Prelim 2, Q4 Solution

Thursday, April 8, 2021 8:06 PM

Solution by Michael Zakoworotny

4) Pendulum. A uniform block (width = w, height = h, mass = m) is suspended by two massless rigid rods AB and CD with lengths L that have ideal bearings at their ends (A,B,C & D). Gravity g causes the pendulum to swing. The pendulum is released from rest (all velocities at $t = 0$ are $\vec{0}$) at $\theta = \theta_0$.

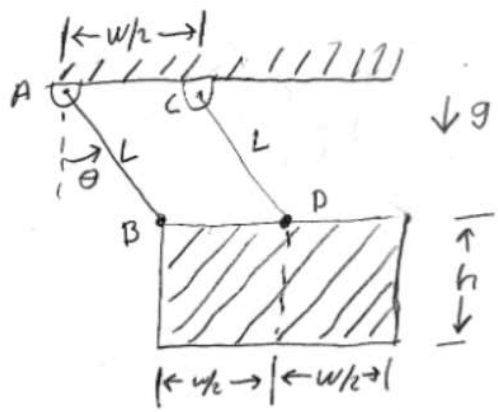
* All answers below should be in terms of some or all of w, h, m, L, g and θ_0.

* Hint: all points on the block have the same acceleration.

a) Immediately after release, find tension AB or CD (your choice)?

b) Find the equations of motion (*e.g.*, find differential equations that, if you solved them would give you the motion.)

c) Find tensions AB and CD (both of them) when the pendulum has swung down to $\theta = 0$.

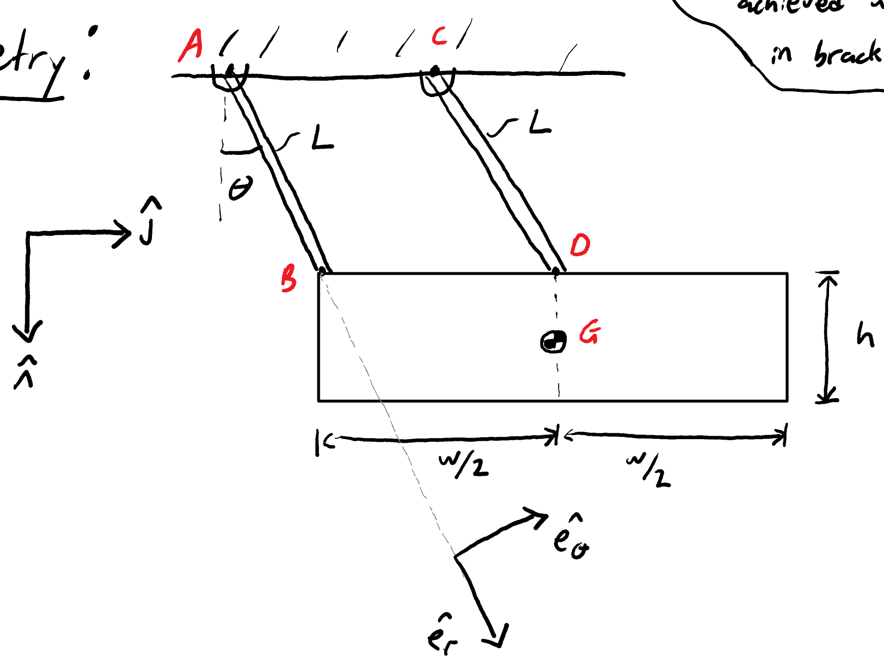


[Given: L, w, h, m, g

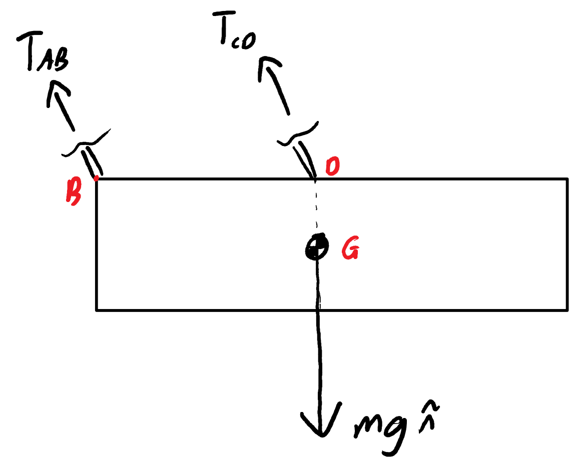
$\theta(0) = \theta_0, \quad \dot{\theta}(0) = 0$]

