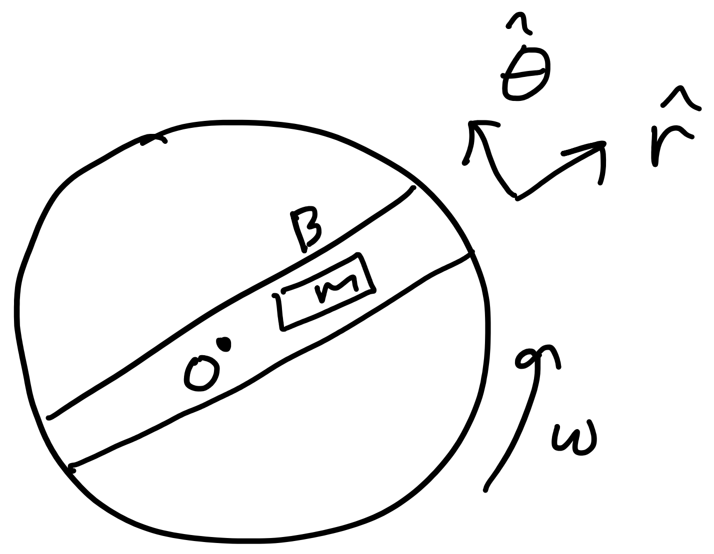
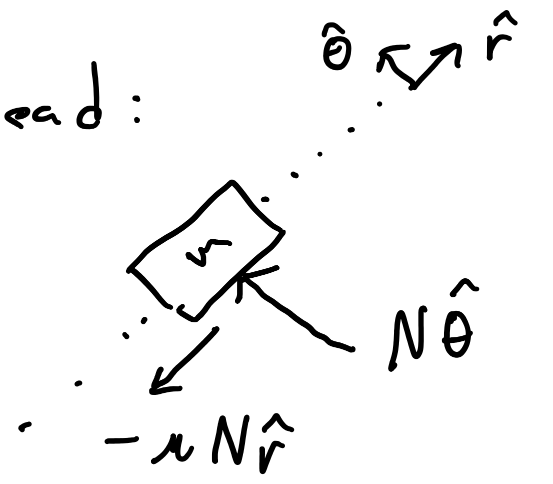
Problem 8 Solution, by A. Rodriguez-Gonzalez

Sunday, May 9, 2021 7:38 PM



FBDs:

bead:



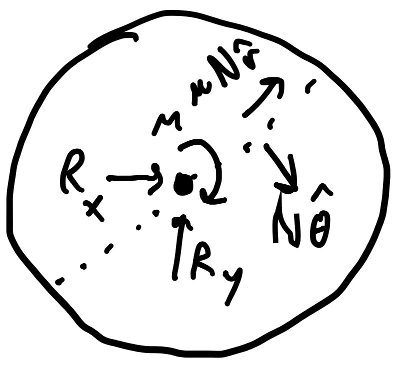
LMB for the bead:

$\vec F = m\vec a$, $\vec F = N\hat\theta - \mu N\hat r$

$\vec a = (\ddot r - r\dot\theta^2)\hat r + (r\ddot\theta + 2\dot\theta\dot r)\hat\theta$

$= (\ddot r - r\omega^2)\hat r + 2\dot r\omega\hat\theta$

disk:



So, $\vec F = m\vec a \rightarrow N\hat\theta - \mu N\hat r = (\ddot r - r\omega^2)\hat r + 2\dot r\omega\hat\theta$

Dotting with $\hat\theta$,

$N = 2\dot r\omega$ (Eq. 1)

