec{V}_G = \vec{P}/m \quad (2)$$

AMB/G

$$\Sigma \int M\,dt = \int I_G\,\alpha\,dt$$
$$\vec{r}_{B/G} \times \vec{P} = I\Delta\vec{\omega} \Rightarrow \Delta\vec{\omega} = \frac{\vec{r}_{B/G}\times\vec{P}}{I_G} \quad (3)$$

ICs

$$\vec{\omega}(0^-) = \vec{0}, \quad \vec{V}_G(0^-) = \vec{0} \Rightarrow \Delta\vec{\omega} = \vec{\omega}(0^+), \quad \Delta\vec{V}_G = \vec{V}_G(0^+) \quad (4)$$

substitute 2,3 into 1 $\Rightarrow \vec{0} = \vec{P}/m + \dfrac{\vec{r}_{B/G}\times\vec{P}}{I_G} \times \vec{r}_{A/G}$

$\vec{P} = P\hat{j}$, $\vec{r}_{B/G} = d\hat{i}$, $\vec{r}_{A/G} = -\ell/2\,\hat{i}$

so $\vec{0} = \frac{P}{m}\hat{j} + \left(\frac{dP}{I_G}\right)\hat{k} \times \left(\frac{-\ell}{2}\right)\hat{i} = \frac{P}{m}\hat{j} - \frac{dP\ell}{2I_G}\hat{j}$

$\{\}\cdot\hat{j} \to 0 = \frac{P}{m} - \frac{dP\ell}{2I_G} \to d = \frac{2I_G}{m\ell}$

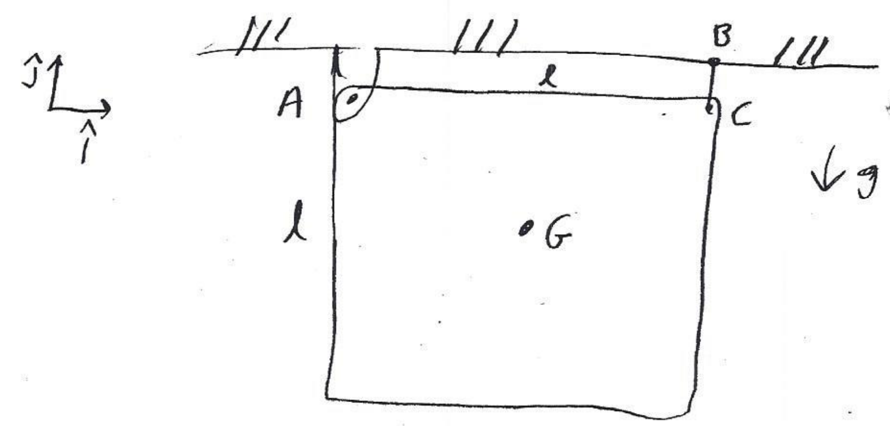
for a rod $I_G = \frac{m\ell^2}{12} \Rightarrow d = \frac{2}{m\ell}\left(\frac{m\ell^2}{12}\right) = \frac{\ell}{6} \Rightarrow \boxed{d = \frac{\ell}{6}}$

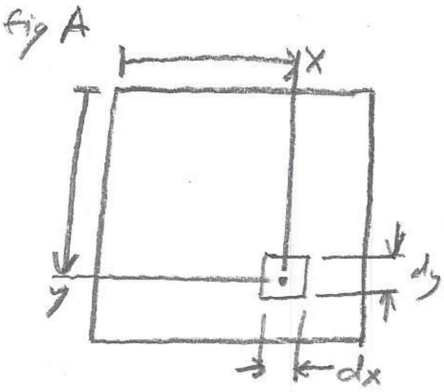
8) 2D. A uniform square plate of mass m and sides ℓ is suspended by a frictionless hinge at A and a string at C from a flat horizontal ceiling. At $t = 0$ the string BC is cut.

a) Calculate the moment of inertia I^G of the block about its center of mass in terms of m and ℓ.

b) Immediately after the string is cut, at $t = 0^+$, what is the acceleration of G, $\vec{a}_G$?
(Answer in terms of ℓ, m, g, I^G and the base vectors shown. Do not use your I^G from part a.)

c) Immediately after the string is cut, at $t = 0^+$, the vertical reaction force at A, $\vec{F}_A \cdot \hat{\jmath} = mg/2$, or not? Please give a clear reasoning for a yes or no answer without any long calculations. Full credit if there is precise reasoning with no algebra.





a) $I_G = \int r^2 dm$

$dm = \rho\, dx\, dy = \frac{m}{\ell^2} dx\, dy$

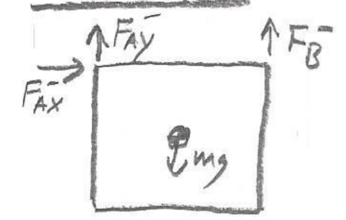
$r^2 = x^2 + y^2$ by figure A

$I_G = \int_{-\ell/2}^{\ell/2}\int_{-\ell/2}^{\ell/2} (x^2+y^2)\left(\frac{m}{\ell^2} dx\, dy\right) = \frac{m}{\ell^2} \iint x^2+y^2\, dx\, dy$

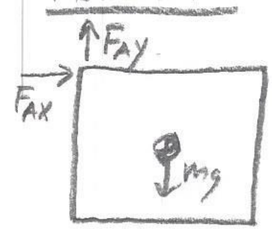
$\iint x^2+y^2\, dx\, dy = \int \left(\frac{x^3}{3} + y^2 x \Big|_{-\ell/2}^{\ell/2}\right) dy = \int \frac{\ell^3}{12} + \ell y^2\, dy = \frac{\ell^3 y}{12} + \frac{\ell y^3}{3}\Big|_{-\ell/2}^{\ell/2} = \frac{\ell^4}{12} + \frac{\ell^4}{12} = \frac{\ell^4}{6}$

$I_G = \frac{m}{\ell^2}\left(\frac{\ell^4}{6}\right) = \frac{m\ell^2}{6} \rightarrow \boxed{I_G = \frac{m\ell^2}{6}}$

b) FBD before



FBD after



$F_{AX} = 0$ by statics

AMB/A

$$\Sigma \vec{M}_{/A} = I_A \vec{\alpha} \rightarrow \Sigma M_{/A} = \frac{-mg\ell}{2}\hat{k} = I_A \alpha \hat{k}$$

$$\alpha = \frac{-mg\ell}{2I_A} \quad ①$$

Parallel Axis Thm

$$I_A = I_G + m|\vec{r}_{A/G}|^2 = I_G + m\left(\left(\frac{\ell}{2}\right)^2+\left(\frac{\ell}{2}\right)^2\right) = I_G + \frac{m\ell^2}{2} \quad ②$$

combining 1 and 2 $\rightarrow \alpha = \dfrac{-mg\ell}{2I_G + m\ell^2}$

Rigid body $\vec{a}_A = \vec{\alpha}\times\vec{r}_{A/G} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{A/G}) + \vec{a}_G$

$\vec\omega=\vec0$ at instant rope is cut $\rightarrow \vec\omega\times(\vec\omega\times\vec r_{A/G})=\vec0$

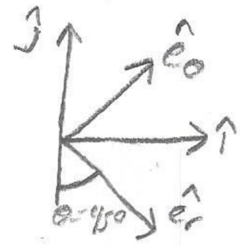
point A is hinge $\rightarrow \vec a_A=\vec0$

$\therefore \vec a_G + \vec\alpha\times\vec r_{A/G} = 0 \rightarrow \vec a_G = -\vec\alpha\times\vec r_{A/G}$ ③

sub 2 into 3 $\rightarrow \vec a_G = -\left(\dfrac{-mg\ell}{2I_G+m\ell^2}\right)\hat k \times \left(\dfrac{-\ell}{2}\hat\imath + \dfrac{\ell}{2}\hat\jmath\right)$

$$\boxed{\vec a_G = \frac{-mg\ell^2}{4I_G+2m\ell^2}(\hat\jmath+\hat\imath)}$$

note: if used $\hat e_r, \hat e_\theta \rightarrow \vec a_G = \dfrac{-mg\ell^2\sqrt2}{4I_G+2m\ell^2}\hat e_\theta$



c) Realize the force at A, the angular and linear acceleration and the force at B all change instantaneously when B is cut.

This means you cannot argue $F_{Ay}^- = \frac{mg}{2} \rightarrow F_{Ay}^+ = \frac{mg}{2}$

Mathematically

LMB: $\Sigma\vec F = F_{Ax}\hat\imath + F_{Ay}\hat\jmath - mg\hat\jmath = m\vec a_G$ (see FBD "after" on previous page)

$\{3\}\cdot\hat\jmath \rightarrow F_{Ay} = m\vec a_G\cdot\hat\jmath + mg = \dfrac{-mg\ell^2}{4I_G+2m\ell^2} + mg = mg\left(1-\dfrac{m\ell^2}{4I_G+2m\ell^2}\right)$

from part a: $I_G = \dfrac{m\ell^2}{6}$

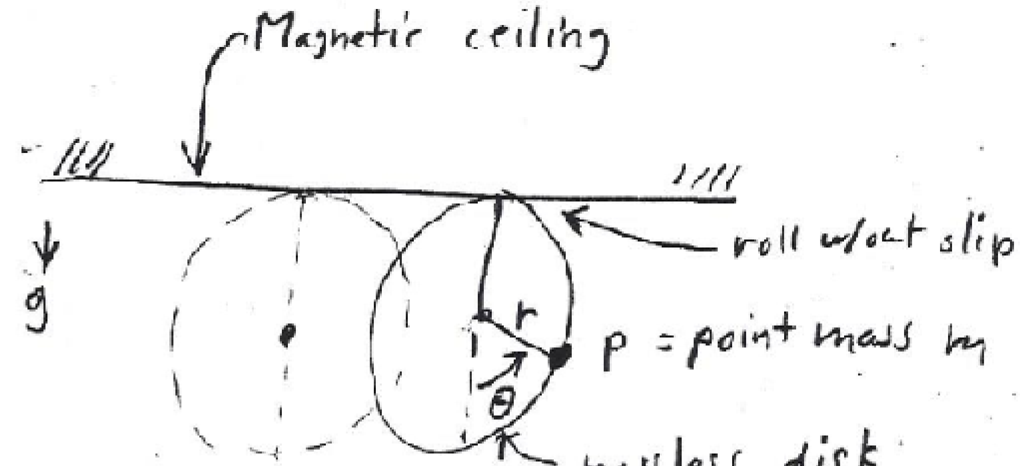
$$F_{Ay} = \left(1-\frac{m\ell^2}{4\left(\frac{m\ell^2}{6}\right)+2m\ell^2}\right)mg = \left(1-\frac{1}{2/3+2}\right)mg = \frac58 mg \Rightarrow \boxed{F_{Ay}=\frac58 mg \neq \frac{mg}{2}}$$

9) 2D. A massless disk with radius r rolls without slip on the ceiling, held up by magnetic forces. On the periphery is attached a point mass m. Gravity g acts. An equilibrium position is with $\theta = 0$ and the mass hanging a distance $2r$ from the ceiling.

a) Given r, θ and g find $\ddot{\theta}$.

