—Print your netID on the top of every side of every sheet—
—Print clearly (for computer text recognition)—

Net ID (don't include '@cornell.edu')
"SOLUTION"

Your name:
ANDY RUINA

Cornell
ME 2030

Prelim 3

April 16, 2019
3 Problems, 90 minutes (+ up to 90 minutes extra)

No calculators, books or notes allowed.

How to get the highest score?

Please do these things:

- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation.**

A+ Be (I) neat, (II) clear and (III) well organized.

□ TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).

>> Make appropriate `Matlab code` clear and correct.
You can use shortcut notation like "$\phi_7 = 2\pi$" instead of, say, "`phi(7) = 2*pi;`".
Small syntax errors will have small penalties.

Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}} \ldots$) and signs ($\pm$) with sketches, equations or words.

→| **Justify** your results so a grader can distinguish an informed answer from a guess. If you quote a fact that a grader might doubt your understaning of, explain it. Especially if it is not commonly used.

If a problem seems *poorly defined*, clearly state any reasonable assumptions (that do not oversimplify the problem). If you are referred here during the exam it means that answering your question would be telling you something you are being tested on.

≈ Work for **partial credit** (from 50% –100%, depending on the problem)

- Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
- Reduce the problem to a clearly defined set of equations to solve.
- Provide Matlab code which would generate the desired answer (and explain the nature of the output).

Extra sheets. The last page is blank for you. Ask for more if you need it. Put your name on each sheet, and fold it in. Refer to extra pages (We have to find them on a computer scan.)

Problem 7: ____ /25

Problem 8: ____ /25

Problem 9: ____ /25

Prelim 3, Q1: Reflections #399

Hey guys,

Problem 1 has been graded. The mean is 12.57; the median is 11; the standard deviation is 5.57. It's a really tough problem, so don't lose your faith if you didn't do well. :)

There are a couple things to pay attention to:

1. **Draw the free body diagram (FBD)!** I can't emphasize this enough, yet more than half of the class forgot to draw the FBD and lost 4 points. The FBD for collision is easy - equal and opposite forces/impulses on two objects - but this doesn't mean you can skip it. If you draw the FBD correctly, you can see that there's no force and thus no acceleration in the tangential direction (denoted by $\hat{\lambda}$ in the solutions), and you immediately get two LMB equations $\vec{v}_A^+ \cdot \hat{\lambda} = \vec{v}_A^- \cdot \hat{\lambda}$ and $\vec{v}_B^+ \cdot \hat{\lambda} = \vec{v}_B^- \cdot \hat{\lambda}$. Remember: The first step to approach a dynamics/mechanics problem is always drawing the FBD.
2. **Linear momentum balance (LMB) equations for collision.** According to my grading rubric, I'm expecting 3 scalar LMB equations and 1 restitution equation, because you need 4 equations to solve for 4 unknowns - $\vec{v}_{A,i}^+$, $\vec{v}_{A,j}^+$, $\vec{v}_{B,i}^+$ and $\vec{v}_{B,j}^+$. If you choose to include the force $\vec{F}$ or the impulse $\vec{P}$ in your equations, then you need 5 equations because now you have 5 unknowns. The solution written by Prof. Ruina used 5 equations - 2 vector LMB equations plus 1 restitution equation. Note that 1 vector equation counts as 2 scalar equations.
   There are many ways to write the LMB equations:
   (1) $P\hat{n} = m\vec{v}_B^+ - m\vec{v}_B^-$ (vector equation)
   (2) $-P\hat{n} = m\vec{v}_A^+ - m\vec{v}_A^-$ (vector equation)
   (3) $m\vec{v}_A^+ + m\vec{v}_B^+ = m\vec{v}_A^- + m\vec{v}_B^-$ (vector equation)
   (4) $\vec{v}_A^+ \cdot \hat{\lambda} = \vec{v}_A^- \cdot \hat{\lambda}$ (scalar equation)
   (5) $\vec{v}_B^+ \cdot \hat{\lambda} = \vec{v}_B^- \cdot \hat{\lambda}$ (scalar equation)
   Any good combination would work.
3. **Notation of vectors.** You need to have different notations for vectors and scalars. If ***x*** is a vector, you can either write it as $\vec{x}$ or as x̲ (less common).
   For the LMB equation $m\vec{v}_A^+ + m\vec{v}_B^+ = m\vec{v}_A^- + m\vec{v}_B^-$,
   you can also write it as 2 scalar equations: $mv_{A,i}^+ + mv_{B,i}^+ = mv_{A,i}^- + mv_{B,i}^-$ and $mv_{A,j}^+ + mv_{B,j}^+ = mv_{A,j}^- + mv_{B,j}^-$.
   You cannot write $mv_A^+ + mv_B^+ = mv_A^- + mv_B^-$. I deducted 0.5 points for each LMB equation with wrong vector notations in "Point Adjustment".
4. **Restitution/collision equation.** The restitution equation is a scalar equation and it only works in the normal direction (denoted by $\hat{n}$ in the solutions). The equation is written as $(\vec{v}_A^+ - \vec{v}_B^+) \cdot \hat{n} = -e_r(\vec{v}_A^- - \vec{v}_B^-) \cdot \hat{n}$ If you got the direction wrong (e.g. along $\hat{j}$) or wrote a vector equation, you wouldn't get any points.
5. **Don't use conservation of energy when solving for velocities.** Conservation of energy works well in 1D but its algebra in 2D gets so complicated that it won't do you any good.

