—Print your netID on the top of every side of every sheet—
—Print clearly (for computer text recognition)—

Net ID (don't include '@cornell.edu') "SOLUTIONS"

Your name: ANDY RUINA

Cornell ME 2030

Prelim 1

Feb. 21, 2019

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes over time)

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation.**
- A+ Be (I) neat, (II) clear and (III) well organized.
- TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct.
  You can use shortcut notation like "$\phi_7 = 2\pi$" instead of, say, "`phi(7) = 2*pi;`".
  Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions $(\ell, h, d, \ldots)$, coordinates $(x, y, r, \theta \ldots)$, variables $(v, m, t, \ldots)$, base vectors $(\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots)$ and signs $(\pm)$ with sketches, equations or words.
- **Justify** your results so a grader can distinguish an informed answer from a guess. If you quote a fact that a grader might doubt your understaning of, explain it. Especially if it is not commonly used.
- If a problem seems ***poorly defined***, clearly state any reasonable assumptions (that do not oversimplify the problem).
- ≈ Work for **partial credit** (from 50% –100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, label it clearly fold it in, and refer to back or extra pages, by label. (We have to find the extra pages on a computer scan.)

Problem 1: ____/25

Problem 2: ____/25

Problem 3: ____/25

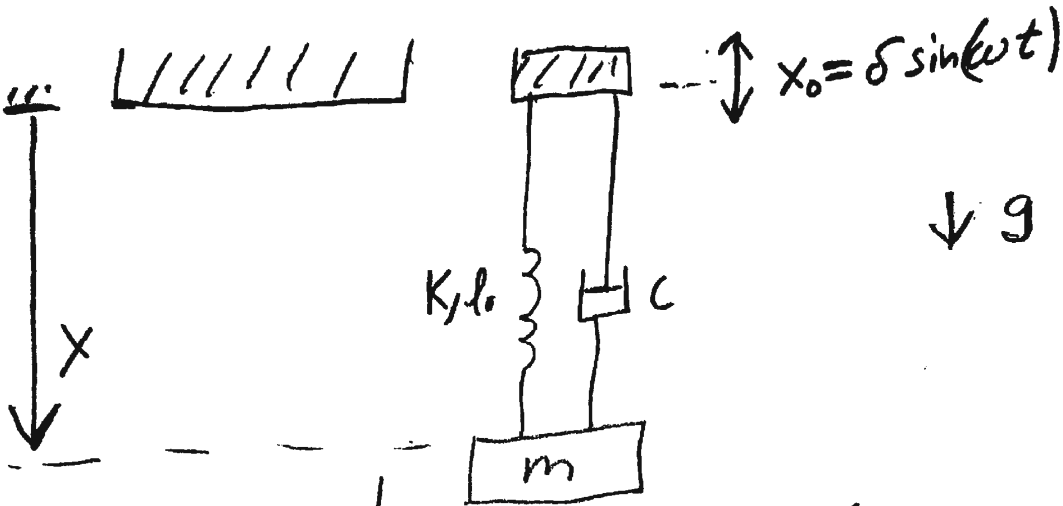
1) Shaken spring and mass. 1D. There is gravity. The rest length of the spring $\ell_0 \neq 0$.

You are given the constants $k, \ell_0, c, m, g, \delta$ and $\omega \neq \sqrt{k/m}$. All constants are positive (> 0).

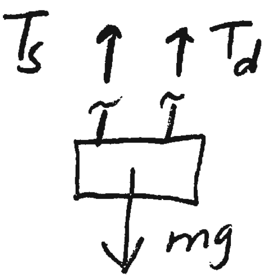
More details. The mass is suspended by a spring and dashpot. Gravity pulls down. A machine, not shown, shakes the support point by moving it sinusoidally up and down with amplitude δ and frequency ω. The the forcing frequency ω is known to not happen to be $\sqrt{k/m}$.

a) *Find the equations of motion.* (That is, find one or more differential equations whose solutions would describe motions of the system).

b) For the special case $c = 0$, find any solution to the governing equations. You need not find the general solution. Your solution, "$x(t) = \ldots$", should be in terms of some or all of $k, \ell_0, c, m, g, \delta, \omega$ and t.