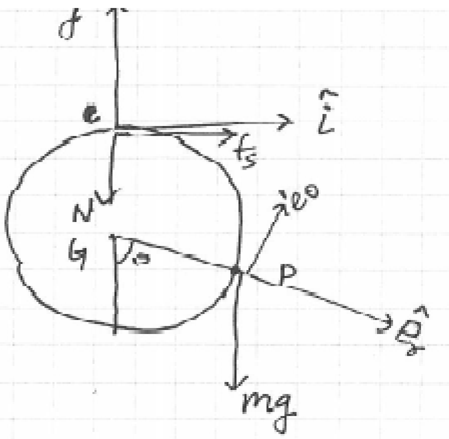
b) Find the period of small oscillation?

c) Does the period depend on amplitude? (genuinely hard question)



$\ddot{\theta} = ?$

FBD



$C \equiv (0,0)$

$G \equiv (0,-R)$

$P \equiv (R\sin\theta, -R(1+\cos\theta))$

$\vec{r}_{PC} = R\sin\theta\,\hat{i} - R(1+\cos\theta)\,\hat{j}$

$\vec{a}_P = \vec{a}_G + \vec{a}_{PG}$

$\vec{a}_G = R\ddot{\theta}\,\hat{i}$ ROLLING

$\vec{a}_{PG} = -\dot{\theta}^2 R\,\hat{e}_r + \ddot{\theta} R\,\hat{e}_\theta$

$\hat{e}_r = \sin\theta\,\hat{i} - \cos\theta\,\hat{j}$

$\hat{e}_\theta = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\vec{a}_P = (-\dot{\theta}^2 R\sin\theta + \ddot{\theta}R(1+\cos\theta))\,\hat{i} + (\dot{\theta}^2 R\cos\theta + \ddot{\theta}R\sin\theta)\,\hat{j}$

AMB

$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$

$\sum \vec{M}_{/C} = (R\sin\theta\,\hat{i} - R(1+\cos\theta)\,\hat{j}) \times (-mg\,\hat{j})$

$= -mgR\sin\theta\,\hat{k}$

$\dot{\vec{H}}_{/C} = m\,\vec{r}_{PC} \times \vec{a}_{cm} + \underbrace{I_{cm}}_{=0}\vec{\alpha}$ POINT MASS

$= m\,\vec{r}_{PC} \times \vec{a}_P$

$= m(R\sin\theta\,\hat{i} - R(1+\cos\theta)\,\hat{j}) \times [(\ddot{\theta}R(1+\cos\theta) - \dot{\theta}^2 R\sin\theta)\,\hat{i}$

$+ (\dot{\theta}^2 R\cos\theta + \ddot{\theta}R\sin\theta)\,\hat{j}\,]$

$$= mR^2\left[\sin\theta\,(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta) + (1+\cos\theta)(\ddot{\theta}(1+\cos\theta) - \dot{\theta}^2\sin\theta)\right]\hat{k}$$

$$= mR^2\left[\ddot{\theta}\left(\sin^2\theta + (1+\cos\theta)^2\right) + \dot{\theta}^2\left(\sin\theta\cos\theta - \sin\theta(1+\cos\theta)\right)\right]\hat{k}$$

$$= mR^2\left[\ddot{\theta}(2+2\cos\theta) + \dot{\theta}^2(-\sin\theta)\right]\hat{k}$$

$$\Rightarrow \quad -mgR\sin\theta = mR^2\left[2\ddot{\theta}(1+\cos\theta) - \sin\theta\,\dot{\theta}^2\right]$$

$$\Rightarrow \quad \frac{g}{R}\sin\theta + 2\ddot{\theta}(1+\cos\theta) - \dot{\theta}^2\sin\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} - \dot{\theta}^2\frac{\sin\theta}{2(1+\cos\theta)} + \frac{\sin\theta}{2(1+\cos\theta)}\frac{g}{R} = 0$$

$$\frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$$

(a)

$$\Rightarrow \quad \boxed{\ddot{\theta} - \frac{1}{2}\tan\frac{\theta}{2}\,\dot{\theta}^2 + \frac{1}{2}\tan\frac{\theta}{2}\,\frac{g}{R} = 0}$$

(b) SMALL ANGLE APPROXIMATION:

$$|\theta| \ll 1 \equiv \tan\frac{\theta}{2} \approx \frac{\theta}{2}$$

EQUATION (a) BECOMES:

$$\ddot{\theta} - \frac{\theta}{4}\dot{\theta}^2 + \frac{g}{4R}\theta = 0$$

IGNORING $\dot{\theta}^2$ TERM ALSO,

$$\boxed{\ddot{\theta} + \frac{g}{4R}\theta = 0}$$ EQUATION OF MOTION

TIME PERIOD $= 2\pi\sqrt{\frac{4R}{g}}$

CHECK IF APPROXIMATION IS FAIR :

$$\dot{\theta} \sim \frac{\theta}{T} = \frac{\theta}{2\pi}\sqrt{\frac{g}{4R}}$$

$$\ddot{\theta} \sim \frac{\theta}{T^2} = \frac{\theta}{4\pi^2}\left(\frac{g}{4R}\right)$$

WE FIND THAT

$$\left|\frac{\theta}{4}\dot{\theta}^2\right| \sim \left|\frac{1}{4}\frac{\theta^3}{4\pi^2}\left(\frac{g}{4R}\right)\right| \ll |\ddot{\theta}|, \left|\frac{g}{4R}\theta\right|$$

SO OUR APPROXIMATION IS OK!

(c) EQUATION FROM (a) CAN BE REWRITTEN AS

$$\cos\frac{\theta}{2}\,\ddot{\theta} - \frac{1}{2}\sin\frac{\theta}{2}\,\dot{\theta}^2 + \frac{g}{2R}\sin\frac{\theta}{2} = 0 \quad — (1)$$

SUBSTITUTE $\alpha = R\sin\frac{\theta}{2}$

$$\dot{\alpha} = \frac{R}{2}\cos\frac{\theta}{2}\,\dot{\theta}$$

$$\ddot{\alpha} = -\frac{R}{4}\sin\frac{\theta}{2}\,\dot{\theta}^2 + \frac{R}{2}\cos\frac{\theta}{2}\,\ddot{\theta}$$

SO EQUATION (1) BECOMES

$$\frac{2\ddot{\alpha}}{R} + \frac{g\alpha}{2R^2} = 0$$

$$\Rightarrow \left.\ddot{\alpha} + \frac{g}{4R}\alpha = 0 \quad\right\} \text{ SHM WITH } T = 2\pi\sqrt{\frac{4R}{g}}$$

WHICH HAPPENS TO BE EQUATION OF SIMPLE HARMONIC MOTION, WHICH SHOWS INDEPENDENCE BETWEEN TIME PERIOD & AMPLITUDE. ALSO THIS TIME PERIOD IS SAME AS FOR SMALL ANGLE APPROXIMATION!

—Print your netID on the top of every side of every sheet—
—Print clearly (for computer text recognition)—

Net ID (don't include '@cornell.edu')

"SOLUTIONS"

Your name:

ANDY RUINA

Cornell
ME 2030

Prelim 1

Feb. 21, 2019

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes over time)

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- >> Make appropriate `Matlab code` clear and correct.
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Problem 1: ____/25

Problem 2: ____/25

Problem 3: ____/25

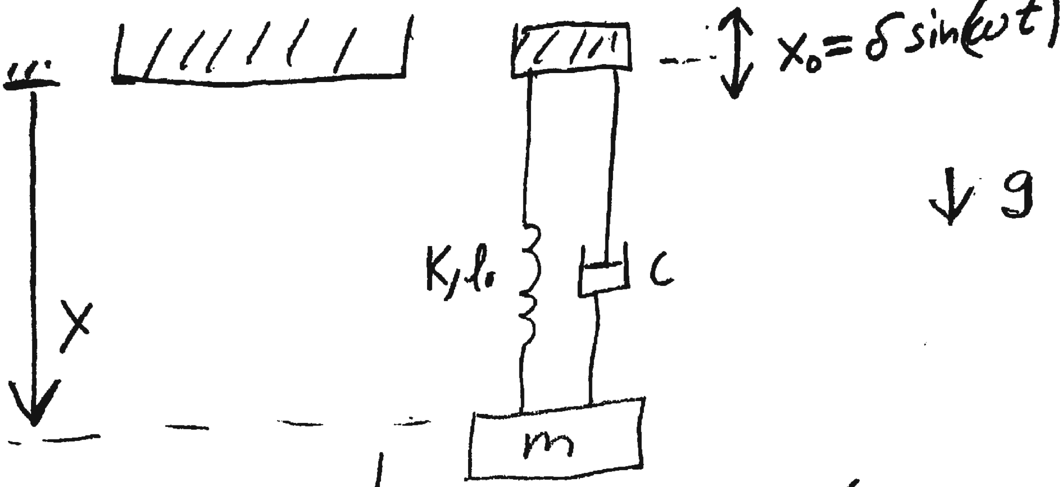
1) Shaken spring and mass. 1D. There is gravity. The rest length of the spring $\ell_0 \neq 0$.

You are given the constants $k, \ell_0, c, m, g, \delta$ and $\omega \neq \sqrt{k/m}$. All constants are positive (> 0).

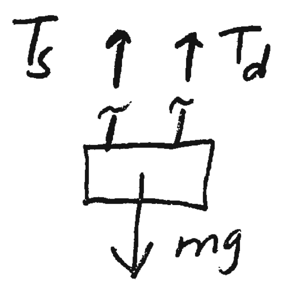
More details. The mass is suspended by a spring and dashpot. Gravity pulls down. A machine, not shown, shakes the support point by moving it sinusoidally up and down with amplitude δ and frequency ω. The the forcing frequency ω is known to not happen to be $\sqrt{k/m}$.

a) *Find the equations of motion.* (That is, find one or more differential equations whose solutions would describe motions of the system).

b) For the special case $c = 0$, find any solution to the governing equations. You need not find the general solution. Your solution, "$x(t) = \ldots$", should be in terms of some or all of $k, \ell_0, c, m, g, \delta, \omega$ and t.



FBD



$$T_s = k(\ell - \ell_0) = k((x - x_0) - \ell_0)$$
$$= k(x - \delta \sin(\omega t) - \ell_0) \quad (1)$$

$$T_D = c\dot{\ell} = c(\dot{x} - \dot{x}_0)$$
$$= c(\dot{x} - \delta\omega \cos(\omega t)) \quad (2)$$

LMB: $\quad F_{TOT} = ma \quad$ (1D, so skip vector notation)

$$\Rightarrow \quad mg - \underbrace{T_s}_{(1)} - \underbrace{T_D}_{(2)} = m\ddot{x}$$

$$\Rightarrow \quad \boxed{m\ddot{x} + c\dot{x} + kx = k\ell_0 + mg + k\delta \sin(\omega t) + k\delta\omega \cos(\omega t)} \quad \boxed{a}$$

(1) cont'd

set $c=0 \Rightarrow \quad m\ddot{x} + Kx = \underbrace{(k\ell_0 + mg)}_{①} + \underbrace{k\delta \sin(\omega t)}_{②}$

"Any" soln. $\Rightarrow$ steady state soln. is o.k

$\Rightarrow$ simplest particular soln. is o.k

$\Rightarrow$ solve ①, then ② then add.

①: guess $\quad x = x_1 = \text{const.}$ (static soln.)

$\Rightarrow \quad Kx_1 = k\ell_0 + mg \Rightarrow x_1 = \ell_0 + mg/k$

②: guess $\quad x = c_2 \sin(\omega t)$

$\Rightarrow -\omega^2 m c_2 \cancel{\sin(\omega t)} + c_2 K \cancel{\sin(\omega t)} = k\delta \cancel{\sin(\omega t)}$

$\Rightarrow c_2(-\omega^2 m + K) = k\delta$

$\Rightarrow \quad c_2 = \dfrac{\delta}{1 - \frac{m}{K}\omega^2}$

$\Rightarrow \quad x_2 = \underbrace{\dfrac{\delta}{1 - \frac{m}{K}\omega^2}}$ $\sin\omega t$

2 cases: [* in phase for $\omega < \sqrt{\frac{k}{m}}$
* out of phase for $\omega > \sqrt{k/m}$

$x(t) = x_1 + x_2$

$$\boxed{x(t) = \underbrace{\left(\ell_0 + \frac{mg}{K}\right)}_{①} + \underbrace{\left(\frac{\delta}{1 - \frac{m}{K}\omega^2}\right) \sin(\omega t)}_{②}}$$ b

Prelim 1, Q1: Some Reflection (Duan Li)

The statistics of the grades for Q1 are: Mean: 13.05, median: 11.0, standard deviation: 5.56. Please refer to my grading rubric on Gradescope to find out where you got points off and to decide whether you need regrading.
Below are a few common misperceptions I noticed and some advice to add onto Walker's and Ryan's a retrospective.

