Your Name: Joe Burns

Your TA: Andy Ruina

T&AM 203 FINAL EXAM

Wednesday May 17, 2000

Draft May 9, 2000

4 problems, 100 points, and 150 minutes.

Please follow these directions to ease grading and to maximize your score.

a) No calculators, books or notes allowed. Six pages of formulas from the front and back of the text are provided. The back of the test can be used for tentative scrap work. Ask for extra scrap paper if you need it.

b) Full credit if

- ↖•↗ →free body diagrams← are drawn whenever linear or angular momentum balance is used;
- $\underline{\bullet}$ correct vector notation is used, when appropriate;
- ↑→ any dimensions, coordinates, variables and base vectors that you add are clearly defined;
- ± all signs and directions are well defined with sketches and/or words;
- ⊣ reasonable justification, enough to distinguish an informed answer from a guess, is given;
- ∗ you clearly state any reasonable assumptions if a problem seems **poorly defined**;
- • work is I.) neat,
  II.) clear, and
  III.) well organized;
- • your answers are TIDILY REDUCED (Don't leave simplifiable algebraic expressions.);
- ☐ your answers are $\boxed{\text{boxed}}$ in; and
- ≫ unless otherwise stated, you will get full credit for, instead of doing a calculation, presenting Matlab code that would generate the desired answer. To ease grading and save space, your Matlab code can use shortcut notation like "$\dot\theta_7$ = 18" instead of, say, "`theta7dot = 18`".

c) Substantial partial credit if your answer is in terms of well defined variables and you have not substituted in the numerical values. Substantial partial credit if you reduce the problem to a clearly defined set of equations to solve.

Problem 1: ____/25

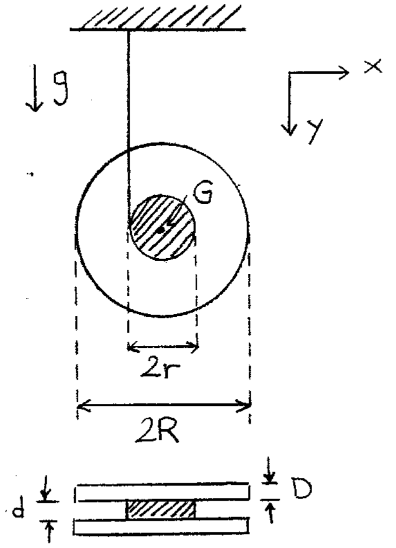
Problem 2: ____/25

Problem 3: ____/25

Problem 3: ____/25

TOTAL: enough/100
to pass

1)(25 pts) Yo-yo. A yo-yo of mass $2m$ is made of two identical disks (mass m, radius R, thickness D) glued on either side of a massless spindle (radius r, thickness d). A string is wrapped around the spindle and unwinds without friction. The string has total length L and is infinitesimally thin and massless. G is at the yo-yo's center of mass.



a) (3 pts) Does G move in the x-direction as the yo-yo falls and unwinds? Why or why not?

b) (6 pts) Find G's vertical acceleration. Comment on the two cases:
i) $R << r$ and ii) $R >> r$,

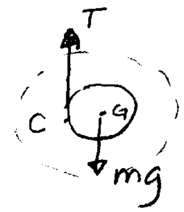
c) (4 pts) Find the tension in the string.

d) (5 pts) Write an expression for the total kinetic energy of the yo-yo when G's speed is v.

e) (5 pts) If $r << L$ and the yo-yo starts from rest, find v when the string is fully unwound.

f) (2 pts) Under what circumstances will the yo-yo rewind completely?

FBD



a.) Since $\Sigma \underline{F}$ is only in the $\hat{j}$ direction, $\boxed{\underline{a}_G \text{ is only in } \hat{j}}$, $\underline{v}_0 = 0$ $\therefore$ no x motion.

b.) Method I

$$\Sigma \underline{M}_{/c} = \dot{\underline{H}}_c$$
$$r\hat{i} \times mg\hat{j} = \dot{\underline{H}}_G + \underline{r} \times m\underline{a}$$
$$\cancel{m}rg\,\hat{k} = \tfrac{1}{2}\cancel{m}R^2\dot{\omega}\,\hat{k} + r\hat{i} \times \cancel{m}a\hat{j}$$
$$rg = \tfrac{1}{2}R^2\frac{a}{r} + ra = \left(\tfrac{1}{2}\frac{R^2}{r} + \frac{r^2}{r}\right)a$$
$$\therefore \quad \boxed{a = \frac{g}{1 + \frac{1}{2}\frac{R^2}{r^2}}}$$