Note: Solution is more thorough than what is expected on exam. Full credit can be achieved w/o including items in brackets [ ]

Geometry:



FBD



The block only moves translationally because it is constrained to never rotate about its C.G. Therefore, the velocity and acceleration of all points on the block are equal: $\vec{a}_B = \vec{a}_D = \vec{a}_G$

a) Find T_{AB} or T_{CD} at $t=0$

Let: $\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

LMB: $\sum \vec{F} = m\,\vec{a}_{G/F} = m\,\vec{a}_{B/F}$

$$\vec{a}_{B/F} = -L\dot{\theta}^2\,\hat{e}_r + L\ddot{\theta}\,\hat{e}_\theta$$

$$-T_{AB}\,\hat{e}_r - T_{CD}\,\hat{e}_r + mg\,\hat{\imath} = m\left(-L\dot{\theta}^2\,\hat{e}_r + L\ddot{\theta}\,\hat{e}_\theta\right) \quad (1)$$

$(1)\cdot\hat{e}_\theta$: $\cancel{m}g\hat{\imath}\cdot\hat{e}_\theta = \cancel{m}L\ddot{\theta}$ $\quad \hat{\imath}\cdot\hat{e}_\theta = -\sin\theta$

$$-g\sin\theta = L\ddot{\theta} \qquad \text{At } t=0,\ \theta=\theta_0$$

$$\ddot{\theta} = -\frac{g}{L}\sin\theta_0 \quad (2)$$

<u>AMB</u>: $\sum \vec{M}_{/D} = \dot{\vec{H}}_{/D}$

$$= \int \vec{r}_{/D} \times \vec{a}\, dm$$

$\hookrightarrow = a_{B/F}$, constant for all pts on block

$$= \int \vec{r}_{/D}\, dm \times \vec{a}_{B/F}$$

$$\sum \vec{M}_{/D} = \vec{r}_{G/D}\, m_{Tot} \times \vec{a}_{B/F}$$

$$\vec{r}_{B/D} \times -T_{AB}\hat{e}_r + \cancel{\vec{r}_{D/D}}^{\vec{0}} \times -T_{CD}\hat{e}_r + \underbrace{\vec{r}_{G/D} \times mg\hat{\imath}}_{=\vec{0},\ \text{parallel}} = \vec{r}_{G/D} \times m\vec{a}_{G/F}$$

$$-\frac{w}{2}\hat{\jmath} \times -T_{AB}\hat{e}_r = \frac{h}{2}\hat{\imath} \times m\left(-L\dot{\theta}^2\hat{e}_r + L\ddot{\theta}\hat{e}_\theta\right)$$

$\hookrightarrow$ At $t=0$, $\dot{\theta}=0$

$$\left\{ -\frac{w}{\cancel{2}} T_{AB} \cancel{\cos\theta_0}\hat{k} = m\frac{h}{\cancel{2}} L\ddot{\theta}\cancel{\cos\theta_0}\hat{k} \right\}$$

$\{\}\cdot\hat{k}$: $-wT_{AB} = mhL\ddot{\theta}$ $\leftarrow$ Apply (2)

$$-wT_{AB} = mh\cancel{L}\cdot -\frac{g}{\cancel{L}}\sin\theta_0$$

$$\boxed{T_{AB} = \frac{mgh}{w}\sin\theta_0} \quad (3)$$

[T_{CD} can be found by instead performing AMB about B ... or] by plugging (3) into (1) $\cdot \hat{e}_r$