1. Pay attention to the coordinate system defined in the problem. You can define your own if there isn't one. In part a, the x axis is pointing downward, not upward, so $m\ddot{x}=mg-F_{spring}-F_{dashpot}$ if F_{spring} and $F_{dashpot}$ are defined to be pointing up in the FBD.
2. Have a clear mind of which force is applied on which object. There are only two components (the spring and the dashpot) connected directly to the mass, so there are only two forces (spring force and damping), besides gravity, applied on the mass. Some students had the driving force on the machine as part of their linear momentum balance (LMB) equation. This is wrong in several ways: (a) The machine drives the mass through the spring and the dashpot, so there is no direct force between the machine and the mass. (b) The total force on the machine should be $m_{machine}\ddot{x}_0$ and $m_{machine}$ is unknown.
3. Spring and dashpot respond to the net effect caused by all components directly connected to them. Here, both the mass and the machine are affecting the length of the spring and the velocity of the dashpot. The spring should respond to the net change in its length and the dashpot should respond to its net velocity.
4. Use physical intuition to check your result. At first glimpse of part b, you should realize this is a purely oscillatory system with no damping so the final solution should consists only of $\sin$, $\cos$ and constants. Some students end up having a t, $t2$ or e^t term in their solution, which means the components are moving further away from each other as time goes on and the system is breaking apart. If you get such a

solution, it's very likely that something is wrong with either your initial EoM or your way of solving the ODE. One case that could lead to a e^t term is if you made the mistake in #1, that is if you happen to set the x axis pointing up and get $m\ddot{x}-kx=something$ as your EoM instead of $m\ddot{x}+kx=something$. Mistakes are often correlated. If you identify one of them, you might to able to correct multiple of them.

2) **Mystery fluid.** 1D. No gravity. A bullet is shot into a mystery fluid. The bullet has mass m and initial velocity v_0. In the experiment it is observed that the bullet approaches a total penetration distance A according to:

$$x(t) = A - B/t$$

where A and B are measured. This is a strange mystery fluid with unknown properties. The goal is to use this data to find drag as a function of velocity.

$$F_D = f(v).$$

(Starts at $t > 0$ $\Rightarrow$ don't worry about $B/0 = \infty$)

Find $f(v)$.

Your formula for $f(v)$ can have some or all of the measure quantities m, v_0, A, and B.

FBD: $F_D \leftarrow \boxed{m}$

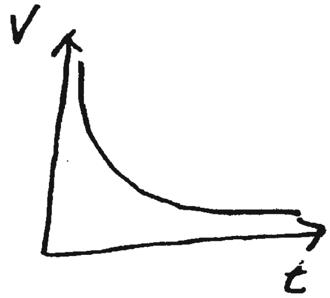
$\longrightarrow x$

Kinematics:

$V = \dot{X}$

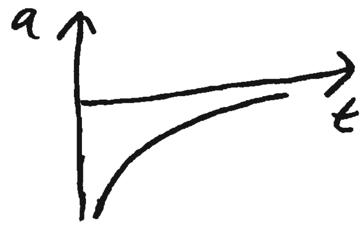
$= \frac{d}{dt}(A - B/t)$

$= B/t^2$



$a = \dot{V}$

$= -2B/t^3$



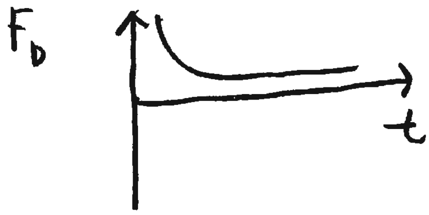
LMB

$F = ma$

$-F_D = ma$

$F_D = 2mB/t^3$

eliminate t $\Rightarrow$ $\boxed{F_D = 2m\, V^{3/2}/\sqrt{B}}$



note: $V^{3/2} = \left(\frac{B}{t^2}\right)^{3/2}$

$= \frac{B^{3/2}}{t^3} = \frac{B\sqrt{B}}{t^3}$

Note: as $V \to 0$

$V^{3/2} <$ linear drag for which $X \sim e^{-t}$

$V^{3/2} >$ quadratic drag for which $X \to \infty$

$\Rightarrow$ $V^{3/2}$ has $X \sim 1/t$ which is in between $X \sim e^{-t}$ and $X \sim \infty$

Prelim 1, Q2: A Retrospective (Walker Lee)

Hey all! I was responsible for grading problem 2 on the first prelim. The median score on the problem was a flat 15/25 (exam grades should be posted within the next few days). A lot of people made the same errors on this problem, so I thought it would be helpful to share a few common mistakes made when solving it. If you got less than 20/25, it is HIGHLY likely that one or more of the following went wrong:

1. Read the problem statement! For this problem, the problem statement explicitly stated that the system is in 1D and that there is no gravity. Despite this, I came across several 2D and 3D free-body diagrams, and several people included gravity on their FBDs or when writing out $F=ma$. Take a few seconds to *carefully* read the problem statement before starting *any* question, especially on an exam, as it will save you a huge headache later.
2. Draw your free-body diagrams! Speaking of FBDs, there were several students who would have scored a perfect 25 on this question, but they lost 5 points for not drawing an FBD. While this was physically painful for me to grade, the instructions on the front of the exam were clear: if you ever use linear momentum balance ($F=ma$) to solve a problem, you MUST draw a free-body diagram, even if there is only 1 force on the body in question.
3. Function notation. The problem asked for a final answer in the form $f(v)$. Whenever a function is written in this notation, the output (in this case, the drag force) is a function of only scalars (numbers such as 2 or π), constants (parameters that don't change, such as m), and the variables inside the parenthesis (in this case, v). Any variables NOT inside the parenthesis (here, t and x) cannot appear in the final answer, because the answer would no longer be $f(v)$ - it would be $f(v,t)$ or $f(v,x)$ instead.

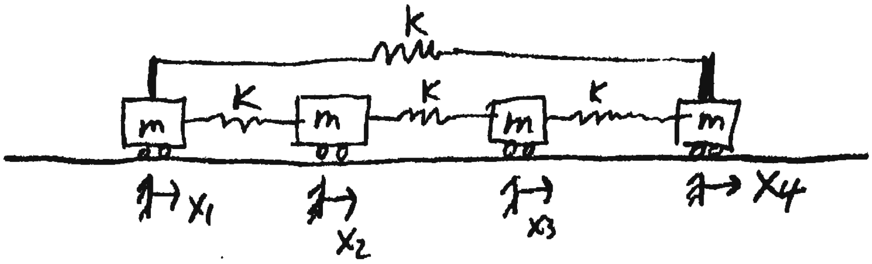
4. Assumptions. This was probably the most common mistake: many students assumed that $f(v)$ would have the form $-cv$ or $-cv^2$. However, for this problem, that is impossible - you already are given $x(t)=A+B/t$, and if you integrate a linear or quadratic drag twice, you get $x(t) \sim e^{-t}$ or $-\ln|t|$, respectively, as you've seen on the homework. Therefore, the drag in this problem cannot possibly be linear nor quadratic. In general, making assumptions that are not given in the problem statement is extremely dangerous, so if you do, always be prepared to back them up.
5. Equation validity. Many students tried to use one or more of these equations without realizing they wouldn't work:

- Some variation of $v=\Delta x/t$
- Some variation of $a=\Delta v/t$
- $\Delta x = v_0 t + \frac{1}{2} at^2$
- $v_f^2 - v_0^2 = 2a\Delta x$
- $\int F dt = Ft$
- $W=Fd$

These equations only hold when velocity (first row), acceleration (rows 2-4), and force (rows 5-6) are constant, respectively. In this problem, none of these are constant, so using these equations wouldn't help you. Before you use an equation, make sure you understand what assumptions were made when it was derived so you don't use it under invalid conditions.

6. Energy methods. Several students attempted to use energy methods to solve this problem. While I applaud your creativity, any answer derived using energy methods would have to contain v_0, and v_0 was a trap: $x(t)=A+B/t$ gives $v(t)=B/t^2$, which in turn means v_0 is undefined.

3) Four masses in a line. Four equal masses m are in a line connected by 4 equal springs k, as shown. You can assume that the springs are all relaxed when $x_1 = x_2 = x_3 = x_4 = 0$.



a) Find the matrices M and K so that $M[\ddot{x}] + K[x] = [0]$, where $[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

b) Assume these have been stored in `p.M` and `p.K`. Write a file, suitable for use with, say, your midpoint solver, that has the differential equations in Matlab language. That is, fill in the middle lines of code (possibly more or less than 3 lines) of this Matlab function:

```
function zdot = myrhs(t,z,p)
...
...
...
zdot = ...
end
```

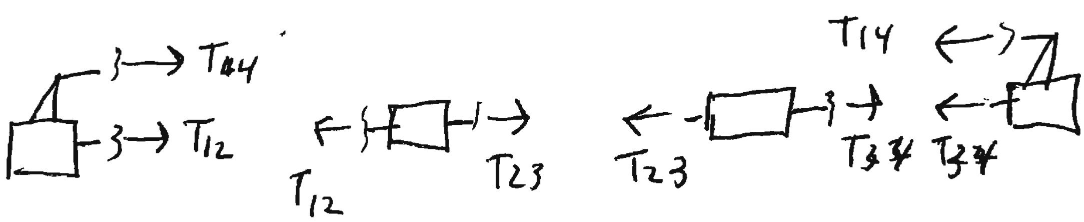
c) The matlab command `[V,D] = eig(M^(-1)*K)` gives two matrices. Using any reasoning, intuition or guesses that you like, *specify as much as you can about the entries in those matrices*. That is, find as many columns of `V` and associated diagonal elements of `D`? (For fairness to those who are now taking Math 2940, and to save you from time doing tedious algebra, do not find and solve a characteristic polynomial.)

If you find lots and lots of normal modes, be clear about which ones are independent and which are just combinations of others.
Hints:

1) Check that one column of `V` is proportional to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and has associated diagonal element of `D` of 0.

2) This problem has lots of symmetry, hence at least some normal modes can be found by inspection;

FBDs



$T_{12} = k(x_2 - x_1)$ $\qquad T_{23} = k(x_3 - x_2)$ $\qquad T_{34} = k(x_4 - x_3)$

$T_{14} = k(x_4 - x_1)$

(3) (cont'd)

LMB

$$m\ddot{x}_1 = -2kx_1 + kx_2 + 0x_3 + kx_4$$
$$m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3 + 0\cdot x_4$$
$$m\ddot{x}_3 = 0\cdot x_1 + kx_2 - 2kx_3 + kx_4$$
$$m\ddot{x}_4 = kx_1 + 0x_2 + kx_3 - 2kx_4$$

Matrix Form:

$$\underset{4\times4}{M}\ \underset{4\times1}{\ddot{x}} + \underset{4\times4}{K}\ x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

a

$$K = k\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$M = m\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{8\times1}{z} = \begin{bmatrix} x_1 \\ \vdots \\ x_4 \\ v_1 \\ \vdots \\ v_4 \end{bmatrix}$$

MATLAB

b

```
function zdot=myrhs(t,z,p)
x= z(1:4);  v= z(5:8);

xdot= v;  % Kinematics

vdot= -inv(p.M)* p.K*x;  % F=ma
% vdot = -p.M\p.K*x;  % Alt. method

zdot= [xdot; vdot];
end
```

③ (cont'd)

Each col. of V is a vector w such that

$$A\,w = \lambda\,w$$

w ← eigenvector

λ ← eigen value = diag. element of D

A = $M^{-1}K$

* Matlab uses unit vectors. Here we'll skip that step (simpler looking).
* Matlab makes arb. choices for linear combs. if one e-value has 2 or more e-vectors. Here we'll make it simple looking.
* Note symmetry: each mass connected to 2 neighbours (even end masses)

Guess w	Picture	Keff	Aw	λ	good?
[1 1 1 1]'	→ → → →	0	0·w	0	✓
[1 0 0 0]'	→ • • •	nonsense	≠ λw	X	X
[1 1 -1 -1]'	→ → ← ←	2K	$\frac{2K}{m}$ w	$\frac{2K}{m}$	✓
[1 -1 -1 1]'	→ ← ← →	2K	$\frac{2K}{m}$ w	2K/m	✓
[1 0 -1 0]'	→ • ← •	2K	$\frac{2K}{m}$ w	2K/m	✓
[0 1 0 -1]'	• → • ←	2K	$\frac{2K}{m}$ w	2K/m	✓
[1 -1 1 -1]'	→ ← → ←	4K	$\frac{4K}{m}$ v	4K/m	✓

any lin. independent lin. combs of *

$$D = \frac{K}{m}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}; \quad V = \begin{bmatrix} 1/2 & & & 1/2 \\ 1/2 & & & -1/2 \\ 1/2 & & & 1/2 \\ 1/2 & & & -1/2 \end{bmatrix}$$

Prelim 1: Q3: Common Mistakes (Ryan Elandt)

Part A (8 Points): If you get 6 points it was likely because you wrote down the negative of the stiffness matrix. For our problems it also needs to be symmetric due to Newton's third law. [In more general advanced vibration problems it is also symmetric, for other reasons. Ask if you want to know.] With practice you should be able to find the components of the stiffness matrix visually. At least for problems with masses in a line. Then you can check your work by comparing with what visual inspection tells you the answer has to be.