$\approx g$ if $R \ll r$

$a \ll g$ if $r \ll R$

Method II

Kinematics: $\dot{\omega} r = a$ —(1)

$(LMB)\cdot\hat{j} \Rightarrow -T + mg = ma$ —(2)

$AMB_G \Rightarrow \tfrac{1}{2}mR^2\dot{\omega} = Tr$ —(3)

3 eqns for the 3 unknowns $\dot{\omega}, a, T$

c.) Plug a back into (2) $\quad T = m(g-a) = mg\left(\dfrac{1 + \frac{1}{2}\frac{R^2}{r^2} - 1}{1 + \frac{1}{2}\frac{R^2}{r^2}}\right) = \boxed{\dfrac{mg}{1 + \frac{2r^2}{R^2}} = T}$

d.) For planar motion

$$KE = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_G\omega^2 \quad \left[\text{or} = \tfrac{1}{2}I_C\omega^2 \text{ since in pure rotation about } C\right]$$
$$= \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left[\tfrac{1}{2}mR^2\frac{v^2}{r^2}\right] = \boxed{\frac{mv^2}{2}\left[1 + \frac{1}{2}\frac{R^2}{r^2}\right] = KE}$$

e.) Use conservation of energy with PE measured from top

$$PE + KE = -mgR + 0 = -\cancel{m}gL + \tfrac{1}{2}\cancel{m}v^2\left(1 + \tfrac{1}{2}\frac{R^2}{r^2}\right)$$

E@ TOP E@ BOTTOM

We ignore r & R compared to L

$$\Rightarrow \boxed{v = \sqrt{\frac{2g(L-R)}{1 + \frac{1}{2}\frac{R^2}{r^2}}}}$$

could also get by $v = \sqrt{2aL}$

f.) As long as energy is conserved (which it never will be in a real system), the yo-yo will rewind perfectly.
loss in string, slip & "bounce" at bottom.

2)(25 pts) Particle on a springy leash. A particle with mass m slides on a rigid horizontal frictionless plane. It is held by a string which is in turn connected to a linear elastic spring with constant k. The string length is such that the spring is relaxed when the mass is on top of the hole in the plane. The position of the particle is $\vec{r} = x\hat{\imath} + y\hat{\jmath}$. For each of the statements below, state the circumstances in which the statement is true (assuming the particle stays on the plane). Justify your answer with convincing explanation and/or calculation.

a) (2 pts) The force of the plane on the particle is $mg\hat{k}$.

b) (2 pts) $\ddot{x} + \frac{k}{m}x = 0$

c) (2 pts) $\ddot{y} + \frac{k}{m}y = 0$

d) (3 pts) $\ddot{r} + \frac{k}{m}r = 0$, where $r = |\vec{r}|$

e) (2 pts) $r =$ constant

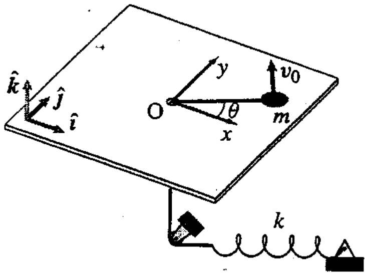
f) (3 pts) $\dot{\theta} =$ constant

g) (3 pts) $r^2\dot{\theta} =$ constant

h) (2 pts) $m(\dot{x}^2 + \dot{y}^2) + kr^2 =$ constant

i) (3 pts) The trajectory is a straight line segment.

j) (3 pts) The trajectory is a circle.

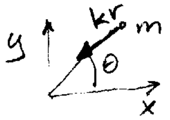


a.) Since the particle has no vertical ($\hat{k}$) motion, $\Sigma F_z = 0$

$\Rightarrow N = mg$

Always true

b.)
c.) Writing LMB in horizontal plane and using Cartesian coords,



$$\Sigma \underline{F} = m\underline{a} \Rightarrow -k\underline{r} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) = -k(x\hat{\imath} + y\hat{\jmath})$$

$$\therefore \boxed{\begin{aligned}\ddot{x} + \tfrac{k}{m}x = 0 \\ \ddot{y} + \tfrac{k}{m}x = 0\end{aligned}}$$