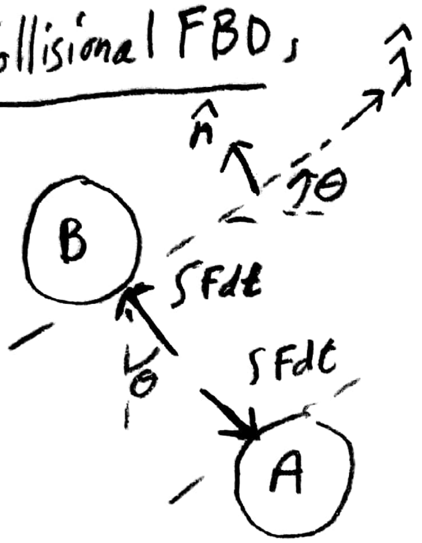
1) Particle collisions. 2D. No gravity. Two equal particles A and B have mass m. Initially, $\vec{v}_A^- = v_0\hat{\jmath}$ and $\vec{v}_B^- = \vec{0}$. They have a frictionless collision with restitution $e_r = 1$.

a) Find the velocities of A and B after the collision. Answer in terms of some or all of m, v_0, $\hat{\imath}$, $\hat{\jmath}$, and θ.

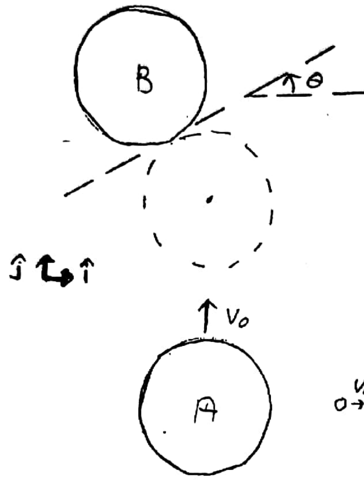
b) Energy loss. What is the total energy of the system of two particles after the collision (small partial credit for an incorrect answer based on correct algebra using incorrect answers to the question above).

c) Challenge: Assume a chain of 100 collisions in sequence, involving 101 balls. Each collision has angle $\theta = \pi/100$, always deflected to the left. The balls are arranged in a semi-circle. Each collision is frictionless with $e_r = 1$. All balls are initially stationary except the first which has $\vec{v}_1^- = v_0\hat{\imath}$. Estimate (using small angle approximations, *etc.*) the velocity of the last ball. A numerical answer is desired (like, *e.g.*, $-v_0\hat{\imath}/2$ or $-v_0 e^{-\pi}\hat{\imath}$, *etc.*).