Part B (9 Points): Students lost points in a wide variety of ways. Students lost three points for each major mistake. Mistakes usually either: produce incorrect answers, produce right answers for the wrong reasons or do something in a way that is excessively long. Some common mistakes included:

- Not realizing that K and M are matrices
- Not realizing that z and zdd are column vectors
- Assigning into an unallocated array z(1:4) (this creates a column vector)
- The correct code needs a line that is equivalent to

"vdot = - inv(p.M) * p.K * x".

That is, solving the equation $M\ddot{x}=-Kx$ for $\ddot{x}$ gives $\ddot{x}=-M^{-1}Kx$.

Advice for the future: Although not required, using inv(M) to represent M^{-1} is a good way to avoid mistakes. This way you do not have to remember which of "M \ K", "M / K", "K \ M", "K / M" and "K * (1 / M)" is right. Although I didn't take off points you should not write "inv(M)" as "M^-1" as it will not work in many languages, you should write "M^(-1)" although, as mentioned, that's not as good as inv(M) or pinv(M).

- Writing out matrix multiplication component by component (this is the point of linear algebra, to *not* do this)

Part C (8 Points but 12 Points possible): The eigenvalues of this matrix are the frequencies of the normal modes squared. A lot of students did not realize this. I did not take off points for this. A lot of students thought that things like [1; -1; 0; 0] or [0; 1; -1; 0] were eigenvectors, they are not. You can see that they are not two ways: 1) pre-multiply by inv(M)*K and see that you don't get the same vector back; or 2) notice that there are forces on the supposedly stationary masses, which is not consistent (that is, you assumed that they don't move even though in your proposed solution there is a force on the masses).

—Print your netID on the top of every side of every sheet—
—Print clearly (for computer text recognition)—

Net ID (don't include '@cornell.edu')

SOLUTION

Your name:

ANDY RUINA

Cornell

ME 2030

No calculators, books or notes allowed.

Prelim 2

March 19, 2019

3 Problems, 90 minutes (+ up to 90 minutes extra)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation.**
- A+ Be (I) neat, (II) clear and (III) well organized.
- TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$\phi_7 = 2\pi$" instead of, say, "`phi(7) = 2*pi;`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- →| **Justify** your results so a grader can distinguish an informed answer from a guess. If you quote a fact that a grader might doubt your understaning of, explain it. Especially if it is not commonly used.
- If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem). If you are referred here during the exam it means that answering your question would be telling you something you are being tested on.
- ≈ Work for **partial credit** (from 50% – 100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the ~~desired~~ answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for you. Ask for more if you need it. Put your name on each sheet, and fold it in. Refer to extra pages (We have to find them on a computer scan.)

Problem 1: ______ /25

Problem 2: ______ /25

Problem 3: ______ /25

1) Towed trailer. 2D. There is gravity. A rope pulls with force F on the top of a trailer with mass m, length 2ℓ, and height h. The center of mass G is in the middle. There are good wheels under the center of mass at A and it skids with coefficient of friction μ at B. The car is already moving to the right when F is applied. Positive acceleration a is to the right. Note whether applying $F > 0$ speeds up or slows down the trailer compared to $F = 0$.

a) Find the acceleration of the block. If relevant, assume that $\mu < \ell/h$. Answer in terms of some or all of F, m, h, ℓ, g and μ.

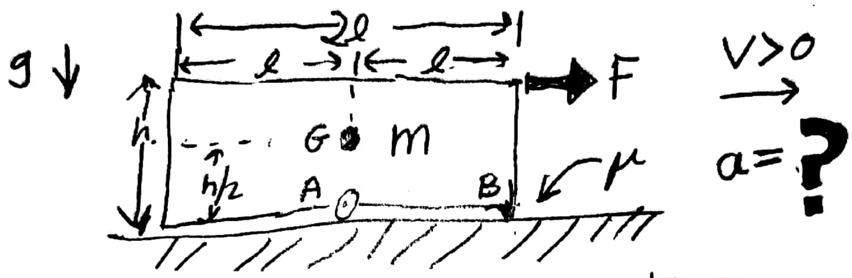
b) Challenges: Find the acceleration of the block, and/or explain what is funny or interesting for various larger values of μ, for example these three cases:

i) $\mu = \ell/h$;

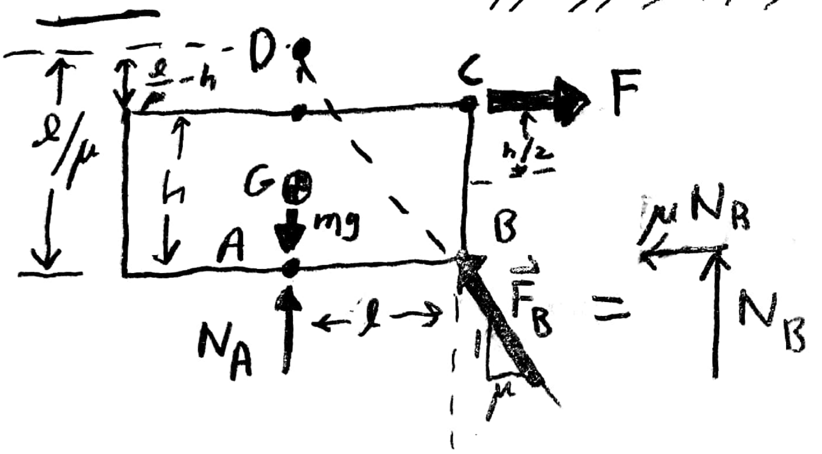
ii) $\ell/h < \mu < 2\ell/h$;

iii) $2\ell/h < \mu$;

$\boxed{\vec{a}_G = a_G \hat{\imath}}$ ← ASSUME NO ROTATION



FBD



AMB/$_D$: $\left\{ \sum \vec{M}_{/D} = \vec{r}_{G/D} \times m\vec{a}_G \right\} \cdot \hat{k}$

$\Rightarrow F \cdot (\ell/\mu - h) = (\ell/\mu - h/2)\, m a_G$

(a) $\boxed{a_G = \frac{F}{m} \frac{(\ell/\mu - h)}{(\ell/\mu - h/2)}}$

for $\ell/\mu > 0$ this is +. F changes a_G from 0 to greater than 0.

bi) $\mu = \ell/h \Rightarrow$!

$a_G = 0$

Force has no effect on accel.

bii) $\boxed{\mu > \ell/h \Rightarrow a_G < 0}$

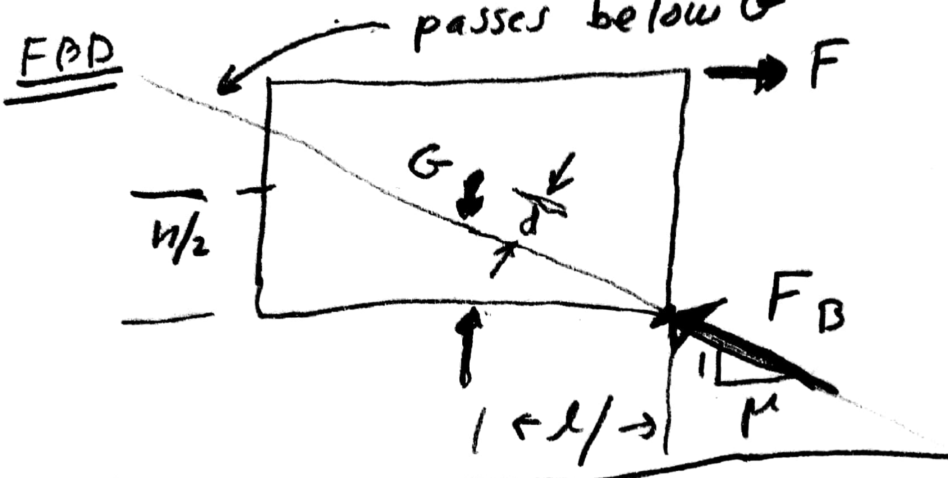
Pulling to right slows trailer down!!

biii) $\mu > 2\ell/h$

changes sign again?

Too Weird $\Rightarrow$

b iii) cont'd $\mu > 2\ell/h$



AMB/G

$-Fh/2 - dF_B = 0$

$\Rightarrow \boxed{F_B < 0}$

Nonsense!

Summary

$F = 0 \Rightarrow$ no accel.

$\mu < \ell/h$	F causes $+a$
$\frac{\ell}{h} < \mu < 2\ell/h$	F causes negative a $\Rightarrow$ pulling causes slowing
$\mu = 2\ell/h$	stopping accel $\to \infty$
$\mu > 2\ell/h$	NONSENSE $\Rightarrow$ problem is nonsense. What would happen? Ans: Tipping

Prelim 2, Q1: Post-Mission Briefing #303

Hey all! I graded Q1 of prelim 2 (the towed trailer problem), and I have returned to explain how I scored the problem and to share some common mistakes that students made while attempting it.

Scoring

Part (a) was scored out of 15 points and each part of (b) was worth 4, for a total of 27 points possible if you did the entire problem perfectly. In general, people did very well on this problem; the median was 19, and over 40% of the class got a 23 or higher.

I was incredibly generous with partial credit for (b). This was because if I weren't, and a mistake in (a) meant an automatic 0 for part (b), then a small algebraic mistake or sign error in part (a) could cost you up to 16 points, which is ridiculous. As a result, I awarded some credit for a reasonable interpretation of your answers, even if your answer to part (a) was wrong. Even so, small mistakes in part (a) were more punishing than usual, because you not only didn't get the acceleration formula correct, but it was also impossible to get full marks for (b).

Common Mistakes

1. **Angular Momentum Balance:** the most common mistake, by far, was to set the sum of the moments about a point equal to 0 when you shouldn't have. In 2D, the formula for AMB is this: $\sum \vec{M}_{/C} = \vec{r}_{G/C} \times m\vec{a}_G + I^G\alpha\hat{k}$ where G is the center of mass and C can be any point of your choice. In this problem, α (angular acceleration) is 0, but you still have to deal with the rest of the right hand side. If you choose C = G, or if you chose C to be another point on the horizontal line containing G, then the right hand side simplifies to 0, but if you chose C to be A or B, then it does not.
2. **Cross Products:** don't get lazy with your cross products! Many of you attempted to "eyeball it" without writing out the unit vectors and got a sign error for your trouble. Because of the sequential nature of this problem - you needed to do (a) perfectly to get full credit on (b) - you paid a heavy price for your mistake, probably on the order of 5-7 points.
3. **Counter-Clockwise is Positive:** additionally, it is worth noting that $\hat{\imath} \times \hat{\jmath}$ equals positive $\hat{k}$; therefore, in this problem, where positive $\hat{\imath}$ points to the right and positive $\hat{\jmath}$ points up, a positive moment $(\hat{k})$ must be counter-clockwise, and a clockwise moment will be negative $(-\hat{k})$. Some of you defined clockwise moments as positive and counter-clockwise as negative. I didn't take any points off this time because the equations you wrote were technically still correct (if you just flip the signs on every term, the equation doesn't change), but in the future, especially in 3D problems, you might end up writing an equation that isn't true if you make that kind of assumption.