always true as long as a frictionless

d.) If we express $\underline{a}$ in polar coords

$$-kr\hat{e}_r = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta\right] \quad *$$

$$\hat{e}_r \cdot (\text{LMB}) \Rightarrow -kr = m(\ddot{r} - r\dot{\theta}^2) \Rightarrow \boxed{\ddot{r} + \tfrac{k}{m}r = 0 \text{ if } \dot{\theta} = 0 \; \underline{\text{motion is purely radial}}}$$

e.) $r =$ constant when $m(-r\dot{\theta}^2) = -kr$ (part of radial $*$)

$$\therefore \dot{\theta} = \sqrt{\frac{k}{m}} \text{ and is const}$$

f.) This is a variant of e.), $\dot{\theta}$ will be constant if it is $= \sqrt{\frac{k}{m}}$ and r is constant

g.) $r^2\dot{\theta}$ is angular momentum per unit mass (i.e., for planar motion $H = mr(r\dot{\theta})$; or $r^2\dot{\theta} = \frac{H}{m}$)

From $*$, we see the $\hat{e}_\theta$ component is $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{2\cancel{r}\dot{r}\dot{\theta} + r^{\cancel{2}}\ddot{\theta}}{\cancel{r}} = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

Since there is no $\hat{e}_\theta$ force component, this is always true $\Rightarrow r^2\dot{\theta} = \text{const.}$

h.) This is twice total energy $2\left(\frac{1}{2}mv^2 + \frac{1}{2}kr^2\right)$. Thus always true for this conservative system.

i.) As part d) showed, get straight line motion if purely radial motion

j.) As parts e) and f) mentioned, circular motion is possible when $\dot{\theta} = \sqrt{\frac{k}{m}}$

3)(25 pts) The problems below (a-d) are independent.

a) (5 pts) A falling wire. A U-shaped wire (of the given dimensions) falls from a height h without rotation and strikes a tabletop at T completely inelastically. Briefly defend your answers to the following questions.

During impact:

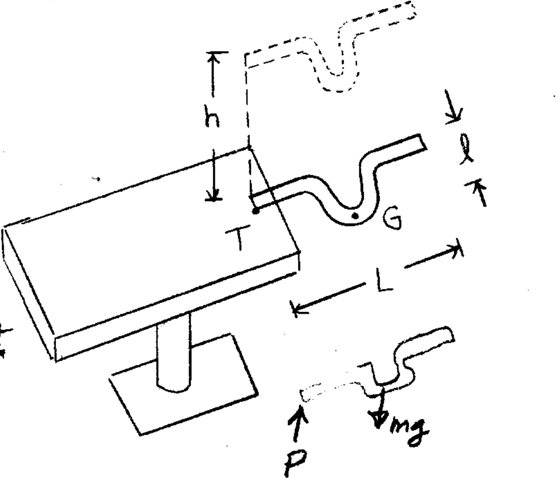
i) (1 pt) Is the wire's linear momentum conserved?

ii) (1 pt) Is the wire's angular momentum about G constant?

iii) (1 pt) Is the wire's angular momentum about T constant?

iv) (1 pt) Is the wire's angular velocity conserved?

v) (1 pt) Is the wire's total mechanical energy (kinetic + potential) constant?



i.) $\underline{\text{LMB}} \Rightarrow \underline{F} = \frac{d}{dt} m\underline{v}$, External forces act $\therefore$ linear momentum not const

ii.) $\underline{\text{AMB}} \Rightarrow \underline{M}_{/G} = \dot{\underline{H}}_{/G}$ $\quad \underline{P}$ has moment about G $\therefore \underline{H}_{/G} \neq$ constant

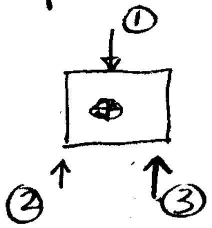
iii) The impulsive force $\underline{P}$ has no moment about T; we can ignore mg in comparison to P and thus $\underline{H}_T$ is approximately constant

iv.) It wasn't rotating beforehand and will afterward (due to $\underline{M}_G$). $\underline{\omega}$ not conserved.

v.) Energy is lost in collisions generally unless they are elastic. This is an inelastic collision $\therefore$ not constant E
(heating, sound)

b) (5 pts) Mars Polar Lander. Last December when the Mars Polar Lander was lost, some blamed its simple thrusters (devices that eject gas and thus are capable of providing an impulse in a single direction). Argue using dynamical principles, the minimum number of thrusters required to have complete three-dimensional control.