FBD



$$T_s = k(\ell - \ell_0) = k((x - x_0) - \ell_0)$$
$$= k(x - \delta \sin(\omega t) - \ell_0) \qquad (1)$$

$$T_D = c\dot{\ell} = c(\dot{x} - \dot{x}_0)$$
$$= c(\dot{x} - \delta\omega \cos(\omega t)) \qquad (2)$$

LMB: $F_{TOT} = ma$ (1D, so skip vector notation)

$$\Rightarrow \underset{(1)}{mg - T_s} - \underset{(2)}{T_D} = m\ddot{x}$$

$$\Rightarrow \boxed{m\ddot{x} + c\dot{x} + kx = k\ell_0 + mg + k\delta \sin(\omega t) + \cancel{k}\, c\,\delta\omega \cos(\omega t)} \quad \boxed{a}$$

(1) cont'd

$$\text{set } c=0 \Rightarrow m\ddot{x} + Kx = \underbrace{(K l_0 + mg)}_{①} + \underbrace{K\delta \sin(\omega t)}_{②}$$

"Any" soln. ⇒ steady state soln. is o.k.
⇒ simplest particular soln. is o.k.
⇒ solve ①, then ② then add.

①: guess $x = x_1 = \text{const.}$ (static soln.)

$$\Rightarrow Kx_1 = K l_0 + mg \Rightarrow x_1 = l_0 + mg/K$$

②: guess $x = c_2 \sin(\omega t)$

$$\Rightarrow -\omega^2 m c_2 \cancel{\sin(\omega t)} + c_2 K \cancel{\sin(\omega t)} = K\delta \cancel{\sin(\omega t)}$$

$$\Rightarrow c_2(-\omega^2 m + K) = K\delta$$

$$\Rightarrow c_2 = \frac{\delta}{1 - \frac{m}{K}\omega^2}$$

$$\Rightarrow x_2 = \underbrace{\frac{\delta}{1-\frac{m}{K}\omega^2}}\sin\omega t$$

2 cases:
- * in phase for $\omega < \sqrt{\frac{K}{m}}$
- * out of phase for $\omega > \sqrt{K/m}$

$$x(t) = x_1 + x_2$$

$$\boxed{x(t) = \underbrace{\left(l_0 + \frac{mg}{K}\right)}_{①} + \underbrace{\left(\frac{\delta}{1-\frac{m}{K}\omega^2}\right)}_{②}\sin(\omega t)}$$ b

Prelim 1, Q1: Some Reflection (Duan Li)

The statistics of the grades for Q1 are: Mean: 13.05, median: 11.0, standard deviation: 5.56. Please refer to my grading rubric on Gradescope to find out where you got points off and to decide whether you need regrading.

Below are a few common misperceptions I noticed and some advice to add onto Walker's and Ryan's a retrospective.

1. Pay attention to the coordinate system defined in the problem. You can define your own if there isn't one. In part a, the x axis is pointing downward, not upward, so $m\ddot{x} = mg - F_{spring} - F_{dashpot}$ if F_{spring} and $F_{dashpot}$ are defined to be pointing up in the FBD.
2. Have a clear mind of which force is applied on which object. There are only two components (the spring and the dashpot) connected directly to the mass, so there are only two forces (spring force and damping), besides gravity, applied on the mass. Some students had the driving force on the machine as part of their linear momentum balance (LMB) equation. This is wrong in several ways: (a) The machine drives the mass through the spring and the dashpot, so there is no direct force between the machine and the mass. (b) The total force on the machine should be $m_{machine}\ddot{x}_0$ and $m_{machine}$ is unknown.
3. Spring and dashpot respond to the net effect caused by all components directly connected to them. Here, both the mass and the machine are affecting the length of the spring and the velocity of the dashpot. The spring should respond to the net change in its length and the dashpot should respond to its net velocity.
4. Use physical intuition to check your result. At first glimpse of part b, you should realize this is a purely oscillatory system with no damping so the final solution should consists only of $\sin$, $\cos$ and constants. Some students end up having a t, $t2$ or e^t term in their solution, which means the components are moving further away from each other as time goes on and the system is breaking apart. If you get such a

solution, it's very likely that something is wrong with either your initial EoM or your way of solving the ODE. One case that could lead to a e^t term is if you made the mistake in #1, that is if you happen to set the x axis pointing up and get $m\ddot{x}-kx=something$ as your EoM instead of $m\ddot{x}+kx=something$. Mistakes are often correlated. If you identify one of them, you might to able to correct multiple of them.

2) **Mystery fluid.** 1D. No gravity. A bullet is shot into a mystery fluid. The bullet has mass m and initial velocity v_0. In the experiment it is observed that the bullet approaches a total penetration distance A according to:

$$x(t) = A - B/t$$

where A and B are measured. This is a strange mystery fluid with unknown properties. The goal is to use this data to find drag as a function of velocity.

$$F_D = f(v).$$

Find $f(v)$.