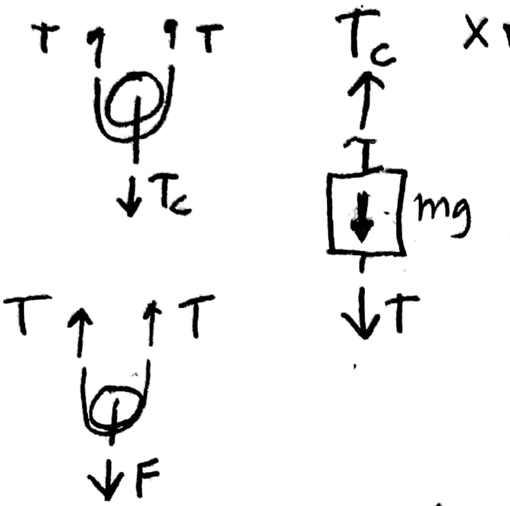
2) Pulleys. 1D. There is gravity. All parts have negligible mass and weight except m. Make the usual assumptions about pulleys and ropes.

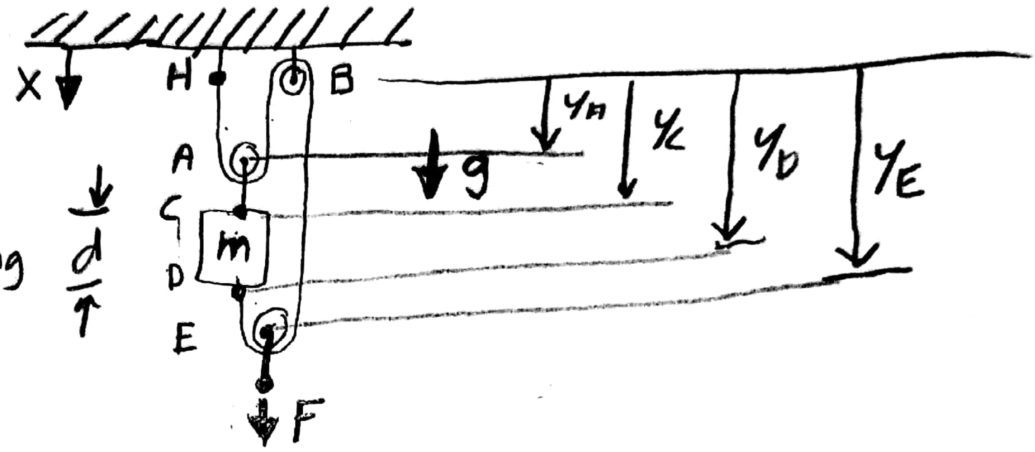
a) Find F for this system to be in static equilibrium?

b) For other values of $F > 0$, find a_E.

Radii all $= R$

FBDs





Statics: $T = F/2$ & $T_c = 2T = F$

(a)

a) Equilib: $\underset{\uparrow F/2}{T} + mg - \underset{\uparrow F}{T_c} = 0 \Rightarrow \boxed{F = 2mg}$ ← equilibrium

b) Kinematics

$$\{2y_A + (\pi R)\cdot 3 + y_E + y_E - y_D = 0\} \quad (1)$$

$$y_D - y_C = d = \text{const}, \quad y_C - y_A = \text{const}$$

$$\frac{d^2}{dt^2}\{\ \} \Rightarrow 2\ddot{y}_C + 2\ddot{y}_E - \underset{\uparrow = \ddot{y}_D}{\ddot{y}_C} = 0 \Rightarrow \boxed{\ddot{y}_C = -2\ddot{y}_E} \ *$$

LMB for mass

$$\sum F_y = m\ddot{y}$$

$$\underset{F/2}{mg + T} - \underset{F}{T_c} = m\ddot{y}_c \Rightarrow \boxed{\ddot{y}_c = g - F/2m}$$

$$* \Rightarrow \ddot{y}_E = \frac{-\ddot{y}_c}{2}$$

$$\boxed{a_E = \ddot{y}_E = \frac{-g}{2} + F/4m}$$ (b)

Same as lecture rotated 90°, gravity added.

check against part (a)

$$\underbrace{F = 2mg}_{\text{equilib soln}} \Rightarrow a_E = \frac{-g}{2} + \frac{g}{2} = 0$$ ✓

Prelim 2 Q2: Summary #308

The whole class did really well on question 2. Everybody drew their free body diagrams. One third of the class got the problem fully correct. The mean is 20.49; the median is 23.0; the standard deviation is 4.86. The grading rubric can be found on Gradescope. There are only a few things to pay attention to:

1. **Consistency of coordinate systems.** Make sure your coordinate system is consistent throughout the problem. A lot of students took positive x as pointing downward when they wrote their string length equation, but flipped the x coordinate when they wrote their linear momentum balance equation (LMB) and ended up with an inverted sign error. I deducted 2 points for each flip of the coordinate system. Remember: $\ddot{\vec{x}} = \vec{a}$, so your positive x, positive $\ddot{x}$, and positive a should all point in the same direction.

2. **Motion of massless objects.** Since the mass of the pulley is negligible, you can't use linear momentum balance (LMB, i.e. $\vec{F} = m\vec{a}$) on it. To treat this kind of problems, you need to relate the motion of the massless object directly to the motion of a mass by considering the linearity between them. In this problem, we get $\ddot{\vec{x}}_A = \ddot{\vec{x}}_m$ because pulley A is rigidly connected to the mass and $\ddot{\vec{x}}_E = -\ddot{\vec{x}}_m/2$ by taking the second derivative of the constant string length equation.

3) Three masses in space. 2D. Assume consistent units. The only concern is one instant in time.
- All three masses are mutually attracted to each other with inverse-square gravity G.
- A linear negligible-mass dashpot c connects m_1 and m_2.
- A given external force $\vec{F}_3^{ext}$ acts on mass 3.
- There are no other forces.
- **Short problem statement:** Write matlab commands to find the acceleration $\vec{a}_1$ of m_1.

Givens. For the instant of interest, assume that these things have been defined (*i.e.*, assigned numerical values):

- **The system state,** the positions and velocities of the three masses (each one is a 2-elementMATLAB column vector): `r1, r2, r3, v1, v2, & v3`, relative to an inertial fixed origin;
- **The three masses:** `m1, m2, & m3`;
- **Other parameters:** The universal gravity constant `G`; the constant `c` of the one dashpot; and the two-component applied force as a column vector `F3_ext`.

Your solution can use any or all of these.

Full problem statement: Find the components of $\vec{a}_1$ by filling in this MATLAB function:

```
function [a1] = my_three_body(m1,m2,m3,r1,r2,r3,v1,v2,v3,F3_ext,G,c)
...
end
```

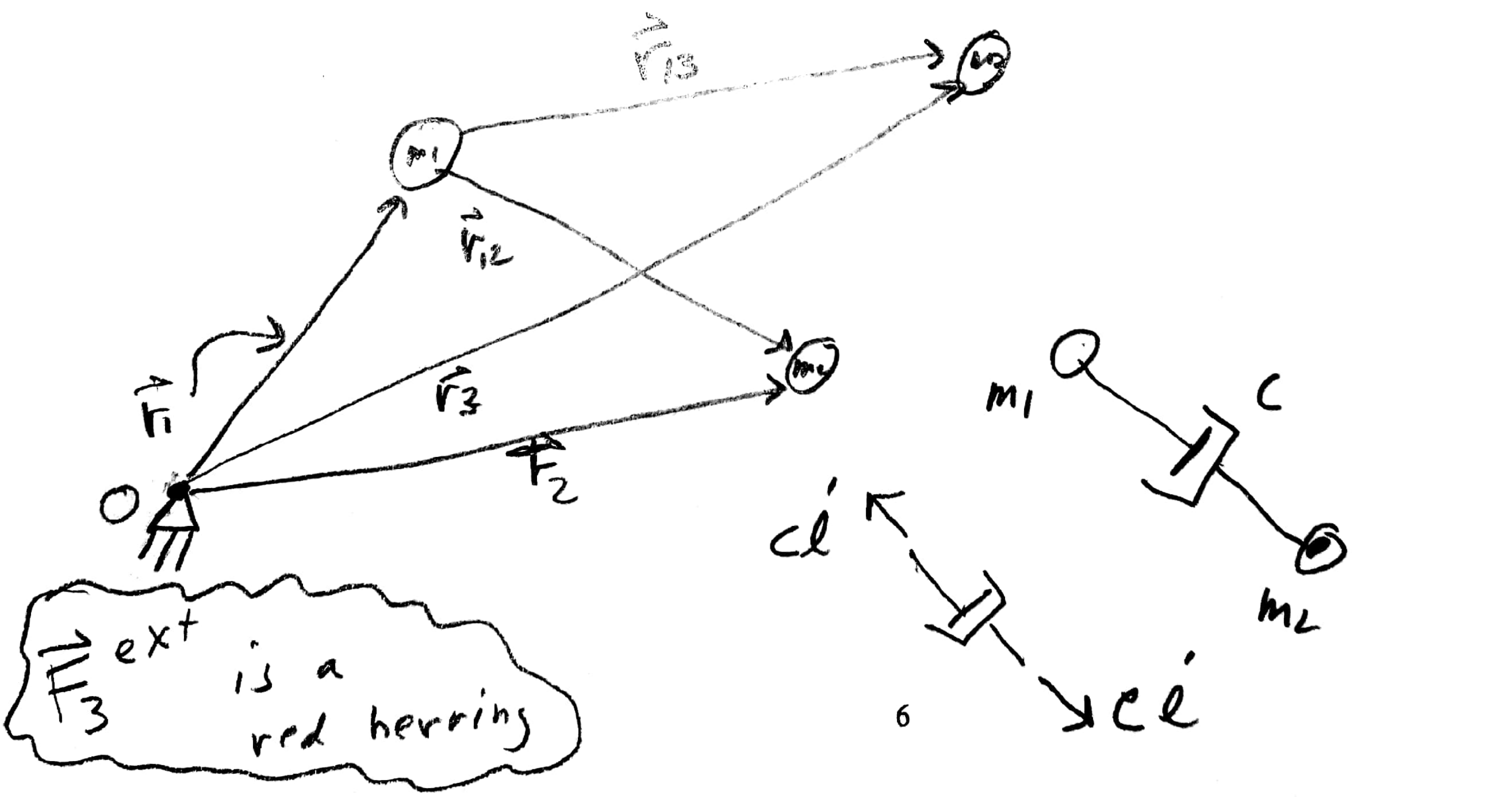
(Note: the accelerations $\vec{a}_2$ and $\vec{a}_3$ are of no concern.)

Problem restatement. Given all of the parameters listed, write MATLAB commands to find the acceleration `a1` of mass 1. The last line of your code should start with:

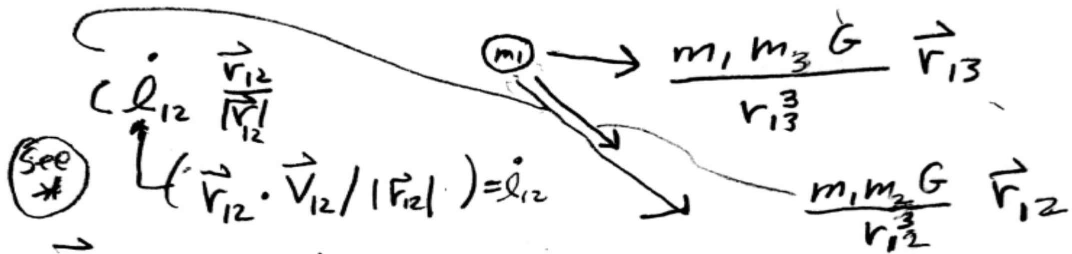
```
a1 = ,
```

where `a1` is a 2 element MATLAB column vector.