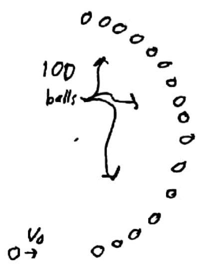
Collisional FBDs

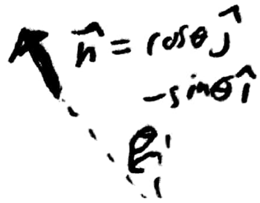


(a & b)



(c)





$\vec V_B^+$ is $\hat n$ dir

LMB for B $\Rightarrow$ $\int F dt\,\hat n = m\vec V_B^+ - \cancel{m\vec V_B^-}$ $\quad \vec V_B^- \to 0$ (1)

" " A $\Rightarrow$ $-\int F dt\,\hat n = m\vec V_A^+ - m\underbrace{V_0\hat j}_{\vec V_A^-}$ (2)

(1)+(2) $\Rightarrow$ $\cancel{m}\underbrace{V_B\hat n}_{\vec V_B^+} = \cancel{m}V_0\hat j - \cancel{m}\vec V_A^+$ (3)

Coeff of rest.

(sep. speed) = e_r (approach speed)

$$(\vec V_B^+ - \vec V_A^+)\cdot\hat n = -e_r(\vec V_B^- - \vec V_A^-)\cdot\hat n$$

(3) $\Rightarrow$ $\left(V_B\hat n - (-V_B\hat n + V_0\hat j)\right)\cdot\hat n = +e_r V_0\hat j\cdot\hat n$

$$2V_B - V_0\cos\theta = e_r V_0\cos\theta$$

$$V_B = V_0\cos\theta$$ (4)

1) (cont'd)

$$\Rightarrow \boxed{\vec{V}_B^{+} = V_B^{+}\hat{n} = \underbrace{V_0\cos\theta}_{V_B^{+}}\underbrace{(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})}_{\hat{n}}} \quad (a)$$

③ $\Rightarrow$

$$\vec{V}_A^{+} = V_0\hat{\jmath} - \vec{V}_B^{+}$$

$$= V_0\left(\cos\theta\sin\theta\,\hat{\imath} + (1-\cos^2)\hat{\jmath}\right)$$

$$\boxed{\vec{V}_A^{+} = V_0\sin\theta\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right)} \quad (1)$$

check: $\theta = 0$: $\vec{V}_B^{+} = V_0\hat{\jmath}$ & $\vec{V}_A^{+} = \vec{0}$ ✓

$\theta = \pi/2$ $\vec{V}_B^{+} = \vec{0}$ & $\vec{V}_A^{+} = V_0\cos\theta\,\hat{\jmath}$

b) Energy?

$$E_K^{-} = \frac{1}{2}mV_0^2$$

$$E_K^{+} = \frac{1}{2}m\vec{V}_A^{+}\cdot\vec{V}_A^{+} + \frac{1}{2}m\vec{V}_B^{+}\cdot\vec{V}_B^{+}$$

$$= \frac{1}{2}mV_0^2\left[(\cos^2\sin^2 + \cos^4) + (\cos^2\sin^2 + \sin^4)\right]$$

$$= \frac{1}{2}mV_0^2(\cos^2 + \sin^2)^2 = \frac{1}{2}mV_0^2 = \boxed{E_K^{-}}$$

(We know already that $e_r = 1 \Rightarrow E_K^{+} = E_K^{-}$)

c) 100 collisions, for each $V_n^{+} = \cos\theta\, V_n^{-}$

$$\Rightarrow V_{100} = (\cos\theta)^{100}V_0 \approx \left(1 - \left(\frac{\pi}{100}\right)^2/2\right)^{100} V_0$$

$$\approx \left(1 - \frac{(\pi^2/200)}{100}\right)^{100} V_0 \approx e^{-\pi^2/200}\,V_0$$

$$V_{100} \approx (1 - \pi^2/200)\,V_0$$

$$\boxed{\vec{V}_{100} \approx -(1 - \pi^2/200)\,V_0\,\hat{\imath}}$$

$\Rightarrow$ $\pi^2 \approx 10 \Rightarrow \pi^2/200 \approx 1/20$

5% loss of speed

Prelim 3, Q2: An Autopsy #395

Hey all - problem 2 of the 3rd prelim has been graded. In general, the class struggled with the problem; the median score was 12/25, with only 13% of students scoring above a 20 and no students answering the challenges (d, e, and f) correctly. Here are a few common mistakes that were made by students attempting to solve this problem:

1. **Acceleration.** A lot of people tried to solve part (a) as a statics problem, setting the sum of the forces equal to 0. However, just because an object isn't moving doesn't mean it has no acceleration. Think back to high school physics: if I hold an apple in my hand and then drop it, at the moment of release, it isn't moving, but it is still accelerating at a rate of 9.8 m/s^2 toward the ground. When you're debating whether to set acceleration equal to 0, consider not only the velocity right now, but also what the velocity will be in a few moments (both speed and direction). Does the velocity change? If the two velocities aren't exactly the same, the object must have some kind of acceleration.
2. **Angular Momentum Balance.** Regarding this equation: on the exam, lots of people were sloppy and didn't write the arrows over the vectors or the subscripts underneath them. Some people wrote the equation wrong, such as writing $\dot{\theta}$ instead of $\ddot{\theta}$. My advice would be this: memorize this equation, and every time you use AMB, write it down exactly as I'm writing it here: $\sum \vec{M}_{/C} = \vec{r}_{G/C} \times m\vec{a}_G + I^G\ddot{\theta}\hat{k}$. **Even if you're taking AMB about the center of mass, or if a_G is 0, or if $\ddot{\theta}$ is 0, or if there are no external moments, write the whole thing out anyway.** Then, write out all the forces and the position vectors individually, *before* you resove the cross products. Some people skipped these steps, but their results were wrong, and then it wasn't clear if they actually understood AMB or not. Just write the equation out - it's worth your time, trust me.
3. **Reaction forces.** This was probably the trickiest part of the question: the point mass on a stick has a reaction force only in the $\hat{e}_r$ direction, whereas the hoop does not. How do we know this? To understand this conceptually, first draw an FBD for the point mass: gravity pointing down, and some constraint force in the $\hat{e}_r$ direction that keeps the mass from flying off the end of the stick. Now draw an FBD for the stick, starting with the same constraint force that keeps the mass in place. Now, since the stick has no mass, it cannot sustain a force; therefore, it must exert an equal and oppoisite force at the other end so that the forces on the stick sum to 0. If the hoop were swinging from a stick, its reaction force would have the same constraint. However, in this case, the hoop is directly attached to the joint, so the reaction force can point any way it wants to. In this sense, the point mass is the exception to the rule - any shape hanging directly from the joint can have a reaction force component in the $\hat{e}_\theta$ direction, but the point mass at the end of the rod cannot. Another way of thinking about this concept is that any reaction force in the $\hat{e}_\theta$ direction will cause the body to rotate about its center of mass; however, a point mass has no moment of inertia about its center of mass, so that system cannot sustain a force in the $\hat{e}_\theta$ direction.
4. **Equivalent systems**. As mentioned above, nobody answered parts (d), (e), or (f) correctly. Here are the logical steps you need to take to arrive at the correct answer:

- The equation for $\theta(t)$ is derived from the equation for $\ddot{\theta}$, which was derived from AMB about O. Therefore, AMB about O for this new shape must also produce $\ddot{\theta} = -\frac{g}{L}sin(\theta)$.
- Plugging this value for $\ddot{\theta}$ back into AMB about O quickly shows that the new system must have the same m, $\vec{r}_{G/O}$, and I^G as the original system, or else equivalence is impossible. Therefore, our new shape has mass *m,* COM located at a distance L/2 from O, and $I^G = \frac{1}{4}mL^2$.
- Since *m and L* are already fixed, no uniform shape (a rod, a square, a circle, etc.) will work - those shapes have their own moments of inertia, and none of them are equal to that of a hoop. Therefore, we need to come up with a distribution of point masses that satisfies all of our requirements.
- The formula for moment of inertia is the integral of r^2 dm. Since we are dealing with point masses, we can pull the r^2 out of the integral. Integrating dm just produces m, so to get $I^G = \frac{1}{4}mL^2$, we need $r^2 = \frac{1}{4}L^2$. This yields r = L/2; therefore, all of the point masses must be located a distance L/2 from the center of mass.
- In summary: our new shape is a distribution of *n* point masses with total mass m, arranged symmetrically about the center of mass with each one a distance L/2 away from the center, and placed with the COM a distance L/2 from point O. Note that this works for any n > 1, so we have an infinite number of possible solutions.