Now consider AMB for the disk at the pin (O):

$$\underbrace{\cancel{\vec r_{O/G}}}_{\vec 0,\ O=G} \times m\vec a_G + I\underbrace{\cancel{\dot\omega}}_{0}\,k = \sum \vec\tau_{/O} = \vec 0$$

$$\sum\vec\tau_{/O} = \underbrace{\cancel{\vec r_{O/O}}}_{\vec 0} \times R_x\hat\imath + \underbrace{\cancel{\vec r_{O/O}}}_{\vec 0} \times R_y\hat\jmath + \vec r_{B/O} \times (-N\hat\theta)$$
$$+ \vec r_{B/O} \times \mu N\hat r + M\hat k$$

$$\sum\vec\tau_{/O} = r\hat r \times -N\hat\theta + \underbrace{\cancel{r\hat r \times \mu N\hat r}}_{0} + M\hat k = \vec 0$$

Dotting with $\hat k$, $M - Nr = 0$, so:

$M = Nr$ (Eq. 2)

We can now calculate the power exerted by the motor onto the disk:

$$P = \sum\vec F \cdot \underbrace{\cancel{\vec v}}_{0,\text{ pin is fixed}} + \sum\vec M\cdot\vec\omega = \sum\vec M\cdot\vec\omega$$

$$= M\omega = Nr\omega \text{ (from Eq. 1) } = 2m\dot r r\omega^2 \text{ (from Eq. 2)}$$

The total amount of work W is the power integrated over time:

$$W = \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} 2\dot r\omega^2 r m\,dt$$

Recall that $\dot r = \frac{dr}{dt}$, so $\dot r\,dt = dr$. Hence,

$$W = \int_{t_1}^{t_2} 2\dot r\omega^2 r m\,dt = \int_{R_1}^{R_2} 2\omega^2 r m\,dr = m\omega^2 r^2\Big|_{r=R_1}^{r=R_2}$$

$$\boxed{W = m\omega^2(R_2^2 - R_1^2)}$$

9) Block with springs. A uniform square plate (sides L, mass m, moment of inertia I^G) is suspended by two identical springs (spring constant = k). It is in equilibrium.

a) Then, spring CD is cut. Immediately after the cut, find the acceleration (a vector) of point H. Answer in terms of some or all of m, I^G, L, g, k and $\hat{\imath}$ & $\hat{\jmath}$.

b) Calculate I^G in terms of m and L.

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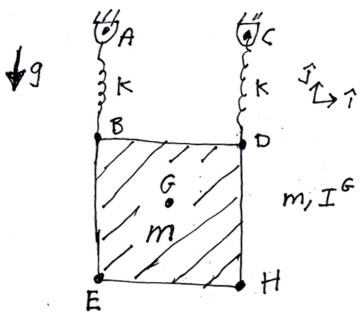
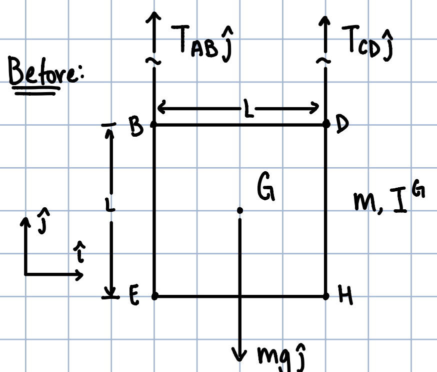


(a)

Before:



By symmetry and inspection $\Rightarrow T_{AB} = T_{CD} = \frac{mg}{2}$ *

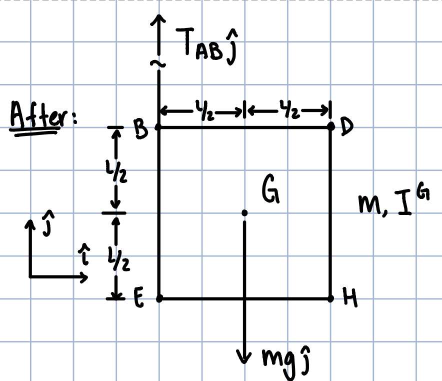
$\Rightarrow$ Or, in detail:

LMB: $\Sigma F = m\vec{a}_G$

↳ Before Cutting: $\cancel{m\vec{a}_G}^{\,0} = T_{AB}\hat{\jmath} + T_{CD}\hat{\jmath} - mg\hat{\jmath}$

$$0 = T_{AB}\hat{\jmath} + T_{CD}\hat{\jmath} - mg\hat{\jmath}$$
$$0 = T_{AB}\hat{\jmath} + T_{AB}\hat{\jmath} - mg\hat{\jmath}$$
$$0 = 2T_{AB}\hat{\jmath} - mg\hat{\jmath}$$
$$\{\}\cdot\hat{\jmath} \Rightarrow 0 = 2T_{AB} - mg$$
$$2T_{AB} = mg$$
$$T_{AB} = \frac{mg}{2} \ *$$

After:



↳ After cutting: $m\vec{a}_G = T_{AB}\hat{\jmath} - mg\hat{\jmath}$

$$m\vec{a}_G = \left[\frac{mg}{2}\right]\hat{\jmath} - mg\hat{\jmath}$$

Divide by m: $\vec{a}_G = \left[\frac{g}{2}\right]\hat{\jmath} - g\hat{\jmath}$

$$\vec{a}_G = -\frac{g}{2}\hat{\jmath} \ *$$

AMB: $\Sigma \vec{M}_{/G} = \vec{r}_{G/G} \times m\vec{a}_G + I^G\alpha\hat{k}$

$$-\frac{L}{2}\hat{\imath} \times T_{AB}\hat{\jmath} = \cancel{-0\hat{\imath} \times m\left(-\frac{g}{2}\hat{\jmath}\right)}^{\,0} + I^G\dot{\omega}\hat{k}$$

$$-\frac{L}{2}\hat{\imath} \times \left(\frac{mg}{2}\right)\hat{\jmath} = I^G\dot{\omega}\hat{k}$$

$$-\frac{mgL}{4}\hat{k} = I^G\dot{\omega}\hat{k}$$

$$\{\}\cdot\hat{k} \Rightarrow -\frac{mgL}{4} = I^G\dot{\omega}$$

$$\dot{\omega} = -\frac{mgL}{4I^G} \ *$$

$$\vec{a}_H = \vec{a}_G + \vec{a}_{H/G}$$
$$\vec{a}_H = \vec{a}_G + \alpha_H\hat{k} \times \vec{r}_{H/G} - \omega^2\vec{r}_{H/G}$$
$$\vec{a}_H = \vec{a}_G + \dot{\omega}\hat{k} \times \vec{r}_{H/G} - \cancel{(0)^2\vec{r}_{H/G}}^{\,0}$$
$$\vec{a}_H = -\frac{g}{2}\hat{\jmath} + \left(-\frac{mgL}{4I^G}\right)\hat{k} \times \left(\frac{L}{2}\hat{\imath} - \frac{L}{2}\hat{\jmath}\right)$$
$$\vec{a}_H = -\frac{g}{2}\hat{\jmath} + \left(-\frac{mgL^2}{8I^G}\hat{\jmath} - \frac{mgL^2}{8I^G}\hat{\imath}\right)$$

$$\boxed{\vec{a}_H = -\frac{mgL^2}{8I^G}\hat{\imath} + \left(-\frac{g}{2} - \frac{mgL^2}{8I^G}\right)\hat{\jmath}}$$

(b) $I^G = \int r^2 \, dm$

$I^G = \int (x^2+y^2)\rho \, dA$

$I^G = \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} (x^2+y^2)\frac{m}{L^2}\, dx\, dy$ (same integral twice)

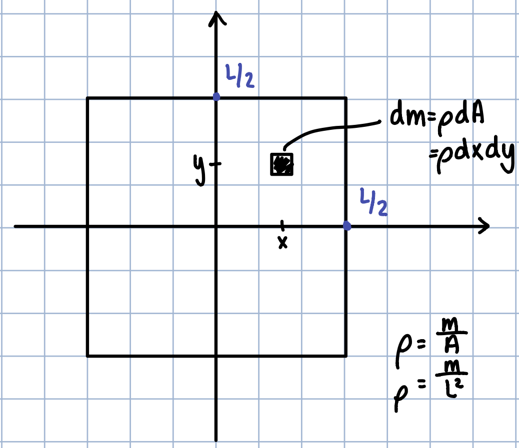
$I^G = (2)\frac{m}{L^2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} x^2 \, dx\, dy$