The full dynamical description of a body is given by the $\overset{\text{LMB}}{\underline{\text{3-D position of the CM}}}$ plus $\underline{\text{three angles }\overset{\text{AMB}}{\text{to give}}\text{ its orientation}}$. For each of these, you need to be able to go + and −. We can control the angular orientations with the same thrusters. For example



① for downward motion

②+③ for pure upward

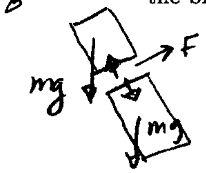
①+③ for + rotation

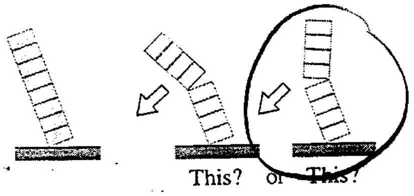
②+① for − rotation

$$\boxed{3 \times 3 = 9 \text{ thrusters}}$$

c) (5 pts) A falling tower. Frequently parents will build a tower of blocks for their children. Just as frequently, kids knock them down. In falling (even when they start to topple aligned), these towers invariably break in two (or more) pieces at some point along their length. Why does this occur? What condition is satisfied at the point of the break? Will the stack bend towards or away from the floor after the break?

$\dot{\theta}, \ddot{\theta}$

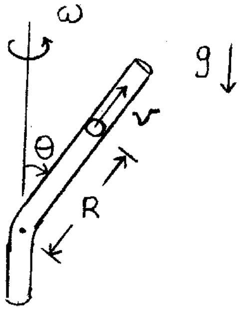


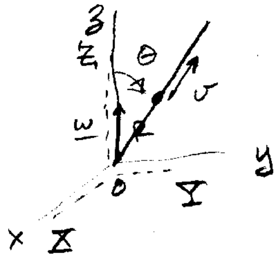


The tower will rotate as a single piece only as long as the forces between the various blocks are compressive and friction sufficient to prevent sliding. Once they become tensile, it will separate since unglued blocks cannot support tension. Top block has (largest r) largest acceleration & least compressive force. Thus the top breaks free first. And then the others follow, The angular motion of the solid stack is given by $\underline{M} = I\dot{\omega}\hat{k}$ where $I \sim h^2$ by $M \sim h$. Thus taller stacks fall more slowly. Hence the break-aways will lag behind & will appear to bend away from the floor.

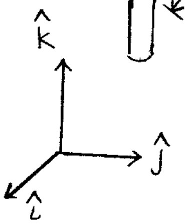
d) (10 pts) A pea shooter. A pea of mass m is being blown out of a tube at constant speed v. The tube itself is at a constant angle θ to the vertical and spins at constant angular velocity $\omega_o\hat{\mathbf{k}}$ (i.e., it sweeps out a cone). At the instant shown, the tube is in the yz-plane and the pea is at a distance R along the tube.

- **i)** (7 pts) What is the pea's acceleration?
- **ii)** (3 pts) What force acts on it?





Choose XYZ, (inertial) as given. Fix moving system xyz such that it spins with the tube and is instantaneously parallel to XYZ (as shown).



$$\underline{a}_{pea} = \underline{a}_{rel} + \underline{a}_o + \underline{\omega} \times (\underline{\omega} \times \underline{R}) + \dot{\underline{\omega}} \times \underline{R} + 2\underline{\omega} \times \underline{v}$$

$\underline{a}_{rel} = 0$ pea moves with constant speed along tube

$\underline{a}_o = 0$ origins are together always

$\dot{\underline{\omega}} = 0$ moving system spins at constant rate

$$\therefore \underline{a}_{pea} = \omega\hat{k} \times (\omega\hat{k} \times R[\sin\theta\hat{j} + \cos\theta\hat{k}]) + 2\omega\hat{k} \times v(\sin\theta\hat{j} + \cos\theta\hat{k})$$