This was a difficult problem, so feel free to come to office hours if you have any questions!

2) Pendula. 2D. There is gravity g. Consider two pendula: a point mass on a massless stick of length L; and a uniform hoop with diameter L. They are both released from rest at angle θ_0 at $t = 0$.

a) Immediately after release find the reaction force (a vector) for both systems (the force on the system at 0).

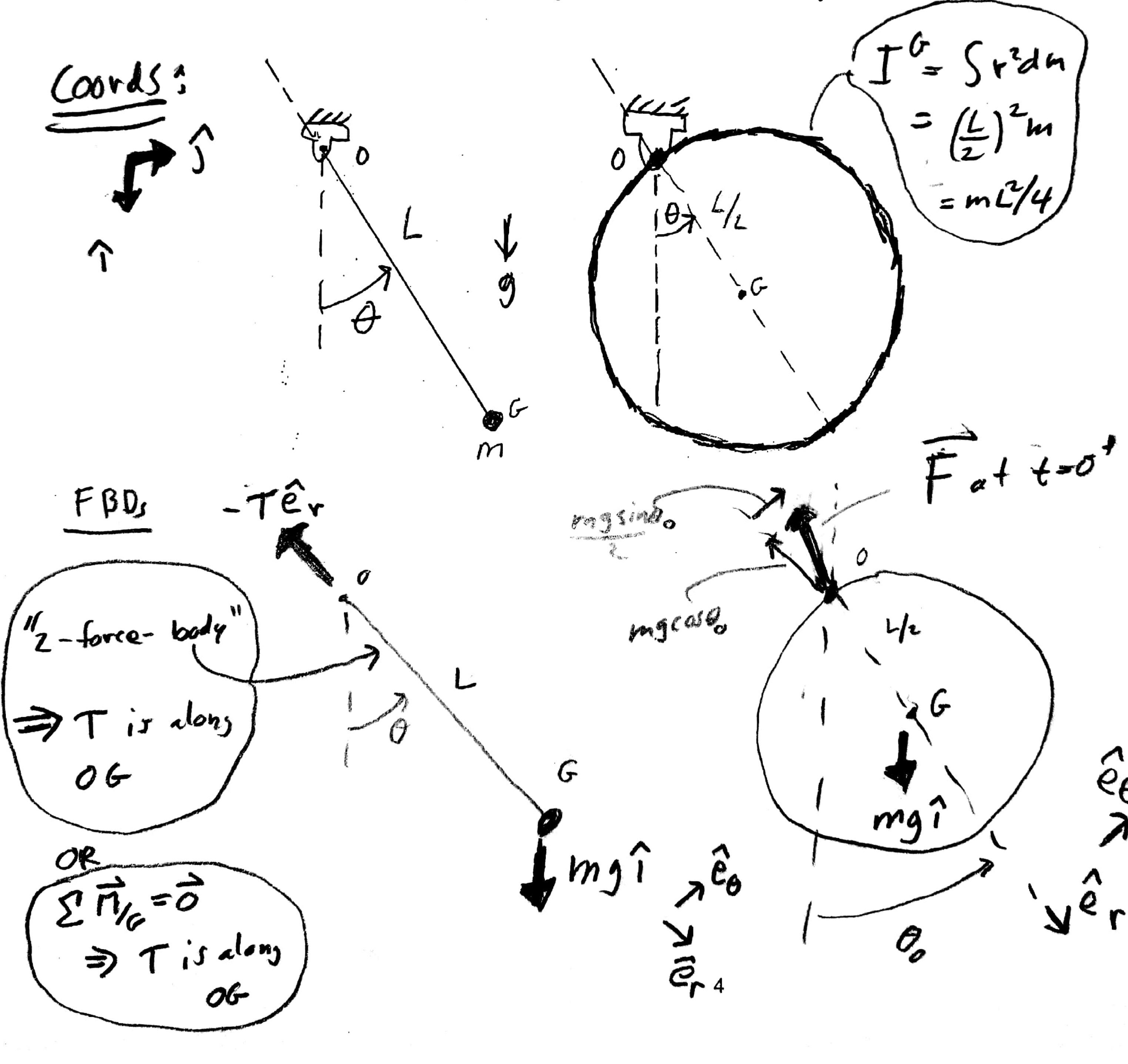
b) Draw the forces carefully so their directions are clear (*e.g.*, are they parallel to $\vec{r}_{G0}$ or $\hat{\imath}$ or $\hat{\jmath}$ or not).