(1) $\cdot \hat{e}_r$: $-T_{AB} - T_{CD} + mg\cos\theta_0 = -mL\cancel{\dot{\theta}_0^2}^{\,0}$

$$T_{CD} = mg\cos\theta_0 - \frac{mgh}{w}\sin\theta_0$$

$$\boxed{T_{CD} = mg\left[\cos\theta_0 - \frac{h}{w}\sin\theta_0\right]}$$

b) [Obtain the EOM:

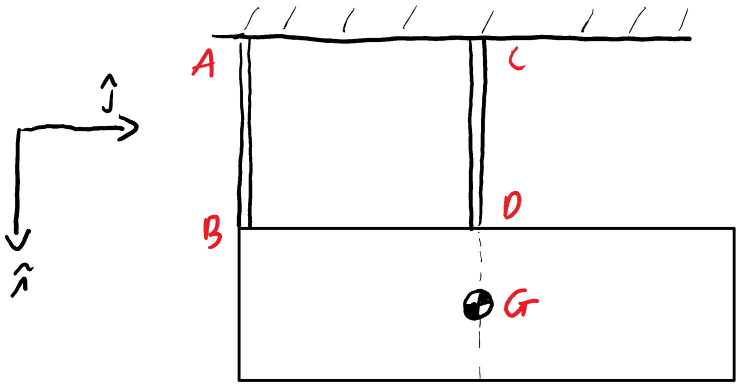
Consider equation (1), from LMB]

$-T_{AB}\hat{e}_r - T_{CD}\hat{e}_r + mg\hat{n} = m\left(-L\dot{\theta}^2\hat{e}_r + L\ddot{\theta}\hat{e}_\theta\right)$ (1)

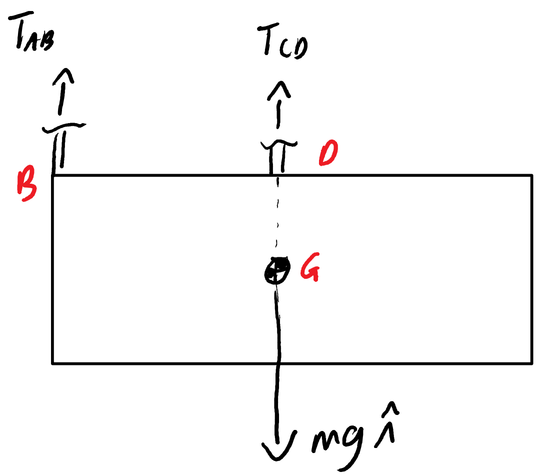
Again, (1) $\cdot \hat{e}_\theta$: $-\cancel{m}g\sin\theta = \cancel{m}L\ddot{\theta}$

$$\boxed{\ddot{\theta} = \frac{-g}{L}\sin\theta}$$

c) Find both T_{AB} and T_{CD} when $\theta = 0$ (bottom)



At $\theta = 0$, $\hat{e}_r = \hat{\imath}$, $e_\theta = \hat{j}$



[In this configuration, G is aligned with rod CD. Rod AB must carry no load, since the block cannot experience any angular accelerations about its C.G.]

$$\vec{M}_{/G} = \dot{\vec{H}}_{/G} = \vec{0} \Rightarrow T_{AB} = 0$$

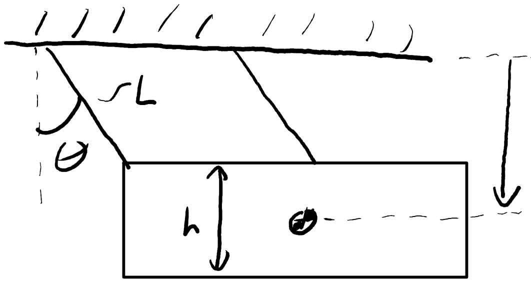
LMB: $\Sigma \vec{F} = m\,\vec{a}_{G/\mathcal{F}}$

$$\{ -T_{CD}\,\hat{\imath} + mg\,\hat{\imath} = m(-L\dot{\theta}^2\,\hat{\imath} + L\ddot{\theta}\,\hat{j}) \}$$

