"SOLUTIONS"

Your Name: Andy Ruina

Section day and time: Tu/Th 9:05-9:55

T&AM 203 Prelim 3

Tuesday Nov 23, 2004

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3 problems, 25 points each, and 90^+ minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. A blank page for tentative scrap work is provided at the back. Ask for extra scrap paper if you need it. If you want to hand in extra sheets, put your name on each sheet and refer to that sheet in the problem book for the relevant problems.

b) Full credit if

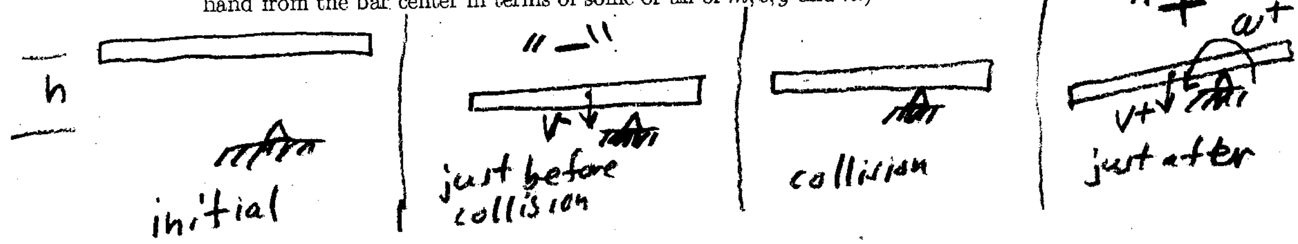
- →free body diagrams← are drawn whenever force, moment, linear momentum, or angular momentum balance is used;
- correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- →| reasonable justification, enough to distinguish an informed answer from a guess, is given; you clearly state any reasonable assumptions if a problem seems **poorly defined**;
- work is I.) neat,
  II.) clear, and
  III.) well organized;
- your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- □ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ Matlab code, if asked for, is clear and correct. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot{\theta}_7$ = 18" instead of, say, "theta7dot = 18". You will be penalized, but not heavily, for minor syntax errors.

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

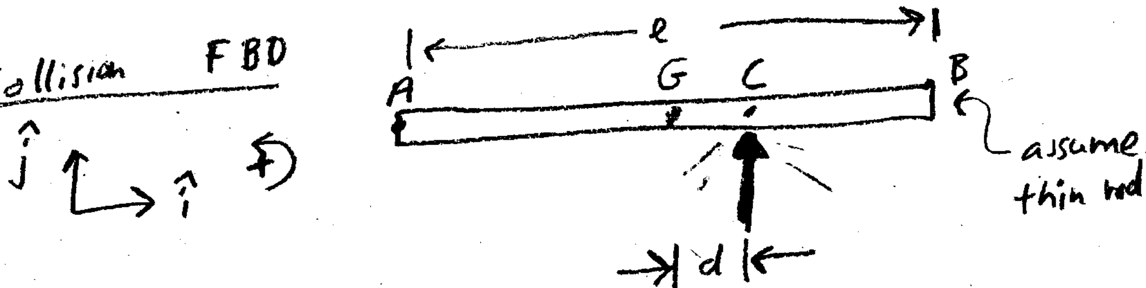
Problem 1:	/25
Problem 2:	/25
Problem 3:	/25

1) (25 pt) A uniform horizontal bar with mass m and length ℓ falls a height h without rotation in gravitational field g. The bar is then caught by a stationary hand. The hand does not move, but imparts an impulse but no torque (it freely allows rotation). *Clearly indicate where on the bar the hand should catch the bar so that one end of the bar (indicate which end) does not have any change of velocity at the impact.* (Indicate the distance d of the hand from the bar center in terms of some or all of m, ℓ, g and h.)