c) Assuming validity of small-angle approximations, find $\theta(t)$ for either system (your choice).

d) Challenge: precisely describe the mass distribution of another shape (other than a hoop or point mass) which has the same period of small oscillation as the point mass pendulum (how much mass distributed in what places at what distances). For example, if you said "a uniform square" you would also have to give it's total mass, its size and where the hinge should be placed).

e) Challenge (1 pt) find (and explain) a different answer to the question above.

f) Challenge (1 pt) precisely describe (and explain) an infinite number of such mass distributions (*e.g.*, "uniform ellipses hung from a focus point so long as the major and minor axis a and b obeyed $L = \sqrt{2ab}$").



2) cont'd   pt. mass    hoop   (5)

AMB/0 ⇒ $-mgL\sin\theta = mL\ddot\theta$

$$\boxed{\ddot\theta = \frac{-g}{L}\sin\theta}$$

⇒ $\dot\theta(0) = 0,\ \theta(0) = \theta_0$

⇒ $\boxed{\theta = \theta_0\cos\sqrt{\frac{g}{L}}\,t}$ (c)

Both systems

small angles

$\vec r_{G/0}\times mg\hat\imath = \vec r_{G/0}\times m\vec a_G + I^G\ddot\theta\hat k$

⇒ $\left\{-\frac{mgL}{2}\sin\theta\hat k = \frac{L}{2}\cdot\frac{L}{2}\ddot\theta m\hat k + \frac{mL^2}{4}\ddot\theta\hat k\right\}$ *

$\{\}\cdot\hat k$ ⇒ $\boxed{\ddot\theta = \frac{-g}{L}\sin\theta}$!

again

LMB $\sum\vec F = m\vec a_G$ Note, no $\dot\theta^2$ terms

$\left\{-T\hat e_r + mg\hat\imath = \ddot\theta L\hat e_\theta m = -\frac{g}{L}\sin\theta L\hat e_\theta\right\}$

$\{\}\cdot\hat e_r$ ⇒ $T = mg\hat\imath\cdot\hat e_r$

$\boxed{T = mg\cos\theta}$

$\boxed{-T\hat e_r = -mg\cos\theta(\cos\theta\hat\imath + \sin\theta\hat\jmath)}$

$\sum\vec F = m\vec a_G$

$\vec F + mg\hat\imath = m\ddot\theta\frac{L}{2}\hat e_\theta$

↑ not nec. in $\hat e_r$ dir.

$\vec F\cdot\hat e_r = -mg\cos\theta$

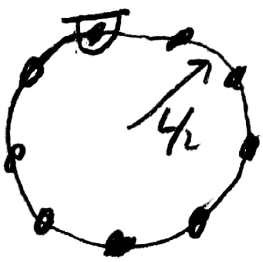
$\vec F\cdot\hat e_\theta = -mg\underbrace{\hat\imath\cdot\hat e_\theta}_{-\sin\theta} + \frac{mL}{2}\frac{-g}{L}\sin\theta$

$= mg\sin\theta - mg\sin\theta/2$

$= mg\sin\theta/2$

$\vec F = -mg(\cos\theta\,\hat e_r - \sin\theta/2\,\hat e_\theta)$

d,e,f) See eqn. * Use shapes w/ $|\vec r_{G/0}| = L/2$ & w all mass at $L/2$



massless hoop w/ n equally spaced pt masses. A special case is 2 pt masses.

masses. The simple pendulum!