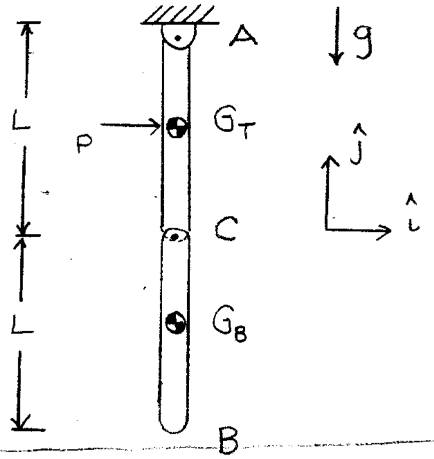
$$= \omega\hat{k} \times \omega R\sin\theta(-\hat{i}) - 2\omega v\sin\theta\hat{i}$$

$$\boxed{\underline{a}_{pea} = -\omega^2 R\sin\theta\hat{j} - 2\omega v\sin\theta\hat{i}}$$

$\underline{LMB} \Rightarrow \boxed{\underline{F}_{EXT} = m\underline{a}_{pea}}$ with $\underline{a}_{pea}$ given above. This force is caused by wall forces (contact) gravity and gas pressure.

4)(25 pts) **Double pendulum.** Two identical homogeneous slender bars (weight W, length L, frictionless hinges) hang vertically in a gravity field. They are initially at rest when a horizontal force $P\hat{\imath}$ is suddenly applied at the center of the top bar.

- a) (10 pts) Write out expressions for the accelerations of the centers of mass G_T and G_B in terms of the angular motions of the bars.
- b) (10 pts) Write down sufficient equations to solve for the reaction forces at A and C, and for the angular motions.
- c) (5 pts) Describe how to use Matlab to solve these equations. You need not solve them.



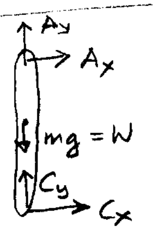
Instantaneously,

a.) $$\underline{a}_{G_T} = \underline{\alpha}_T \times \frac{L}{2}(-\hat{\jmath}) = \alpha_T \frac{L}{2}\hat{\imath}$$

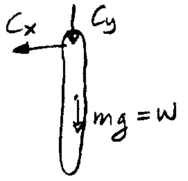
Does not include $\underline{\omega}\times(\underline{\omega}\times\frac{L}{2})$ because $\underline{\omega}=0$ initially

$$\underline{a}_{B_T} = \underline{a}_C + \underline{a}_{B/C} = \alpha_T L\hat{\imath} + \alpha_B \hat{k}\times\frac{L}{2}(-\hat{\jmath})$$
$$= \left(\alpha_T L + \alpha_B\frac{L}{2}\right)\hat{\imath}$$

b.)



We have 6 unknowns : $\underline{A}$ & $\underline{C}$ (2 components each) and 2 αs and 6 eqns (LMB for top and bottom bars - 2 components each) and AMB for each bar.



Top ; LMB

$$\sum \underline{F}_{EXT} = -W\hat{\jmath} + \underline{A} + \underline{C} = m a_{G_T}$$

$$\boxed{\frac{W}{g}\alpha_{G_T}\frac{L}{2}\hat{\imath} = (A_x + C_x)\hat{\imath} + (A_y + C_y - W)\hat{\jmath}}$$

AMB

$$\sum \underline{M}_{G_T} = \dot{\underline{H}}_{G_T} = \frac{1}{12}\frac{W}{g}L^2\dot{\omega}_{G_T}\hat{k}$$

$$\boxed{(C_x - A_x)\cancel{\frac{L}{2}} = \frac{WL^{\cancel{2}}}{\underset{6}{\cancel{12}}g}\alpha_T}$$

Bottom

LMB $$\sum \underline{F}_{EXT} = -W\hat{\jmath} - \underline{C} = m\underline{a}_{GB}$$

$$\boxed{-C_x\hat{\imath} - (+W + C_y)\hat{\jmath} = \frac{W}{g}L\left(\alpha_g + \frac{\alpha_B}{2}\right)\hat{\imath}}$$

AMB $$\sum \underline{M}_{GB} = \dot{\underline{H}}_{G_T} = \frac{W}{12g}L^2\alpha_B\hat{k}$$

$$\boxed{C_x\frac{L}{2} = \frac{W}{12g}L^2\alpha_B}$$

Ⓒ The boxed equations are 6 algebraic eqns for the unknowns $A_x, A_Y, C_x, C_Y, \alpha_T$ & α_B. We write them in matrix form $Ax = b$, where x is a column vector of ↗. and solve $x = A^{-1}B$. In Matlab we would write

entries

$\underline{\underline{A}} = [6\ ;\ 6;6;6;6;6];$

$\mathbf{b} = [1;1;1;1;1;1];$

$x = A\backslash b$