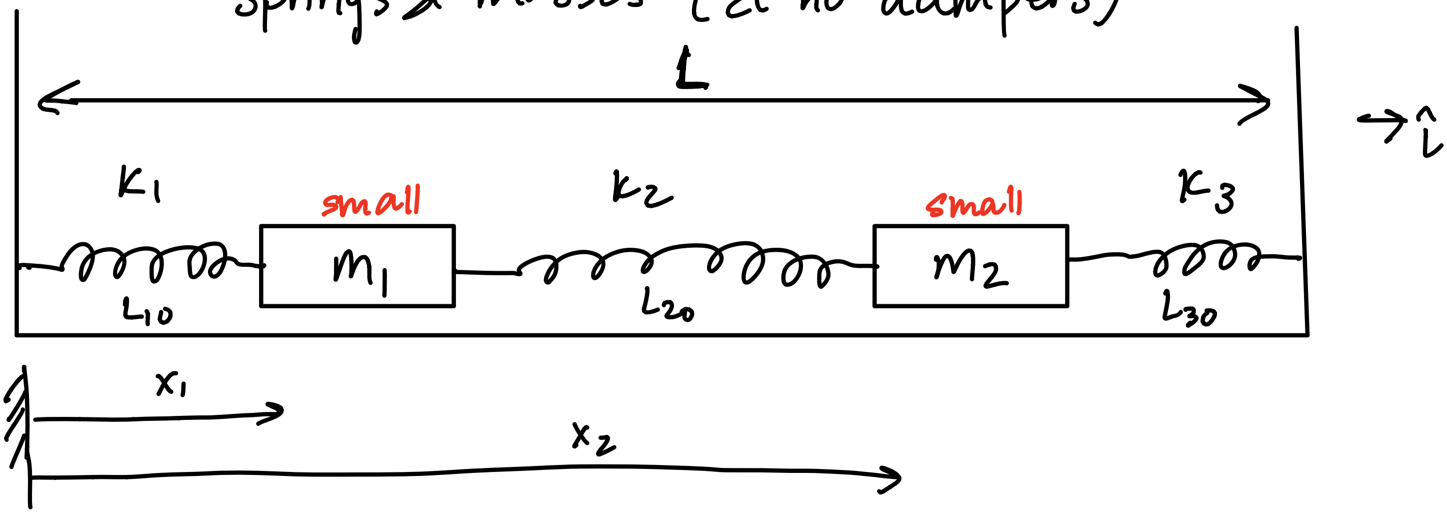
MAE 2030 Lecture #8:

2/24/21

Today: ① Normal modes
② Demos, videos

springs & masses (& no dampers)



Multi-DoF systems:

FBDs:

$T_2 = k_2(L_2 - L_{20})$

$x_2 - x_1$

$T_1 = k_1(L_1 - L_{10})$

x_1

$= k_3(L_3 - L_{30})$

$L - x_2$

LMB 1:

$$\left\{ \sum \vec{F} = m_1 \vec{a}_1 \right\}$$

$$\left\{ \quad \right\} \cdot \hat{i}$$

$$\Rightarrow -k_1(x_1 - L_{10}) + k_2(x_2 - x_1 - L_{20}) = m_1 \ddot{x}_1 \quad 1$$

LMB 2:

$$\left\{ \sum \vec{F} = m_2 \vec{a}_2 \right\}$$

$$\left\{ \quad \right\} \cdot \hat{i}$$

$$\Rightarrow -k_2(x_2 - x_1 - L_{20}) + k_3(L - x_2 - L_{30}) = m_2 \ddot{x}_2 \quad 2$$

1 & 2 are 2 2nd order ODEs (equivalent to 4 1st order ODEs)

Math World:

1 $\Rightarrow m_1 \ddot{x}_1 + (k_1 + k_2) x_1 + (-k_2) x_2 =$ bunch of constants

2 $\Rightarrow m_2 \ddot{x}_2 + (-k_2) x_1 + (k_2 + k_3) x_2 =$ bunch of constants

particular sol'ns $\begin{cases} x_{1p} = C_1 \\ x_{2p} = C_2 \end{cases}$ could figure out by writing out R.H.S.

homogeneous sol'n: 3

Solve

$$m_1 \ddot{x} + (k_1+k_2)x_1 + (-k_2)x_2 = 0$$

$$m_2 \ddot{x}_2 + (-k_2)x_1 + (k_2+k_3)x_2 = 0$$

$\Rightarrow$ homogeneous sol'n

Matrix notation:

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\underline{M}} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}}_{\underline{K}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve $\underline{M}\ddot{z} + \underline{K}z = 0$

$$z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

one sol'n is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\text{constant vector}} \sin \omega_1 t$$

another sol'n is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_{\text{constant vector}} \sin \omega_2 t$$

FACT 1: collections of springs & masses have synchronous sine wave solutions $\Rightarrow$ normal modes

FACT 2: all motions are superpositions of normal modes

demo: Andy shows types of elastic systems (vibrating wine glasses, string instruments, golden gate bridge)

Multi-DoF oscillator eqn:

$$M\ddot{z} + Kz = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (*)$$

$z \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

M: mass matrix

K: Stiffness matrix

$$M = \begin{bmatrix} m_1 & \\ & m_2 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}$$

Solve $(*)$!

How ? : Guess

$$z = \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{V = \text{constant}} \sin(\omega_1 t) = [V] \sin(\omega_1 t)$$

Good guess?

How to find out?: check

$$M\ddot{z} + Kz = \begin{bmatrix}0\\0\end{bmatrix}$$

$$M\left([v]\,\ddot{\sin}(\omega t)\right) + K\left([v]\sin(\omega_1 t)\right) = \begin{bmatrix}0\\0\end{bmatrix}$$

$$-\omega_1^2 M[v]\cancel{\sin(\omega_1 t)} + K\cdot[v]\cancel{\sin(\omega_1 t)} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\longrightarrow$ simplify :

$$\left\{ \boxed{-\underset{?}{\omega_1^2}\,\overset{\checkmark}{M}\,\underset{?}{[v]} + \overset{\checkmark}{K}\,\underset{?}{[v]} = \begin{bmatrix}0\\0\end{bmatrix}} \right\}$$

$$M^{-1}\{\ \} \Rightarrow \left(-\omega_2 \underset{\text{identity}}{\underset{\uparrow}{I}} + M^{-1}K\right)V = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\boxed{M^{-1}K[v] = +\omega^2[v]}$$