Clarifications:
* Write easy-to-understand short lines of code; <u>don't</u> write long complex lines of code.
* Don't make any explicit use of vector components; only partial credit if you refer to, say, `r1(1)` or `r1(i)`.
* Variables on the right side of = signs must be givens (above), or previously defined by you.
* Partial credit for missing, incomplete or incorrect code will be based on:
  sketches, geometry, free body diagrams, math (*e.g.*, eqns using $\vec{a}_1$, $\vec{r}_3$,...), *etc.*



FBD of mass 1



$$\vec F_c = c(\vec r_{12}\cdot\vec V_{12})\,\vec r_{12}/|\vec r_{12}|^2$$

```
function [a1] = my_three_body(m1,m2,m3,r1,r2,r3,v1,v2,v3,F3_ext,G,c)

    r_13 = r3 - r1;  % position from 1 to 3
    r_12 = r2 - r1;  % position from 1 to 2

    v_12 = v2 - v1;  % velocity of 2 wrt 1

    dist_13 = norm(r_13);  % distance from 1 to 3
    dist_12 = norm(r_12);  % distance from 1 to 2

    unit_r_12 = r_12 / dist_12;  % unit vector pointing from 1 to 2
    unit_r_13 = r_13 / dist_13;  % unit vector pointing from 1 to 3

    dot_dist_12 = dot(unit_r_12, v_12);  % change in distance from 1 to 2

    mag_f_grav_12 = m1 * m2 * G / (dist_12^2);  % gravitational force magnitude between 1 and 2
    mag_f_grav_13 = m1 * m3 * G / (dist_13^2);  % gravitational force magnitude between 1 and 3
    mag_f_c_12    = c * dot_dist_12;  % damping force magnitude between 1 and 2

    f_grav_12 = mag_f_grav_12 * unit_r_12;  % gravitational force vector between 1 and 2
    f_grav_13 = mag_f_grav_13 * unit_r_13;  % gravitational force vector between 1 and 3
    f_c_12    = mag_f_c_12    * unit_r_12;  % damping force vector between 1 and 2

    a1 = (1 / m1) * (f_grav_12 + f_grav_13 + f_c_12);  % Newton's 2nd Law
end
```

*Note: $\{\ell_{12}^2 = \vec r_{12}\cdot\vec r_{12}\}$ (as per lecture)

$$\frac{d}{dt}\{\} \Rightarrow 2\ell_{12}\dot{\ell}_{12} = 2\,\vec r_{12}\cdot\dot{\vec r}_{12}$$

$$\dot{\ell}_{12} = \vec r_{12}\cdot\dot{\vec r}_{12}/\ell_{12} = \vec r_{12}\cdot\vec V_{12}/|\vec r_{12}|$$

I. Final Thoughts: Problem 3

This problem was graded on a two-tiered grading system. The first tier is for students who made small number of mistakes in their code. The second tier is for students who wrote problematic code or no code at all. In these cases, I carefully looked at the free body diagram and equations of motion to assess understanding on a case by case basis.

A. *If you scored 16 or more points:*

Most students lost 4 points for incorrectly calculating the damping force. The damper opposes the *change in length* between masses 1 and 2. As covered in lecture, there are multiple ways of deriving this change in length. One way is:

$$l^2 = r_{12} \cdot r_{12} \tag{1}$$

$$2l\dot{l} = 2r_{12} \cdot \dot{r}_{12} \tag{2}$$

$$\dot{l} = \frac{r_{12} \cdot \dot{r}_{12}}{l} \tag{3}$$

The damping force magnitude is $c\dot{l}$ and the damping force vector is $c\dot{l}r_{12}$.

About half of students lost 4 points for calculating unit vectors incorrectly in MATLAB. The MATLAB command norm does $\sqrt{x \cdot x}$ or $\sqrt{x^T x}$. In MATLAB one calculates a unit vector by doing $x/\text{norm}(x)$. This was the only common MATLAB related issue.

B. *If you scored 15 or fewer points:*

If you got 15 points it means you got the equations right, but paid a price for your principled stand against MATLAB. If you got below 10-12 points your work likely indicated that you are confused between scalars and vectors. For the purposes of this class, being confused between scalars and vectors in an emergency. If the words scalar/vector are written on your test, or if you think this might be the case, I implore you to come talk to Dr. Ruina, Ryan Elandt, Duan Li or Walker Lee.

—Print your netID on the top of every side of every sheet—
—Print clearly (for computer text recognition)—

Net ID (don't include '@cornell.edu'): "SOLUTION"

Your name: ANDY RUINA

Cornell
ME 2030

No calculators, books or notes allowed.

Prelim 3

April 16, 2019
3 Problems, 90 minutes (+ up to 90 minutes extra)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation.**

A+ Be (I) neat, (II) clear and (III) well organized.

□ TIDILY REDUCE and box in your answers (Don't leave simplifyable algebraic expressions).

>> Make appropriate `Matlab code` clear and correct.
You can use shortcut notation like "$\phi_7 = 2\pi$" instead of, say, "`phi(7) = 2*pi;`".
Small syntax errors will have small penalties.

Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.

→| **Justify** your results so a grader can distinguish an informed answer from a guess. If you quote a fact that a grader might doubt your understaning of, explain it. Especially if it is not commonly used.

If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem). If you are referred here during the exam it means that answering your question would be telling you something you are being tested on.

≈ Work for **partial credit** (from 50% –100%, depending on the problem)

- Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
- Reduce the problem to a clearly defined set of equations to solve.
- Provide Matlab code which would generate the desired answer (and explain the nature of the output).

Extra sheets. The last page is blank for you. Ask for more if you need it. Put your name on each sheet, and fold it in. Refer to extra pages (We have to find them on a computer scan.)

Problem 7:	/25
Problem 8:	/25
Problem 9:	/25

Prelim 3, Q1: Reflections #399

Hey guys,

Problem 1 has been graded. The mean is 12.57; the median is 11; the standard deviation is 5.57. It's a really tough problem, so don't lose your faith if you didn't do well. :)

There are a couple things to pay attention to:

1. **Draw the free body diagram (FBD)!** I can't emphasize this enough, yet more than half of the class forgot to draw the FBD and lost 4 points. The FBD for collision is easy - equal and opposite forces/impulses on two objects - but this doesn't mean you can skip it. If you draw the FBD correctly, you can see that there's no force and thus no acceleration in the tangential direction (denoted by $\hat{\lambda}$ in the solutions), and you immediately get two LMB equations $\vec{v}_A^+ \cdot \hat{\lambda} = \vec{v}_A^- \cdot \hat{\lambda}$ and $\vec{v}_B^+ \cdot \hat{\lambda} = \vec{v}_B^- \cdot \hat{\lambda}$. Remember: The first step to approach a dynamics/mechanics problem is always drawing the FBD.

2. **Linear momentum balance (LMB) equations for collision.** According to my grading rubric, I'm expecting 3 scalar LMB equations and 1 restitution equation, because you need 4 equations to solve for 4 unknowns - $\vec{v}_{A,i}^+$, $\vec{v}_{A,j}^+$, $\vec{v}_{B,i}^+$ and $\vec{v}_{B,j}^+$. If you choose to include the force $\vec{F}$ or the impulse $\vec{P}$ in your equations, then you need 5 equations because now you have 5 unknowns. The solution written by Prof. Ruina used 5 equations - 2 vector LMB equations plus 1 restitution equation. Note that 1 vector equation counts as 2 scalar equations.
   There are many ways to write the LMB equations:
   (1) $P\hat{n} = m\vec{v}_B^+ - m\vec{v}_B^-$ (vector equation)
   (2) $-P\hat{n} = m\vec{v}_A^+ - m\vec{v}_A^-$ (vector equation)
   (3) $m\vec{v}_A^+ + m\vec{v}_B^+ = m\vec{v}_A^- + m\vec{v}_B^-$ (vector equation)
   (4) $\vec{v}_A^+ \cdot \hat{\lambda} = \vec{v}_A^- \cdot \hat{\lambda}$ (scalar equation)
   (5) $\vec{v}_B^+ \cdot \hat{\lambda} = \vec{v}_B^- \cdot \hat{\lambda}$ (scalar equation)
   Any good combination would work.

3. **Notation of vectors.** You need to have different notations for vectors and scalars. If ***x*** is a vector, you can either write it as $\vec{x}$ or as <u>x</u> (less common).
   For the LMB equation $m\vec{v}_A^+ + m\vec{v}_B^+ = m\vec{v}_A^- + m\vec{v}_B^-$,
   you can also write it as 2 scalar equations: $mv_{A,i}^+ + mv_{B,i}^+ = mv_{A,i}^- + mv_{B,i}^-$ and $mv_{A,j}^+ + mv_{B,j}^+ = mv_{A,j}^- + mv_{B,j}^-$.
   You cannot write $mv_A^+ + mv_B^+ = mv_A^- + mv_B^-$. I deducted 0.5 points for each LMB equation with wrong vector notations in "Point Adjustment".

4. **Restitution/collision equation.** The restitution equation is a scalar equation and it only works in the normal direction (denoted by $\hat{n}$ in the solutions). The equation is written as $(\vec{v}_A^+ - \vec{v}_B^+) \cdot \hat{n} = -e_r(\vec{v}_A^- - \vec{v}_B^-) \cdot \hat{n}$ If you got the direction wrong (e.g. along $\hat{j}$) or wrote a vector equation, you wouldn't get any points.

5. **Don't use conservation of energy when solving for velocities.** Conservation of energy works well in 1D but its algebra in 2D gets so complicated that it won't do you any good.

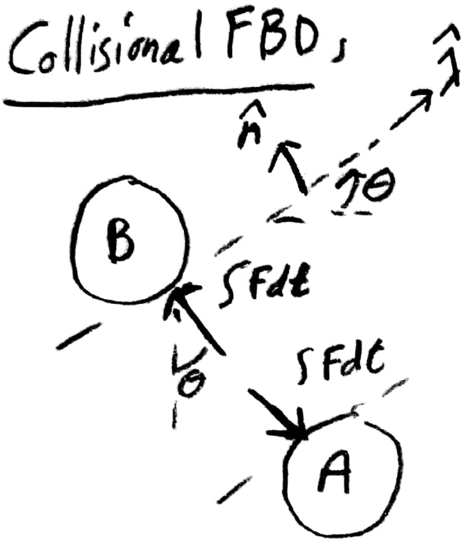
1) Particle collisions. 2D. No gravity. Two equal particles A and B have mass m. Initially, $\vec{v}_A^- = v_0\hat{j}$ and $\vec{v}_B^- = \vec{0}$. They have a frictionless collision with restitution $e_r = 1$.

a) Find the velocities of A and B after the collision. Answer in terms of some or all of m, v_0, $\hat{\imath}$, $\hat{\jmath}$, and θ.

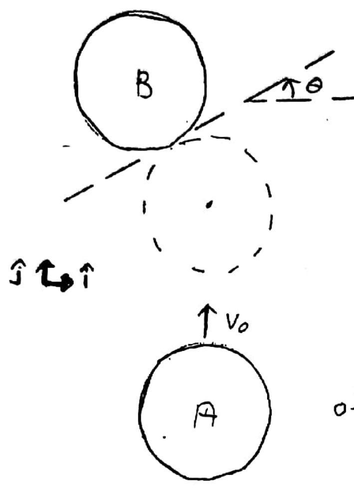
b) Energy loss. What is the total energy of the system of two particles after the collision (small partial credit for an incorrect answer based on correct algebra using incorrect answers to the question above).

c) Challenge: Assume a chain of 100 collisions in sequence, involving 101 balls. Each collision has angle $\theta = \pi/100$, always deflected to the left. The balls are arranged in a semi-circle. Each collision is frictionless with $e_r = 1$. All balls are initially stationary except the first which has $\vec{v}_1^- = v_0\hat{\imath}$. Estimate (using small angle approximations, *etc.*) the velocity of the last ball. A numerical answer is desired (like, *e.g.*, $-v_0\hat{\imath}/2$ or $-v_0 e^{-\pi}\hat{\imath}$, *etc.*).