$\downarrow g$



Collision FBD



AMB /C

$$\underline{H}^-_{/C} = \underline{H}^+_{/C}$$

$$\left\{ v^- m d\,\hat{k} = v^+ m d\,\hat{k} + \overbrace{I^G}^{m\ell^2/12} \omega^+ \hat{k} \right\}$$

$$v^+ = \omega^+ d$$

$$\{\ \}\cdot\hat{k} \Rightarrow v^- \cancel{m} d = \omega^+ \cancel{m}\left[d^2 + \tfrac{1}{12}\ell^2\right]$$

$$\Rightarrow \omega^+ = \frac{v^- d}{d^2 + \frac{1}{12}\ell^2} \Rightarrow v_A^+ = \omega^+\left(d + \tfrac{\ell}{2}\right) = \frac{v^- d(d+\ell/2)}{d^2 + \ell^2/12}$$

Problem statement: $v_A^+ = v_A^-$

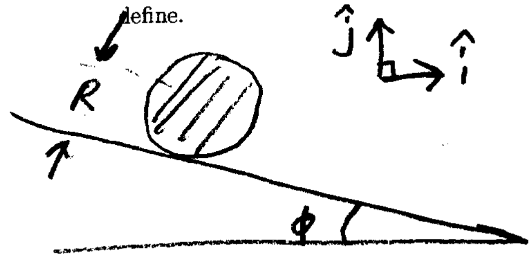
$$\Rightarrow \frac{v^- d(d+\ell/2)}{d^2+\ell^2/12} = v^-$$

$$\Rightarrow \frac{d\ell}{2} = \frac{\ell^2}{12}$$

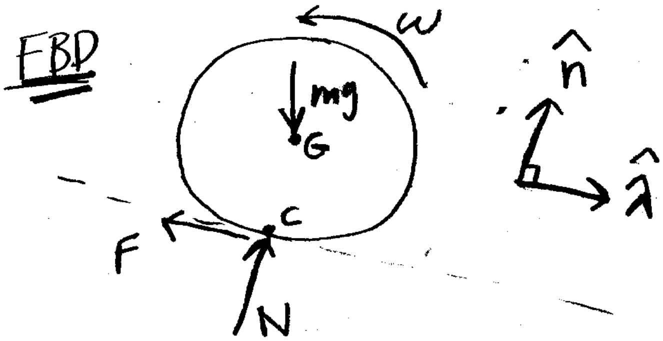
$$\Rightarrow \boxed{d = \ell/6}$$

Pt. C is called the center of percussion. Thinking of AB as a baseball bat its where to hit the ball so the hand end at A has no jump in velocity. An approximation of "the sweet spot".

2) **(25 pt)** A uniform disk of mass m and radius R is released from rest at $t = 0$ to roll-without-slip down a slope ϕ (measured relative to the horizontal) as accelerated by gravity g. *At time t what is the acceleration of the point on the disk that is then touching the ground?* Answer in terms of some or all of m, R, g, ϕ, t and any base vectors that you choose that you clearly define.



FBD



FBD shown is valid at all times.

Kinematics : $\underline{V}_G = \underbrace{V_G \hat{\lambda} = \underbrace{-\omega R}_{}}_{\text{rolling}} \hat{\lambda} \Rightarrow a_G = -\dot{\omega} R$

AMB$_{/C}$: $\sum \underline{M}_{/C} = \dot{\underline{H}}_{/C}$

$$\underline{r}_{G/C} \times (-mg\hat{j}) = \underline{r}_{G/C} \times m\underline{a}_G + I^G \dot{\omega} \hat{k}$$

$\underline{r}_{G/C} = R\hat{n}$ $\quad$ $\underline{a}_G = -\dot{\omega}R\hat{\lambda}$ $\quad$ $I^G = \frac{1}{2}mR^2$ (see next page)

$\hat{n} \times \hat{j} = \sin\phi \hat{k} \Rightarrow \{ -Rmg\sin\phi \hat{k} = R^2 m\dot{\omega}\hat{k} + \frac{1}{2}mR^2\dot{\omega}\hat{k} \}$

$\{ \} \cdot \hat{k} \Rightarrow -Rg\sin\phi = (R^2 + \frac{1}{2}R^2)\dot{\omega}$

$\Rightarrow \boxed{\dot{\omega} = \frac{-2g\sin\phi}{3R}}$ (right hand side = constant)

$\omega_0 = 0 \Rightarrow \boxed{\omega = \frac{-2g\sin\phi}{3R} t}$

What is accel. of pt. C ?

$$\underline{a}_C = \underline{a}_G + \underline{a}_{C/G} = -\dot{\omega}R\hat{\lambda} + \dot{\underline{\omega}} \times \underline{r}_{C/G} + (-\omega^2 \underline{r}_{C/G})$$