Prelim 3 Q3 Reflections #398

This question had the following issues:

1. i and j were switched in the picture
2. the problem failed to explicitly (and arguably implicitly) mention that the initial angular velocity was 0
3. the problem failed to articulate exactly what in the $+i$ direction meant in part C. Students used all three interpretations: $v \cdot i = 1, 0 < v \cdot i, v \cdot j = 0$; two of which are strained.

To complicate things further, some students flipped j in part B when they flipped i and j in the picture some didn't. Similarly, some students flipped i in part C when they flipped i and j in the picture some didn't.

It follows that: issue 1 has 2 interpretations, issue 2 has 2, issue 3 has 3, part B has 2, and part A has 2. This gives a total of 48 possible questions.

Given the 48 total tests, the grader partially determined grades by assessing overall the students understanding. This holistic assessment was done in the context of the difficulty of the test you chose.

Part A (15 points):

Students got 3 points for writing AMB, 3 points for for writing LMB, 3 points for deriving or memorizing I_G and 3 points for writing down $v_a = v_g + w \times r_{a/G}$. Students earned the final 3 points by getting the correct answer (within your axis of choice, of course).

Part B and C (5 points each):

These parts were graded simultaneously in the context of your axis of choice and interpretation of what you think direction means. If you flipped i in B you had to flip j in C. If you flipped in B and not in C this is called flip flopping; and lost points accordingly possibly with the words “lacks internal consistency” on it.

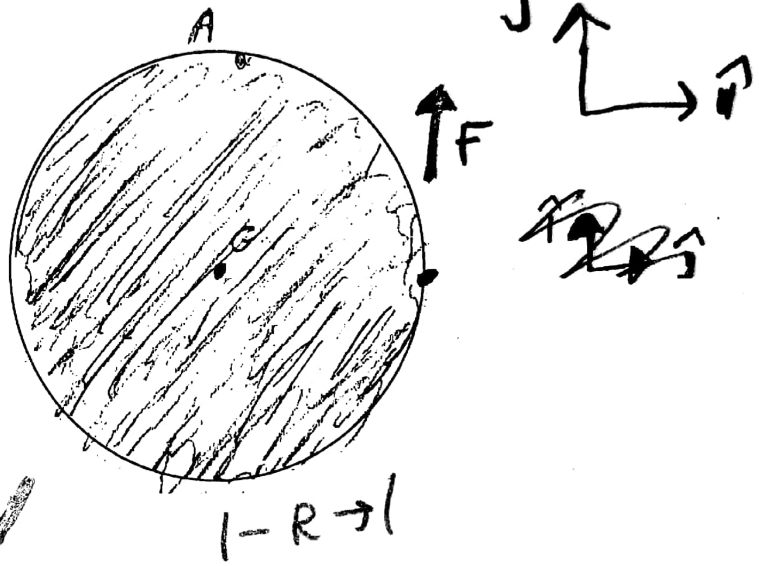
$$I_G = \int r^2 dm = \int_0^R r^2\, 2\pi r \rho\, dr = \frac{\pi R^4}{2}\rho = mR^2/2$$

3) **Force on a disk** 2D. No gravity. A constant vertical force F is suddenly applied.
- There are no other forces, bearings, hinges *etc.*
- The only concern is one instant in time, immediately after force application.
- The disk is uniform with radius R and mass m.
- Clearly define any other variables you need.

a) Find the acceleration of point A.

b) Find any point B on the disk whose acceleration is in the $\hat{\jmath}$ direction (*i.e.*, , has $\vec{a}_B \cdot \hat{\imath} = 0$) and find its acceleration, or prove that no such point exists.

c) Find all points C on the disk (or on a rigid massless extension of the disk that is welded to the disk) that have acceleration in the $+\hat{\imath}$ direction, or prove that no such point exists. (What's a rigid massless extension? Points in space that move rigidly with the disk but that don't have mass. Imagine a large sheet of negligible-mass styrofoam glued to the disk. *e.g.*, the center of mass of the letter C is on a rigid massless extension of the letter C).