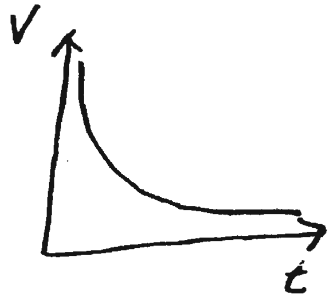
(Starts at $t > 0$ $\Rightarrow$ don't worry about $B/0 = \infty$)

Your formula for $f(v)$ can have some or all of the measure quantities m, v_0, A, and B.

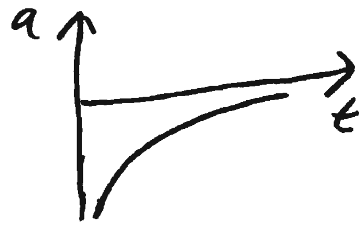
FBD: $F_D \leftarrow \boxed{m}$ $\quad \longrightarrow x$

Kinematics:

$$V = \dot{X} = \frac{d}{dt}(A - B/t) = B/t^2$$



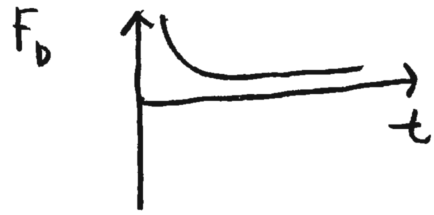
$$a = \dot{V} = -2B/t^3$$



LMB

$$F = ma$$
$$-F_D = ma$$
$$F_D = 2mB/t^3$$

eliminate t $\Rightarrow$ $\boxed{F_D = 2m\, V^{3/2}/\sqrt{B}}$



note: $V^{3/2} = \left(\frac{B}{t^2}\right)^{3/2} = \frac{B^{3/2}}{t^3} = \frac{B\sqrt{B}}{t^3}$

Note: as $V \to 0$

$V^{3/2} <$ linear drag for which $X \sim e^{-t}$

$V^{3/2} >$ quadratic drag for which $X \to \infty$

$\Rightarrow$ $V^{3/2}$ has $X \sim 1/t$ which is in between $X \sim e^{-t}$ and $X \sim \infty$

Prelim 1, Q2: A Retrospective (Walker Lee)

Hey all! I was responsible for grading problem 2 on the first prelim. The median score on the problem was a flat 15/25 (exam grades should be posted within the next few days). A lot of people made the same errors on this problem, so I thought it would be helpful to share a few common mistakes made when solving it. If you got less than 20/25, it is HIGHLY likely that one or more of the following went wrong:

1. Read the problem statement! For this problem, the problem statement explicitly stated that the system is in 1D and that there is no gravity. Despite this, I came across several 2D and 3D free-body diagrams, and several people included gravity on their FBDs or when writing out $F=ma$. Take a few seconds to *carefully* read the problem statement before starting *any* question, especially on an exam, as it will save you a huge headache later.
2. Draw your free-body diagrams! Speaking of FBDs, there were several students who would have scored a perfect 25 on this question, but they lost 5 points for not drawing an FBD. While this was physically painful for me to grade, the instructions on the front of the exam were clear: if you ever use linear momentum balance ($F=ma$) to solve a problem, you MUST draw a free-body diagram, even if there is only 1 force on the body in question.
3. Function notation. The problem asked for a final answer in the form $f(v)$. Whenever a function is written in this notation, the output (in this case, the drag force) is a function of only scalars (numbers such as 2 or π), constants (parameters that don't change, such as m), and the variables inside the parenthesis (in this case, v). Any variables NOT inside the parenthesis (here, t and x) cannot appear in the final answer, because the answer would no longer be $f(v)$ - it would be $f(v,t)$ or $f(v,x)$ instead.