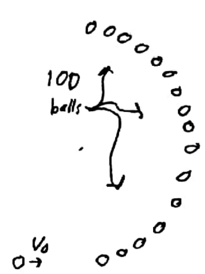
Collisional FBDs



(a & b)



(c)



$\hat{n} = \cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath}$

LMB for B $\Rightarrow$ $\int F dt\,\hat{n} = m\vec{V}_B^+ - \cancel{m\vec{V}_B^-}$ (1)

$\vec{V}_B^+$ in $\hat{n}$ dir.

0

" " A $\Rightarrow$ $-\int F dt\,\hat{n} = m\vec{V}_A^+ - m\underbrace{V_0\hat{\jmath}}_{\vec{V}_A^-}$ (2)

(1)+(2) $\Rightarrow$ $\underbrace{\cancel{m}V_B^+\hat{n}}_{\vec{V}_B^+} = \cancel{m}V_0\hat{\jmath} - \cancel{m}\vec{V}_A^+$ (3)

Coeff of rest.

(sep. speed) = e_r (approach speed)

$$(\vec{V}_B^+ - \vec{V}_A^+)\cdot\hat{n} = -e_r(\vec{V}_B^- - \vec{V}_A^-)\cdot\hat{n}$$

(3) $\Rightarrow$ $\left(V_B\hat{n} - (-V_B\hat{n} + V_0\hat{\jmath})\right)\cdot\hat{n} = +e_r V_0\hat{\jmath}\cdot\hat{n}$

$$2V_B - V_0\cos\theta = e_r V_0\cos\theta$$

$$V_B = V_0\cos\theta$$ (4)

1) (cont'd)

$$\Rightarrow \boxed{\vec{V}_B^+ = V_B^+ \hat{n} = \underbrace{V_0 \cos\theta}_{V_B^+} \underbrace{(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})}_{\hat{n}}} \text{(a)}$$

(3) $\Rightarrow$

$$\vec{V}_A^+ = V_0 \hat{\jmath} - \vec{V}_B^+$$

$$= V_0 \left(\cos\theta \sin\theta\,\hat{\imath} + (1 - \cos^2)\hat{\jmath} \right)$$

$$\boxed{\vec{V}_A^+ = V_0 \sin\theta \left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \right)} \quad (1)$$

<u>check:</u> $\theta = 0$: $\vec{V}_B^+ = V_0 \hat{\jmath}$ & $\vec{V}_A^+ = \vec{0}$

$\theta = \pi/2$ $\vec{V}_D^+ = \vec{0}$ & $\vec{V}_A^+ = V_0 \cos\theta\,\hat{\jmath}$

b) Energy?

$$E_K^- = \frac{1}{2} m V_0^2$$

$$E_K^+ = \frac{1}{2} m \vec{V}_A^+ \cdot \vec{V}_A^+ + \frac{1}{2} m \vec{V}_B^+ \cdot \vec{V}_B^+$$

$$= \frac{1}{2} m V_0^2 \left[(\cos^2 \sin^2 + \cos^4) + (\cos^2 \sin^2 + \sin^4) \right]$$

$$= \frac{1}{2} m V_0^2 (\cos^2 + \sin^2)^2 = \frac{1}{2} m V_0^2 = \boxed{E_K^-}$$

(We know already that $e_r = 1 \Rightarrow E_K^+ = E_K^-$)

c) 100 collisions, for each $V_n^+ = \cos\theta\, V_n^-$

$$\Rightarrow V_{100} = (\cos\theta)^{100} V_0 \approx \left(1 - \left(\frac{\pi}{100}\right)^2 / 2\right)^{100} V_0$$

$$\approx \left(1 - \frac{(\pi^2/200)}{100}\right)^{100} V_0 \approx e^{-\pi^2/200}\, V_0$$

$$V_{100} \approx (1 - \pi^2/200)\, V_0$$

$$\boxed{\vec{V}_{100} \approx -(1 - \pi^2/200)\, V_0\, \hat{\imath}} \Rightarrow$$

$\pi^2 \approx 10 \Rightarrow \pi^2/200 \approx 1/20$

5% loss of speed

Prelim 3, Q2: An Autopsy #395

Hey all - problem 2 of the 3rd prelim has been graded. In general, the class struggled with the problem; the median score was 12/25, with only 13% of students scoring above a 20 and no students answering the challenges (d, e, and f) correctly. Here are a few common mistakes that were made by students attempting to solve this problem:

1. **Acceleration.** A lot of people tried to solve part (a) as a statics problem, setting the sum of the forces equal to 0. However, just because an object isn't moving doesn't mean it has no acceleration. Think back to high school physics: if I hold an apple in my hand and then drop it, at the moment of release, it isn't moving, but it is still accelerating at a rate of 9.8 m/s^2 toward the ground. When you're debating whether to set acceleration equal to 0, consider not only the velocity right now, but also what the velocity will be in a few moments (both speed and direction). Does the velocity change? If the two velocities aren't exactly the same, the object must have some kind of acceleration.
2. **Angular Momentum Balance.** Regarding this equation: on the exam, lots of people were sloppy and didn't write the arrows over the vectors or the subscripts underneath them. Some people wrote the equation wrong, such as writing $\dot{\theta}$ instead of $\ddot{\theta}$. My advice would be this: memorize this equation, and every time you use AMB, write it down exactly as I'm writing it here: $\sum \vec{M}_{/C} = \vec{r}_{G/C} \times m\vec{a}_G + I^G\ddot{\theta}\hat{k}$. **Even if you're taking AMB about the center of mass, or if a_G is 0, or if $\ddot{\theta}$ is 0, or if there are no external moments, write the whole thing out anyway.** Then, write out all the forces and the position vectors individually, *before* you resove the cross products. Some people skipped these steps, but their results were wrong, and then it wasn't clear if they actually understood AMB or not. Just write the equation out - it's worth your time, trust me.
3. **Reaction forces.** This was probably the trickiest part of the question: the point mass on a stick has a reaction force only in the $\hat{e}_r$ direction, whereas the hoop does not. How do we know this? To understand this conceptually, first draw an FBD for the point mass: gravity pointing down, and some constraint force in the $\hat{e}_r$ direction that keeps the mass from flying off the end of the stick. Now draw an FBD for the stick, starting with the same constraint force that keeps the mass in place. Now, since the stick has no mass, it cannot sustain a force; therefore, it must exert an equal and oppoisite force at the other end so that the forces on the stick sum to 0. If the hoop were swinging from a stick, its reaction force would have the same constraint. However, in this case, the hoop is directly attached to the joint, so the reaction force can point any way it wants to. In this sense, the point mass is the exception to the rule - any shape hanging directly from the joint can have a reaction force component in the $\hat{e}_\theta$ direction, but the point mass at the end of the rod cannot. Another way of thinking about this concept is that any reaction force in the $\hat{e}_\theta$ direction will cause the body to rotate about its center of mass; however, a point mass has no moment of inertia about its center of mass, so that system cannot sustain a force in the $\hat{e}_\theta$ direction.
4. **Equivalent systems**. As mentioned above, nobody answered parts (d), (e), or (f) correctly. Here are the logical steps you need to take to arrive at the correct answer:

- The equation for $\theta(t)$ is derived from the equation for $\ddot{\theta}$, which was derived from AMB about O. Therefore, AMB about O for this new shape must also produce $\ddot{\theta} = -\frac{g}{L}sin(\theta)$.
- Plugging this value for $\ddot{\theta}$ back into AMB about O quickly shows that the new system must have the same m, $\vec{r}_{G/O}$, and I^G as the original system, or else equivalence is impossible. Therefore, our new shape has mass *m,* COM located at a distance L/2 from O, and $I^G = \frac{1}{4}mL^2$.
- Since *m and L* are already fixed, no uniform shape (a rod, a square, a circle, etc.) will work - those shapes have their own moments of inertia, and none of them are equal to that of a hoop. Therefore, we need to come up with a distribution of point masses that satisfies all of our requirements.
- The formula for moment of inertia is the integral of r^2 dm. Since we are dealing with point masses, we can pull the r^2 out of the integral. Integrating dm just produces m, so to get $I^G = \frac{1}{4}mL^2$, we need $r^2 = \frac{1}{4}L^2$. This yields r = L/2; therefore, all of the point masses must be located a distance L/2 from the center of mass.
- In summary: our new shape is a distribution of *n* point masses with total mass m, arranged symmetrically about the center of mass with each one a distance L/2 away from the center, and placed with the COM a distance L/2 from point O. Note that this works for any n > 1, so we have an infinite number of possible solutions.

This was a difficult problem, so feel free to come to office hours if you have any questions!

2) Pendula. 2D. There is gravity g. Consider two pendula: a point mass on a massless stick of length L; and a uniform hoop with diameter L. They are both released from rest at angle θ_0 at $t = 0$.

a) Immediately after release find the reaction force (a vector) for both systems (the force on the system at 0).

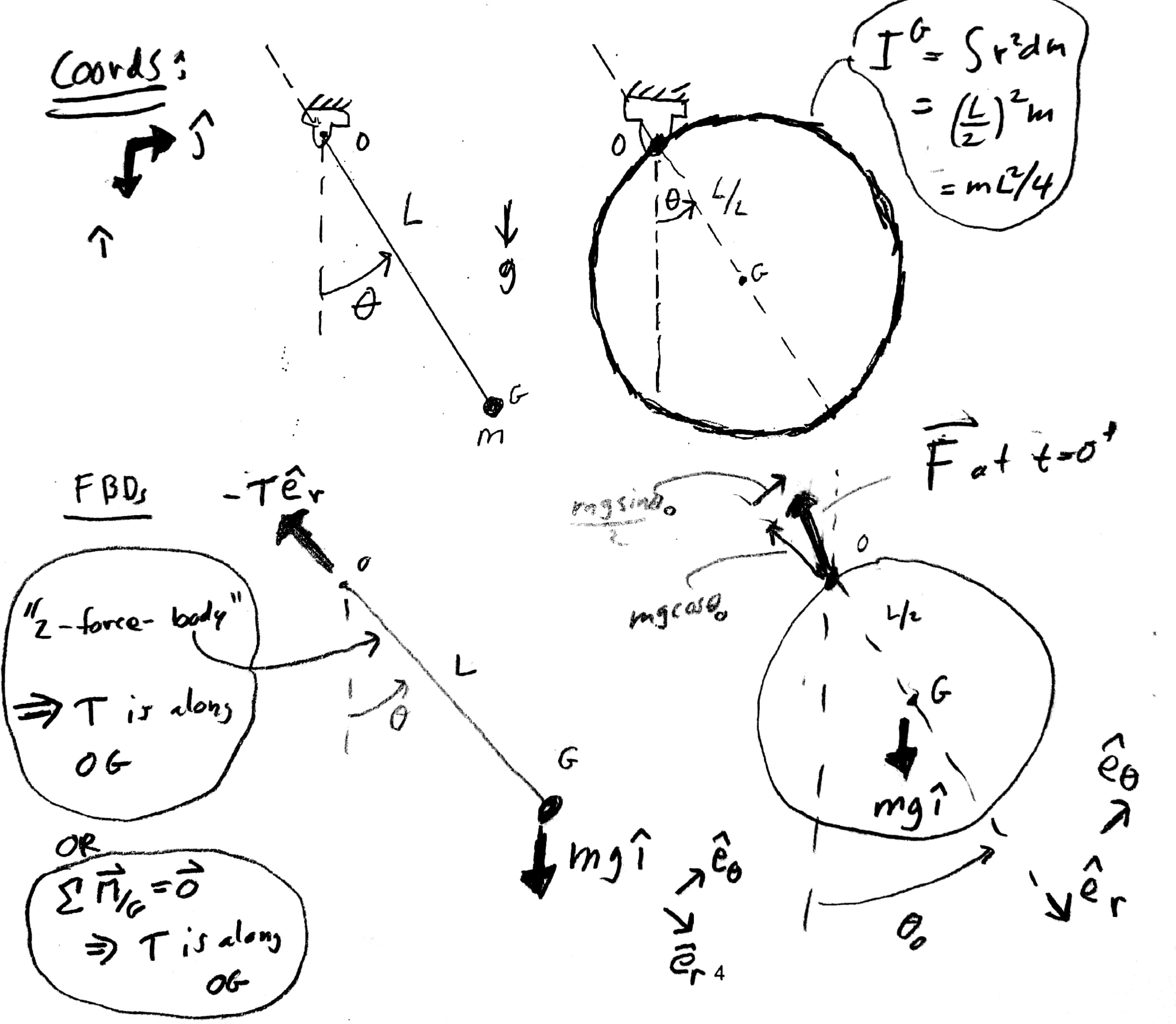
b) Draw the forces carefully so their directions are clear (*e.g.*, are they parallel to $\vec{r}_{G0}$ or $\hat{\imath}$ or $\hat{\jmath}$ or not).

c) Assuming validity of small-angle approximations, find $\theta(t)$ for either system (your choice).

d) Challenge: precisely describe the mass distribution of another shape (other than a hoop or point mass) which has the same period of small oscillation as the point mass pendulum (how much mass distributed in what places at what distances). For example, if you said "a uniform square" you would also have to give it's total mass, its size and where the hinge should be placed).

e) Challenge (1 pt) find (and explain) a different answer to the question above.

f) Challenge (1 pt) precisely describe (and explain) an infinite number of such mass distributions (*e.g.*, "uniform ellipses hung from a focus point so long as the major and minor axis a and b obeyed $L = \sqrt{2ab}$").