$I^G = \frac{2m}{L^2}\left(\frac{x^3}{3}\Big|_{-L/2}^{L/2}\right)(L)$ $\qquad \int_{-L/2}^{L/2} dy = L$

$I^G = \frac{2m}{L}\left[\frac{(L/2)^3}{3} - \frac{(-L/2)^3}{3}\right]$

$I^G = \frac{2m}{L}\left(\frac{L^3}{12}\right)$

$\boxed{I^G = \frac{mL^2}{6}}$



$\rho = \frac{m}{A}$

$\rho = \frac{m}{L^2}$

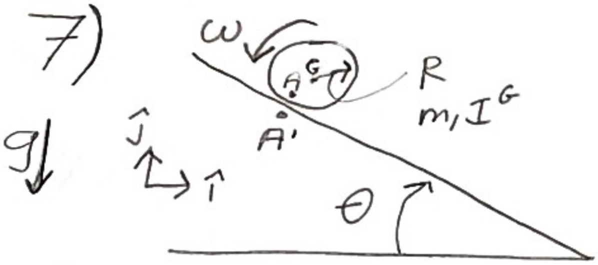
Prelim 3 "Solutions"

MAE 2030

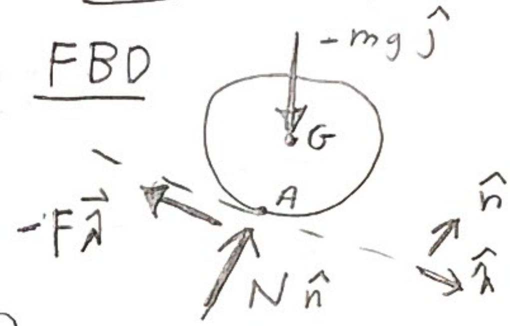
May 6, 2020

— ANDY RUINA

7)



FBD



Rolling: $\vec{V}_A = \vec{V}_{A'}$

① $\boxed{V_G + R\omega = 0}$ $(|F| \le \mu N)$

sliding to right $\Rightarrow V_G + R\omega > 0$

$\Downarrow$

$F = \mu N > 0$

LMB:

$$\sum \vec{F} = m\vec{a}_G$$

② $\boxed{-mg\hat{j} + N\hat{n} - F\hat{n} = m a_G \hat{\lambda}}$

Method:

Assume sliding; Solve ODEs; Find when V_G & ω are such that rolling constraint ① is met.

AMB$_{/G}$

$$\left\{\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}\right\}$$

$\{\}\cdot\hat{k} \Rightarrow \boxed{FR = -I^G\dot{\omega}}$ ③

Assume sliding to right

LMB ② $\Rightarrow \left\{-mg\hat{j} + N\hat{n} - \mu N\hat{\lambda} = m\dot{V}_G\hat{\lambda}\right\}$

$\{\}\cdot\hat{n} \Rightarrow N = mg\,\hat{j}\cdot\hat{n} = mg\cos\theta$

$\{\}\cdot\hat{\lambda} \Rightarrow -\cancel{m}g\,\underbrace{\hat{j}\cdot\hat{\lambda}}_{-\sin\theta} - \mu\cancel{m}g\cos\theta = \cancel{m}\dot{V}_G$

$\Rightarrow \boxed{\dot{V}_G = g(\sin\theta - \mu\cos\theta)}$ ④

unit vectors

AMB$_{/G}$

$\mu N = \mu mg\cos\theta$

③ $\Rightarrow -FR = I^G\dot{\omega} \Rightarrow$ $\dot{\omega} = \dfrac{-\mu mgR\cos\theta}{I^G}$ ⑤