$\{\}\cdot\hat{\imath}$: $-T_{CD} + mg = -mL\dot{\theta}^2$

$$T_{CD} = mg + mL\dot{\theta}^2, \text{ at } \theta = 0 \quad (4)$$

Get $\dot{\theta}^2$ from energy conservation



$$d = L\cos\theta + \frac{h}{2}$$

At $t=0$: $E_0 = -mgd_0$

$$= -mg\left(L\cos\theta_0 + \frac{h}{2}\right)$$

At bottom: $E = -mgd + \frac{1}{2}m|\vec{v}_G|^2$

$d = L + \frac{h}{2}$ at $\theta = 0$

$\hookrightarrow |\vec{v}_G| = |\vec{v}_B|$
$= L\dot{\theta}$

$$E = -mg\left(L + \frac{h}{2}\right) + \frac{1}{2}mL^2\dot{\theta}^2$$

Conservation: $E_0 = E$

$$-\cancel{m}g\left(L\cos\theta_0 + \cancel{\frac{h}{2}}\right) = -\cancel{m}g\left(L + \cancel{\frac{h}{2}}\right) + \frac{1}{2}\cancel{m}L^2\dot{\theta}^2$$

$$\frac{1}{2}L^2\dot{\theta}^2 = gL(1-\cos\theta_0)$$

$$\dot{\theta}^2 = \frac{2g}{L}(1-\cos\theta_0) \qquad (5)$$

Apply (5) to (4):

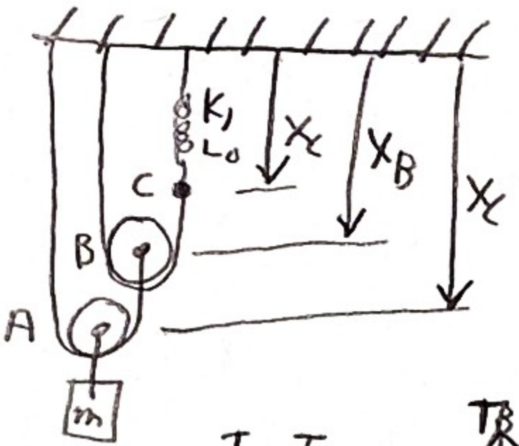
$$T_{CD} = mg + m\cancel{L} \cdot \frac{2g}{\cancel{L}}(1 - \cos\theta_0)$$

$$\boxed{T_{CD} = mg(3 - 2\cos\theta_0), \quad T_{AB} = 0}$$

[Note: If $\theta_0 = \pi/2$, this results in $T_{CD} = 3mg$, which is the same as the particle mass pendulum from lecture.]

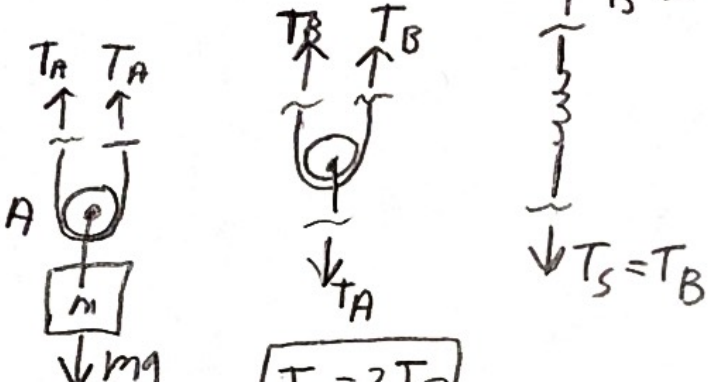
(5)

Neglect pulley radii & masses.



$$\uparrow T_s = K\underbrace{(X_c - L_o)}_{s = \text{spring stretch}}$$

FBDs:



$$\boxed{T_A = 2T_B}$$

Kinematics

String 1: $L_A = X_A + (X_A - X_B)$ ①

Sting 2: $L_B = X_B + X_B - X_C$ ②

Spring: $S = X_C - L_0$ (spring stretch)