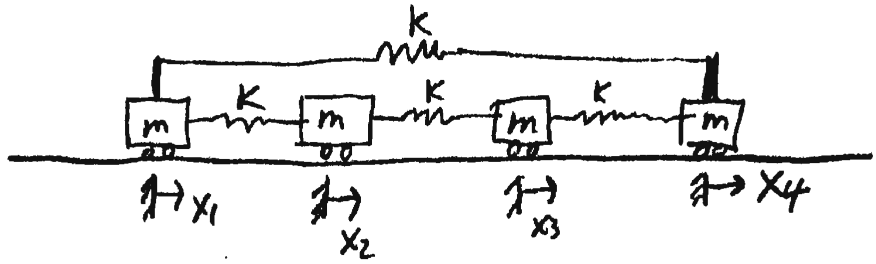
4. Assumptions. This was probably the most common mistake: many students assumed that $f(v)$ would have the form $-cv$ or $-cv^2$. However, for this problem, that is impossible - you already are given $x(t)=A+B/t$, and if you integrate a linear or quadratic drag twice, you get $x(t) \sim e^{-t}$ or $-\ln|t|$, respectively, as you've seen on the homework. Therefore, the drag in this problem cannot possibly be linear nor quadratic. In general, making assumptions that are not given in the problem statement is extremely dangerous, so if you do, always be prepared to back them up.
5. Equation validity. Many students tried to use one or more of these equations without realizing they wouldn't work:

- Some variation of $v=\Delta x/t$
- Some variation of $a=\Delta v/t$
- $\Delta x = v_0 t + \frac{1}{2} at^2$
- $v_f^2 - v_0^2 = 2a\Delta x$
- $\int F dt = Ft$
- $W=Fd$

These equations only hold when velocity (first row), acceleration (rows 2-4), and force (rows 5-6) are constant, respectively. In this problem, none of these are constant, so using these equations wouldn't help you. Before you use an equation, make sure you understand what assumptions were made when it was derived so you don't use it under invalid conditions.

6. Energy methods. Several students attempted to use energy methods to solve this problem. While I applaud your creativity, any answer derived using energy methods would have to contain v_0, and v_0 was a trap: $x(t)=A+B/t$ gives $v(t)=B/t^2$, which in turn means v_0 is undefined.

3) Four masses in a line. Four equal masses m are in a line connected by 4 equal springs k, as shown. You can assume that the springs are all relaxed when $x_1 = x_2 = x_3 = x_4 = 0$.



a) Find the matrices M and K so that $M[\ddot{x}] + K[x] = [0]$, where $[x] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

b) Assume these have been stored in `p.M` and `p.K`. Write a file, suitable for use with, say, your midpoint solver, that has the differential equations in Matlab language. That is, fill in the middle lines of code (possibly more or less than 3 lines) of this Matlab function:

```
function zdot = myrhs(t,z,p)
...
...
...
zdot = ...
end
```

c) The matlab command `[V,D] = eig(M^(-1)*K)` gives two matrices. Using any reasoning, intuition or guesses that you like, *specify as much as you can about the entries in those matrices*. That is, find as many columns of `V` and associated diagonal elements of `D`? (For fairness to those who are now taking Math 2940, and to save you from time doing tedious algebra, do not find and solve a characteristic polynomial.)

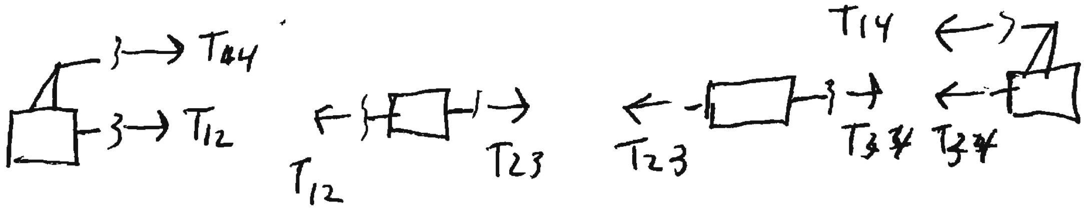
If you find lots and lots of normal modes, be clear about which ones are independent and which are just combinations of others.

Hints:

1) Check that one column of `V` is proportional to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and has associated diagonal element of `D` of 0.

2) This problem has lots of symmetry, hence at least some normal modes can be found by inspection;

FBDs



$T_{12} = k(x_2 - x_1)$ $\quad T_{23} = k(x_3 - x_2)$ $\quad T_{34} = k(x_4 - x_1)$

$T_{14} = k(x_4 - x_1)$

③ (cont'd)

LMB

$$m\ddot{x}_1 = -2kx_1 + kx_2 + 0x_3 + kx_4$$
$$m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3 + 0\cdot x_4$$
$$m\ddot{x}_3 = 0\cdot x_1 + kx_2 - 2kx_3 + kx_4$$
$$m\ddot{x}_4 = kx_1 + 0x_2 + kx_3 - 2kx_4$$