Solve ODEs for $V_G(t)$, $\omega(t)$

IC's: $V_G(0) = 0$, $\omega(0) = 0$

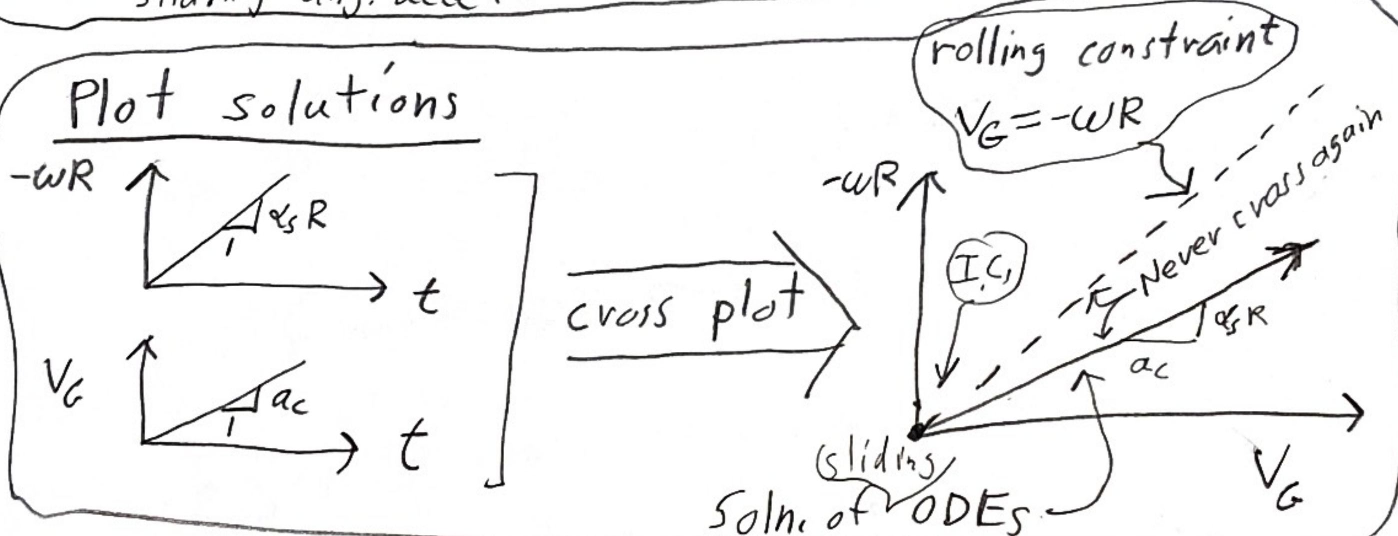
$a_c = g(\sin\theta - \mu\cos\theta) = \text{const} \Rightarrow$

$\dot{\omega} = -\alpha_s = -\mu m g \cos\theta R / I_G = \text{const} \Rightarrow$

$$\boxed{\begin{aligned} V_c &= a_c t \\ \omega &= -\alpha_s t \end{aligned}}$$

↑ sliding ang. accel

Plot solutions



Note: Soln. of ODEs <u>never</u> achieves rolling constraint $\Rightarrow$ sliding goes on $\boxed{\text{forever}}$. (a)
(∞ time, ∞ distance)

Using Equations

Rolling constraint: $V_G(t) + \omega(t) R = 0$

sliding soln $\Rightarrow$ $a_G t - \alpha t R = 0$

can't solve for t!

unless $a_G = \alpha R$ ← critical case

Crit. case: $a_G = \alpha R \Rightarrow g(\sin\theta - \mu\cos\theta = R^2 \mu m g \cos\theta / I_G$ (b)

min. μ for rolling $= \boxed{\mu = \tan\theta / \left(1 + \frac{mR^2}{I_G}\right)}$