LMB:

$m\ddot{X}_A = \sum F$ ③

$= mg - 2T_A$ (FBD A)

$T_A = 2T_B$ (FBD B)

$T_B = T_S$ (FBD spring)

$T_S = k(X_C - L_0)$

② ⇒ $X_C = 2X_B - L_B$

① ⇒ $X_B = 2X_A - L_A$

Back substitute through relations above

(5) $\Rightarrow$ $X_C = 2X_B - L_B = 4X_A - 2L_A - L_B$

$T_S = K(X_C - L_0) = K(4X_A - (\underbrace{2L_A + L_B + L_0}_{\equiv L}))$ (4)

$T_A = 2T_B = 2T_S = 2K(4X_A - L)$

LMB (again)

(3) $m\ddot{X}_A = mg - 2T_A$

$= mg - 16KX_A + 4Lk$

$\Rightarrow \boxed{m\ddot{X}_A + 16KX = mg + 4LK}$

$\boxed{\omega = \sqrt{16K/m} = 4\sqrt{K/m}}$ (a)

(b) $X_A = X_p + X_h$

$X_h = A\cos(\omega t) + B\sin(\omega t)$

$A = 0$, $B = d$

problem statement

$X_p = \frac{mg}{16K} + L/4$ (5)

Want $T_s > 0$. At extreme $T_s = 0$.

" " $X_A = X_p - d$

most up

$T_s = 0$ & (4) $\Rightarrow$ $4X_A - L = 0$

$\Rightarrow$ $4X_p - 4d - L = 0$

(5) $\Rightarrow$ $\left(\frac{mg}{4K} + \cancel{L}\right) - 4d - \cancel{L} = 0$

$\Rightarrow$ biggest possible $\boxed{d = mg/16K}$

(b) Intuitive Solution :

max amplitude = d = displacement of mass due to gravity (starting w/ no slack)

[At that displacement there is no slack & no tension, Any more and there will be slack]

$$\underbrace{\text{stretch of spring}}_{\text{statics}} = \frac{\text{spring tension}}{K}$$

$$= \frac{mg/4}{K}$$

$$= mg/4K$$

Displacement of mass = (spring stretch)/4

2 pulleys

$$\Rightarrow \quad \boxed{d = \frac{mg}{16K} = \text{static disp. of A due to gravity}}$$

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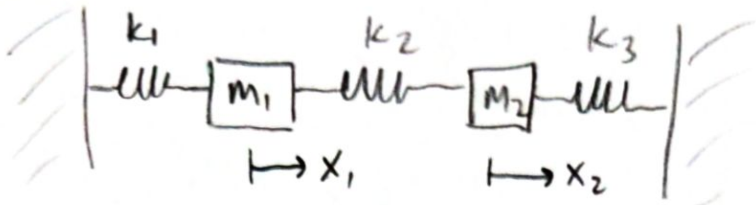
P6: Vibrations

GIVEN:



→ all springs relaxed in eqilibrium position as pictured

PART A: find governing equations, write in matrix form

FBD's:

From this we reciere EOM's:

(1) $m_1\ddot{x}_1 = k_2 x_2 - (k_1 + k_2)x_1$

(2) $m_2\ddot{x}_2 = -(k_2+k_3)x_2 + k_2 x_1$

put into matrix:

$$M\ddot{x} + \cancel{C\dot{x}}^{\,0 \text{ no damping}} + Kx = \cancel{F}^{\,0 \text{ no forcing}}$$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{x}} + \underbrace{\begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = 0$$

PART B: Given $k_2 = k_1$, $k_3 = 4k_1$, $m_2 = 2m_1$

Determine if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sin(\lambda t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a normal mode.