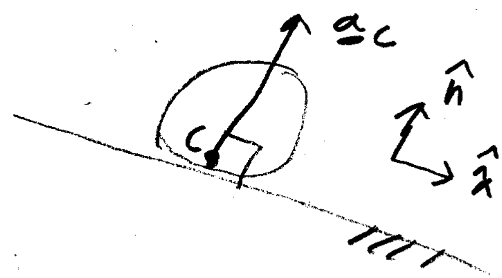
$$\underline{a}_c = -\dot{\omega} R \hat{\lambda} + \dot{\omega} \hat{k} \times (-R\hat{n}) + \omega^2 R \hat{n} \qquad \hat{k} \times \hat{n} = -\hat{\lambda}$$

$$= -\dot{\omega} R \hat{\lambda} + \dot{\omega} R \hat{\lambda} + \omega^2 R \hat{n}$$

$$= \omega^2 R \hat{n}$$

$$= \left(\frac{-3g \sin\phi}{R} t\right)^2 R \hat{n}$$

$$\boxed{\underline{a}_c = \frac{4 g^2 \sin^2\phi \, t^2}{9 R} \hat{n}}$$



Calculate I^G for uniform disk $\qquad \rho = \frac{m}{A} = \frac{m}{\pi R^2}$

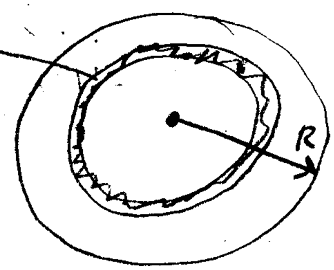
$$I^G = \int r^2 \, dm$$

$$= \int_0^R r^2 (\rho 2\pi r \, dr)$$

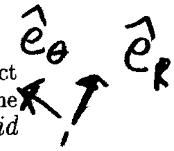
$$= 2\pi\rho \int_0^R r^3 \, dr = 2\pi \frac{m}{\pi R^2} \left(\frac{r^4}{4}\right)\Bigg|_0^R$$

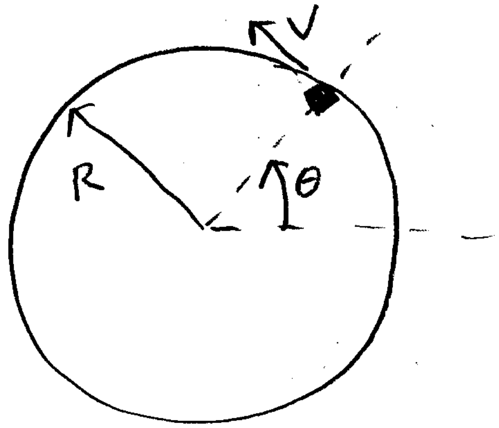
$$= 2\cancel{\pi} \frac{m}{\cancel{\pi} R^2} \frac{R^4}{4}$$

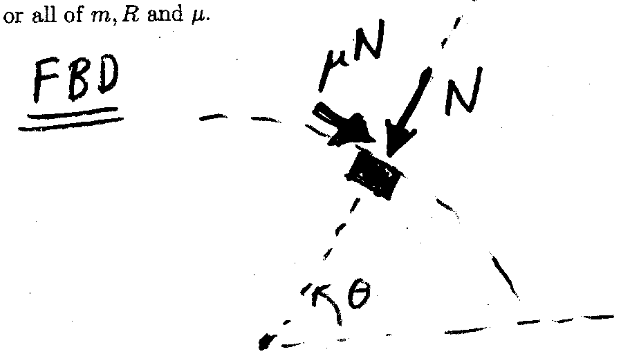
$$\boxed{I^G = mR^2/2}$$



3) (25 pt) A small bead with mass m slides on a rigid stationary circular hoop with radius R. Neglect gravity. The bead slides loosely on the wire (does not pinch it) with coefficient of friction μ. The initial speed of the bead is v_0 (along the circle). *What is the bead speed after it has slid once around the hoop?* Answer in terms of some or all of m, R and μ.