Matrix Form:

$$\underset{4\times4}{M}\,\underset{4\times1}{\ddot{x}} + \underset{4\times4}{K}\,x = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

a

$$K = k\begin{bmatrix}2 & -1 & 0 & -1\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ -1 & 0 & -1 & 2\end{bmatrix}$$

$$M = m\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\underset{8\times1}{z} = \begin{bmatrix}x_1\\ \vdots\\ x_4\\ v_1\\ \vdots\\ v_4\end{bmatrix}$$

MATLAB

b

```
function zdot=myrhs(t,z,p)
x= z(1:4); v= z(5:8);
xdot= v; % Kinematics
vdot= -inv(p.M)* p.K*x; % F=ma
% vdot = -p.M\ p.K* x; % Alt. method
zdot= [xdot; vdot];
end
```

③ (cont'd)

Each col. of V is a vector w such that

$$\underset{L_{M^{-1}K}}{A}\, \underbrace{w}_{\text{eigenvector}} = \lambda w$$

λ ← eigen value = diag. element of D

* Matlab uses unit vectors. Here we'll skip that step (simpler looking).
* Matlab makes arb. choices for Linear combs. if one e-value has 2 or more e-vectors. Here we'll make it simple looking.
* Note symmetry: each mass connected to 2 neighbors (even end masses)

Guess w	Picture	Keff	Aw	λ	good?
[1 1 1 1]'	→ → → →	0	0·w	0	✓
[1 0 0 0]'	→ • • •	nonsense	≠λ w	✗	✗
[1 1 -1 -1]'	→ → ← ←	2K	$\frac{2K}{m}$ w	$\frac{2K}{m}$	✓
[1 -1 -1 1]'	→ ← ← →	2K	$\frac{2K}{m}$ w	2K/m	✓
[1 0 -1 0]'	→ • ← •	2K	$\frac{2K}{m}$ w	2K/m	✓
[0 1 0 -1]'	• → • ←	2K	$\frac{2K}{m}$ w	2K/m	✓
[1 -1 1 -1]'	→ ← → ←	4K	$\frac{4K}{m}$ V	4K/m	✓

any lin. independent lin. combs of *

$$D = \frac{K}{m}\begin{bmatrix} 0&0&0&0\\ 0&2&0&0\\ 0&0&2&0\\ 0&0&0&4 \end{bmatrix}; \quad V = \begin{bmatrix} 1/2 & & & 1/2\\ 1/2 & & & -1/2\\ 1/2 & & & 1/2\\ 1/2 & & & -1/2 \end{bmatrix}$$

Prelim 1: Q3: Common Mistakes (Ryan Elandt)

Part A (8 Points): If you get 6 points it was likely because you wrote down the negative of the stiffness matrix. For our problems it also needs to be symmetric due to Newton's third law. [In more general advanced vibration problems it is also symmetric, for other reasons. Ask if you want to know.] With practice you should be able to find the components of the stiffness matrix visually. At least for problems with masses in a line. Then you can check your work by comparing with what visual inspection tells you the answer has to be.

Part B (9 Points): Students lost points in a wide variety of ways. Students lost three points for each major mistake. Mistakes usually either: produce incorrect answers, produce right answers for the wrong reasons or do something in a way that is excessively long. Some common mistakes included:

- Not realizing that K and M are matrices
- Not realizing that z and zdd are column vectors
- Assigning into an unallocated array z(1:4) (this creates a column vector)
- The correct code needs a line that is equivalent to

"vdot = - inv(p.M) * p.K * x".

That is, solving the equation $M\ddot{x}=-Kx$ for $\ddot{x}$ gives $\ddot{x}=-M^{-1}Kx$.

Advice for the future: Although not required, using inv(M) to represent M^{-1} is a good way to avoid mistakes. This way you do not have to remember which of "M \ K", "M / K", "K \ M", "K / M" and "K * (1 / M)" is right. Although I didn't take off points you should not write "inv(M)" as "M^-1" as it will not work in many languages, you should write "M^(-1)" although, as mentioned, that's not as good as inv(M) or pinv(M).

- Writing out matrix multiplication component by component (this is the point of linear algebra, to *not* do this)

Part C (8 Points but 12 Points possible): The eigenvalues of this matrix are the frequencies of the normal modes squared. A lot of students did not realize this. I did not take off points for this. A lot of students thought that things like [1; -1; 0; 0] or [0; 1; -1; 0] were eigenvectors, they are not. You can see that they are not two ways: 1) pre-multiply by inv(M)*K and see that you don't get the same vector back; or 2) notice that there are forces on the supposedly stationary masses, which is not consistent (that is, you assumed that they don't move even though in your proposed solution there is a force on the masses).