☆ translation: is $\sin(\lambda t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a solution to our EOM?

$$\boxed{\begin{aligned} \text{EOM: } &\text{①}\ m_1\ddot{x}_1 = -(k_1+k_2)x_1 + k_2 x_2 \\ &\text{②}\ m_2\ddot{x}_2 = k_2 x_1 - (k_2+k_3)x_2 \end{aligned}}$$

plug in

$$\boxed{\begin{aligned} x_1 &= \sin\lambda t & \ddot{x}_1 &= -\lambda^2 \sin\lambda t \\ x_2 &= -\sin\lambda t & \ddot{x}_2 &= \lambda^2 \sin\lambda t \end{aligned}}$$

① $m_1(-\lambda^2 \cancel{\sin\lambda t}) = -(k_1+k_2)(\cancel{\sin\lambda t}) - k_2 \cancel{\sin\lambda t}$

$-m_1\lambda^2 = -(k_1+k_2) - k_2$

plug in params:

$-m_1\lambda^2 = -2k_1 - k_1 = -3k_1$

$\lambda^2 = 3k_1/m_1$

② $m_2(\lambda^2 \cancel{\sin\lambda t}) = k_2(\cancel{\sin\lambda t}) - (k_2+k_3)(-\cancel{\sin\lambda t})$

$m_2\lambda^2 = k_2 + (k_2 + k_3)$

plug in params:

$2m_1\lambda^2 = 2k_1 + 4k_1 = 6k_1$

$\lambda^2 = 6k_1/6m_1 = 3k_1/m_1$

both equations ① & ② are satisfied if $\lambda = 3k_1/m_1$, which means that the solution we tried $\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sin(\lambda t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ is a TRUE solution. so YES, the person is right. ☺

★ ★ $\boxed{\text{2 extra methods for Part B: eigenvectors}}$ ★

PART B: Given $k_2 = k_1$, $k_3 = 4k_1$, $m_2 = 2m_1$

Determine if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \sin(\lambda t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a normal mode.

★ alternative translation: is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an eigenvector of $\underbrace{\begin{bmatrix} M \backslash K \end{bmatrix}}_{\text{"A"}}$?

first compute A matrix:

$$A = M \backslash K = M^{-1}K = \begin{bmatrix} (k_1+k_2)/m_1 & -k_2/m_1 \\ -k_2/m_2 & (k_2+k_3)/m_2 \end{bmatrix}$$

setting $k_1 = 1 \ldots$ $m_1 = 1 \ldots$

↳ $k_2 = 1$ ↳ $m_2 = 2$

$k_3 = 4$

then A becomes:

$$\begin{bmatrix} (1+1)/1 & -1/1 \\ -1/2 & (1+4)/2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1/2 & 5/2 \end{bmatrix}$$

TBH could have left it all symbolic but I didn't want to

Q: is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 2 & -1 \\ -1/2 & 5/2 \end{bmatrix}$?

$\boxed{1}$ eigenvector check Method:

if $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = S\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector (& normal mode)

↖ any scalar

compute:

$$\begin{bmatrix} 2 & -1 \\ -1/2 & 5/2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ -1/2 - 5/2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \overset{\checkmark}{=} S\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

yes, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ works as a normal mode :)

2 Marigot's "I hate eigenvectors but love MATLAB" method:

MATLAB:

```
A = [ -2    1;
      1/2  -5/2 ]

[vec, val] = eig(A)            ← I know [vec] will have
                                  the normalized eigenvectors

vec1 = vec(:, 1)
vec2 = vec(:, 2)               ← unpack [vec] into the
                                  separate eigenvectors

mode = [-1 ; 1]                ← mode that we're hoping
                                  either [vec1] or [vec2] is
                                  a multiple of

S1 = vec1 ./ mode              ← [S1] & [S2] will be vectors
S2 = vec2 ./ mode

if S1(1) == S1(2)              ← if the elements in [S1]
    disp('yes')                   are the same, [vec1] is
                                  a scalar multiple of [mode]
elseif S2(1) == S2(2)          ← same for [S2] and [vec2]
    disp('yes')
else                           ← if niether [vec1] nor [vec2]
    disp('no')                    is a scalar multiple of
                                  [mode] ... we will get 'no'
```

(this will end up printing... which you wouldn't necessarily know on the test. that's OK! Nearly full points for code that will give the right answer when run.)