LMB: $m\vec{a}_G = F\hat{\jmath}$

AMB$_{/G}$: $\{\sum \vec{M}_{/G} = I^G \alpha \hat{k}\}$

$\{\}\cdot\hat{k}$ $\quad FR = I^G \alpha$

$\alpha = FR/I_G$

b) $\boxed{\text{pt. } G \text{ has } \vec{a}_G = \frac{F}{m}\hat{\jmath}}$

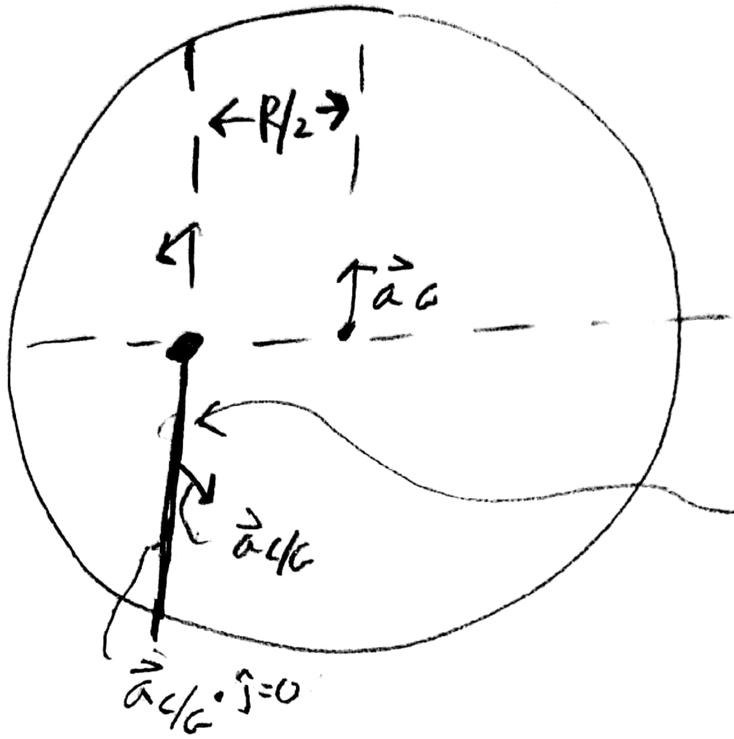
a) $\vec{a}_A = \vec{a}_G + \underbrace{\dot{\vec{\omega}}}_{\alpha\hat{k}} \times \vec{r}_{A/G} - \cancel{\omega^2 \vec{r}_{A/G}}$ $\quad \omega = 0$ at start

$$\vec{a}_A = \frac{F}{m}\hat{\jmath} + \frac{FR}{I_G}\hat{k}\times R\hat{\jmath} = \frac{F}{m}\hat{\jmath} + \frac{-FR^2}{mR^2/2}\hat{\imath} = \boxed{\frac{F}{m}(\hat{\jmath} - 2\hat{\imath})}$$

(a) $\vec{a}_A$

3) cont'd

7



$\vec{a}_c \parallel$ to $+\hat{i}$

$$\boxed{\begin{array}{l} x_c = -R/2 \\ y_c < 0 \end{array}}$$

$\vec{a}_c \parallel \hat{i}$

$$\vec{a}_c \cdot \hat{j} = 0 \Rightarrow \left(\vec{a}_G + \vec{a}_{c/G}\right) \cdot \hat{j} = 0$$

$$\Rightarrow \left(\frac{F}{m}\hat{j} + \dot{\omega}\hat{k} \times (x_c\hat{i} + y_c\hat{j})\right) \cdot \hat{j} = 0$$

$$\Rightarrow \frac{F}{m} + \dot{\omega} x_c = 0$$

$$x_c = \frac{-F}{m\dot{\omega}} = \frac{-F\, I^G}{m\, FR} = \frac{-F\, mR^2/2}{mFR}$$

$$\boxed{x_c = -R/2}$$

For $\vec{a}_c \cdot \hat{i} > 0 \Rightarrow y < 0$