LMB

$$\underline{F} = m\underline{a} \Rightarrow -N\hat{e}_R - \mu N\hat{e}_\theta = m\left[(\cancel{\ddot{R}}^{0} - R\dot{\theta}^2)\hat{e}_R + (R\ddot{\theta} + \cancel{2\dot{R}\dot{\theta}}^{0})\hat{e}_\theta\right]$$

$$\left\{-N\hat{e}_R - \mu N\hat{e}_\theta = -mR\dot{\theta}^2\hat{e}_R + mR\ddot{\theta}\hat{e}_\theta\right\}$$

$$\left.\begin{array}{l} \{\}\cdot\hat{e}_R \Rightarrow \quad N = mR\dot{\theta}^2 \\ \{\}\cdot\hat{e}_\theta = \quad -\mu N = mR\ddot{\theta} \end{array}\right] \Rightarrow mR\ddot{\theta} = -\mu mR\dot{\theta}^2$$

$\boxed{\text{Define } \omega = \dot{\theta}}$

$$\Rightarrow \ddot{\theta} = -\mu\dot{\theta}^2 \Rightarrow \boxed{\dot{\omega} = -\mu\omega^2} *$$

$$\boxed{\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt}}$$

Why? Because we are given final θ not final t.

Now think of ω as $\omega(\theta)$.

$$\Rightarrow \frac{d\omega}{d\theta}\omega = -\mu\omega^2$$

$$\Rightarrow \frac{d\omega}{d\theta} = -\mu\omega$$

$$\Rightarrow \omega = \omega_0 e^{-\mu\theta}$$

[Note $V = \omega R$]

$$\Rightarrow V = \underbrace{\omega_0 R}_{V_0} e^{-\mu\theta}$$

$$\Rightarrow V = V_0 e^{-\mu\theta}$$

$$\underbrace{\theta = 2\pi}_{\uparrow \text{ one revolution}} \Rightarrow \boxed{V = V_0 e^{-2\pi\mu}}$$

Alternative soln. to ODE *, (the long way around)

$$\frac{d\omega}{dt} = -\mu\omega^2 \Rightarrow \frac{d\omega}{\omega^2} = -\mu\, dt \quad \text{(seperable 1st order ODE)}$$

$$\Rightarrow -\omega^{-1} - (-\omega_0^{-1}) = -\mu t \Rightarrow \frac{1}{\omega_0} - \frac{1}{\omega} = -\mu t$$

$$\Rightarrow \frac{1}{\omega} = \frac{1}{\omega_0} + \mu t \Rightarrow \boxed{\omega = \frac{1}{\mu t + 1/\omega_0} = \frac{\omega_0}{1+\omega_0\mu t}} \quad (1)$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\omega_0}{1+\omega_0\mu t} \Rightarrow d\theta = \frac{\omega_0\, dt}{1+\omega_0\mu t} \quad \text{(seperable again)}$$

$$\Rightarrow \int_0^\theta d\theta' = \int_0^t \frac{\omega_0\, dt'}{1+\omega_0\mu t'}$$

substitition; let $u = 1+\omega_0\mu t$
$du = \omega_0\mu\, dt$

$$\theta = \int_1^u \frac{1}{\mu}\frac{du'}{u'} = \frac{1}{\mu}\left[\ln(u) - \ln(1)\right] = \frac{1}{\mu}\ln u$$

$$\theta = \frac{1}{\mu}\ln(1+\omega_0\mu t)$$

$\underline{\underline{\theta = 2\pi}}$ $\Rightarrow$ $2\pi\mu = \ln(1+\omega_0\mu t)$

$$\Rightarrow e^{2\pi\mu} = 1+\omega_0\mu t$$

$$t = \frac{e^{2\pi\mu}-1}{\omega_0\mu} \quad (2)$$

Apply (2) to (1) $\Rightarrow$ $$\omega\Big|_{\theta=2\pi} = \frac{\omega_0}{1+\omega_0\mu\left[\frac{e^{2\pi\mu}-1}{\omega_0\mu}\right]} = \frac{\omega_0}{e^{2\pi\mu}}$$

$$\Rightarrow \omega = \omega_0 e^{-2\pi\mu} \Rightarrow \omega R = \omega_0 R e^{-2\pi\mu}$$

$$\Rightarrow \boxed{V = V_0 e^{-2\pi\mu}} \quad \text{(again)}$$

at $\theta = 2\pi$