2) cont'd

pt. mass

hoop

AMB/O $\Rightarrow -mgL\sin\theta = mL\ddot{\theta}$

$$\boxed{\ddot{\theta} = \frac{-g}{L}\sin\theta}$$

$\Rightarrow \dot{\theta}(0) = 0,\ \theta(0) = \theta_0$

$$\Rightarrow \boxed{\theta = \theta_0 \cos\sqrt{\frac{g}{L}}\,t} \quad (c)$$

Both systems

small angles

$\vec{r}_{G/O} \times mg\hat{\imath} = \vec{r}_{G/O} \times m\vec{a}_G + I^G\ddot{\theta}\hat{k}$

$\Rightarrow$

$\left\{ \frac{-mgL}{2}\sin\theta\hat{k} = \frac{L}{2}\cdot\frac{L}{2}\ddot{\theta}m\hat{k} + \frac{mL^2}{4}\ddot{\theta}\hat{k} \right\}$ *

$\{\}\cdot\hat{k} \Rightarrow \boxed{\ddot{\theta} = \frac{-g}{L}\sin\theta}$!

again

LMB $\sum\vec{F} = m\vec{a}_G$

Note, no $\dot{\theta}^2$ terms

$\{ -T\hat{e}_r + mg\hat{\imath} = \ddot{\theta}L\hat{e}_\theta m$

$= -\frac{g}{L}\sin\theta L\hat{e}_\theta \}$

$\{\}\cdot\hat{e}_r \Rightarrow T = mg\hat{\imath}\cdot\hat{e}_r$

$$\boxed{T = mg\cos\theta}$$

$$\boxed{-T\hat{e}_r = -mg\cos\theta(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})}$$

$\sum\vec{F} = m\vec{a}_G$

$\vec{F} + mg\hat{\imath} = m\ddot{\theta}\frac{L}{2}\hat{e}_\theta$

↑ not nec. in $\hat{e}_r$ dir.

$\vec{F}\cdot\hat{e}_r = -mg\cos\theta$

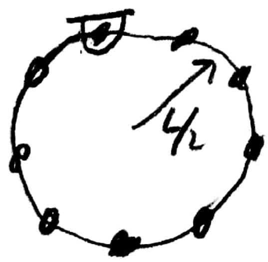
$\vec{F}\cdot\hat{e}_\theta = -mg\underbrace{\hat{\imath}\cdot\hat{e}_\theta}_{-\sin\theta} + \frac{mL}{2}\frac{-g}{L}\sin\theta$

$= mg\sin\theta - mg\sin\theta/2$

$= mg\sin\theta/2$

$\vec{F} = -mg(\cos\theta\,\hat{e}_r - \sin\theta/2\,\hat{e}_\theta)$

d, e, f) See eqn. * Use shapes w/ $|\vec{r}_{G/O}| = L/2$ & w/ all mass at $L/2$



massless hoop w/ n equally spaced pt masses. A special case is 2 masses. The simple pendulum!

Prelim 3 Q3 Reflections #398

This question had the following issues:

1. i and j were switched in the picture
2. the problem failed to explicitly (and arguably implicitly) mention that the initial angular velocity was 0
3. the problem failed to articulate exactly what in the $+i$ direction meant in part C. Students used all three interpretations: $v \cdot i = 1, 0 < v \cdot i, v \cdot j = 0$; two of which are strained.

To complicate things further, some students flipped j in part B when they flipped i and j in the picture some didn't. Similarly, some students flipped i in part C when they flipped i and j in the picture some didn't.

It follows that: issue 1 has 2 interpretations, issue 2 has 2, issue 3 has 3, part B has 2, and part A has 2. This gives a total of 48 possible questions.

Given the 48 total tests, the grader partially determined grades by assessing overall the students understanding. This holistic assessment was done in the context of the difficulty of the test you chose.

Part A (15 points):

Students got 3 points for writing AMB, 3 points for for writing LMB, 3 points for deriving or memorizing I_G and 3 points for writing down $v_a = v_g + w \times r_{a/G}$. Students earned the final 3 points by getting the correct answer (within your axis of choice, of course).

Part B and C (5 points each):

These parts were graded simultaneously in the context of your axis of choice and interpretation of what you think direction means. If you flipped i in B you had to flip j in C. If you flipped in B and not in C this is called flip flopping; and lost points accordingly possibly with the words “lacks internal consistency” on it.

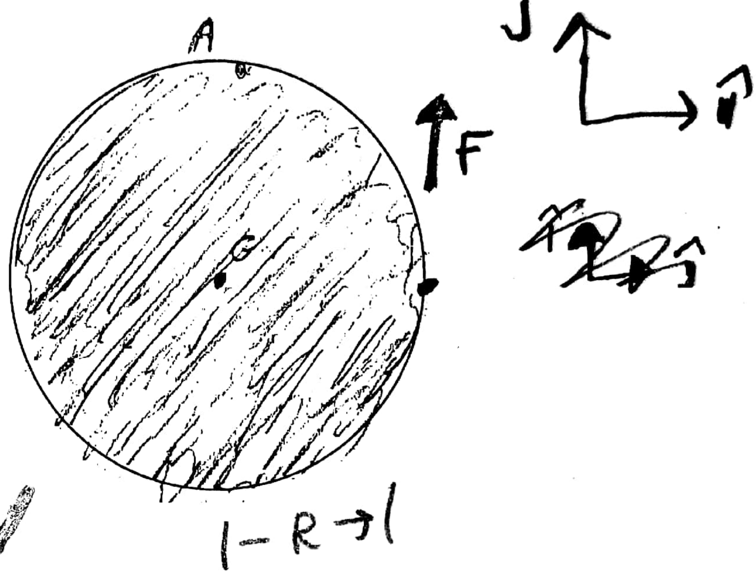
$$I_G = \int r^2 dm = \int_0^R r^2\, 2\pi r \rho\, dr = \frac{\pi R^4}{2}\rho = mR^2/2$$

3) Force on a disk 2D. No gravity. A constant vertical force F is suddenly applied.
- There are no other forces, bearings, hinges *etc.*
- The only concern is one instant in time, immediately after force application.
- The disk is uniform with radius R and mass m.
- Clearly define any other variables you need.

a) Find the acceleration of point A.

b) Find any point B on the disk whose acceleration is in the $\hat{\jmath}$ direction (*i.e.*, , has $\vec{a}_B \cdot \hat{\imath} = 0$) and find its acceleration, or prove that no such point exists.

c) Find all points C on the disk (or on a rigid massless extension of the disk that is welded to the disk) that have acceleration in the $+\hat{\imath}$ direction, or prove that no such point exists. (What's a rigid massless extension? Points in space that move rigidly with the disk but that don't have mass. Imagine a large sheet of negligible-mass styrofoam glued to the disk. *e.g.*, the center of mass of the letter C is on a rigid massless extension of the letter C).



LMB: $m\vec{a}_G = F\hat{\jmath}$

AMB/G: $\{\sum \vec{M}_{/G} = I^G \alpha \hat{k}\}$

$\{\}\cdot\hat{k}$ $\quad FR = I^G\alpha$

$\alpha = FR/I_G$

b) $\boxed{\text{pt. } G \text{ has } \vec{a}_G = \frac{F}{m}\hat{\jmath}}$

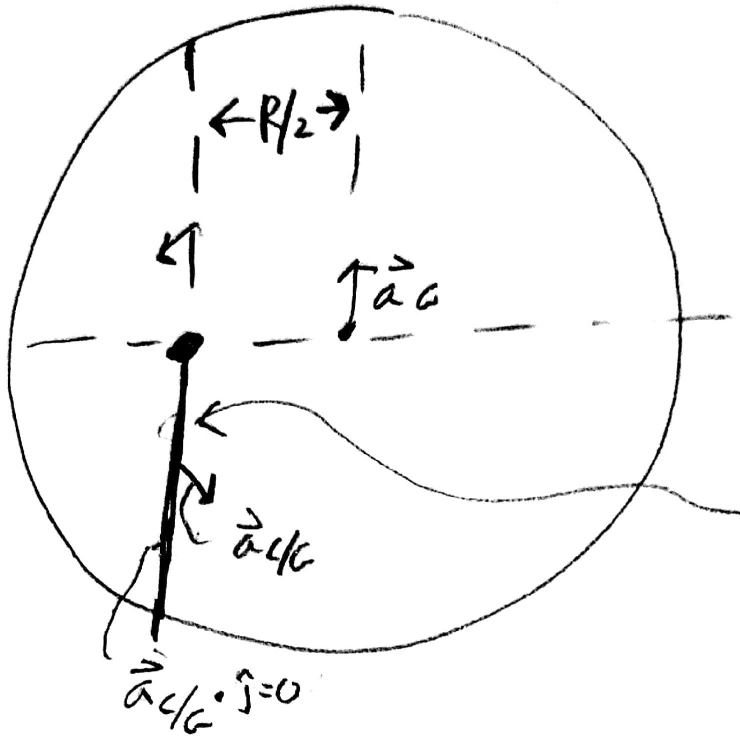
a) $\vec{a}_A = \vec{a}_G + \vec{\dot{\omega}} \times \vec{r}_{A/G} - \cancel{\omega^2 \vec{r}_{A/G}}$ $\quad \omega \to 0$ at start

$\vec{\dot{\omega}} = \alpha\hat{k}$

$\vec{a}_A = \frac{F}{m}\hat{\jmath} + \frac{FR}{I_G}\hat{k}\times R\hat{\jmath} = \frac{F}{m}\hat{\jmath} + \frac{-FR^2}{mR^2/2}\hat{\imath} = \boxed{\frac{F}{m}(\hat{\jmath} - 2\hat{\imath})}$ (a) $\vec{a}_A$

3) cont'd

7



$\vec{a}_C \parallel$ to $+\hat{\imath}$

$$\boxed{\begin{aligned} x_C &= -R/2 \\ y_C &< 0 \end{aligned}}$$

$\vec{a}_C \parallel \hat{\imath}$

$$\vec{a}_C \cdot \hat{j} = 0 \Rightarrow \left(\vec{a}_G + \vec{a}_{C/G}\right) \cdot \hat{j} = 0$$

$$\Rightarrow \left(\frac{F}{m}\hat{j} + \dot{\omega}\hat{K} \times (x_C\hat{\imath} + y_C\hat{j})\right) \cdot \hat{j} = 0$$

$$\Rightarrow \frac{F}{m} + \dot{\omega} x_C = 0$$

$$x_C = \frac{-F}{m\dot{\omega}} = \frac{-F I^G}{m FR} = \frac{-F mR^2/2}{mFR}$$

$$\boxed{x_C = -R/2}$$

For $\vec{a}_C \cdot \hat{\imath} > 0 \Rightarrow y < 0$