Alt. soln. to part b

Method 2: Assume rolling, calculate N & F
check that $\mu \geq |F|/N$

$$① \Rightarrow \dot{v}_c + \dot{\omega} R = 0 \quad \text{(rolling constraint)}$$

$$AMB_{/A} \Rightarrow \sum \vec{M}_{/A} = \vec{r}_{G/A} \times m\vec{a} + I^G \dot{\omega} \hat{k}$$

$$\{ -mgR\sin\theta \hat{k} = -R\dot{v}_c m\hat{k} + I^G \underbrace{\dot{\omega}}_{-\dot{v}_c/R} \hat{k} \}$$

$$\{\} \cdot \hat{k} \Rightarrow -mgR\sin\theta = -(mR + I^G/R)\dot{v}_c$$

$$\underset{\text{(rolling)}}{\Rightarrow} \boxed{\dot{v}_c = \frac{g\sin\theta}{1 + I^G/mR^2}} \quad ⑦$$

$$LMB \cdot \hat{\lambda} \Rightarrow mg\sin\theta - F = m\dot{v}_c$$

$$\Rightarrow F = mg\sin\theta \left(1 - \frac{1}{1 + I^G/mR^2}\right) \quad ⑧$$

$$⑧ \; \& \; ② \Rightarrow \mu \geq \frac{F}{N}$$

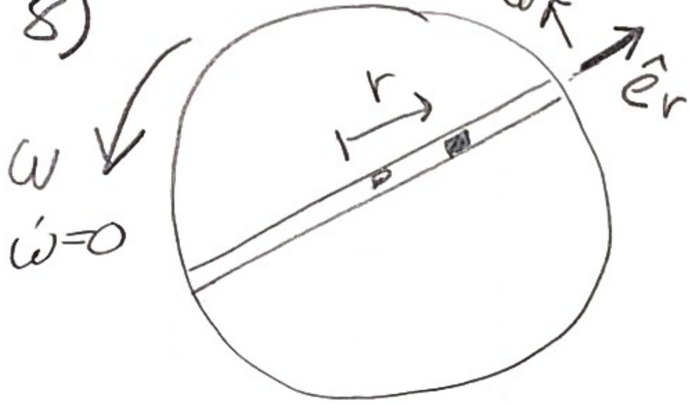
for rolling

$$\Rightarrow \mu \geq \frac{\cancel{mg}\sin\theta \left(1 - \frac{1}{1 + I^G/mR^2}\right)}{\cancel{mg}\cos\theta}$$

$$\Rightarrow \boxed{\mu \geq \frac{\sin\theta}{\cos\theta\,(1 + mR^2/I^G)} \quad (b)}$$

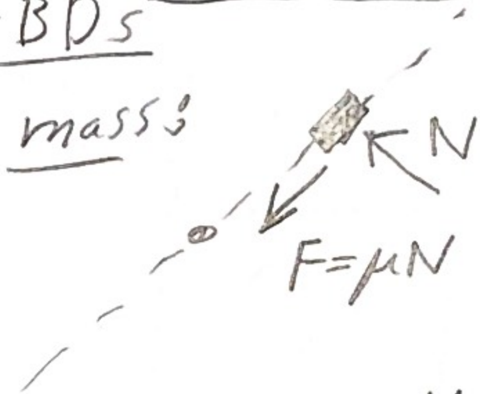
(Same as before)

8)

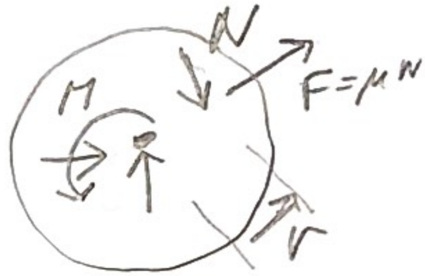


FBDs

mass:



disk:



LMB for mass:

$$\sum \vec{F} = m\vec{a}$$

$$\left\{N\hat{e}_\theta - \mu N\hat{e}_r = \left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\cancel{\ddot{\theta}}^{0} + 2\dot{r}\overset{\omega}{\dot{\theta}})\hat{e}_\theta\right]m\right\}$$

$$\{\}\cdot\hat{e}_r \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = \mu N$$

$$\{\}\cdot\hat{e}_\theta \Rightarrow \quad N = 2m\dot{r}\dot{\theta} \quad (1)$$

AMB for disk $\quad \sum \vec{M}_{/G} = I^G \cancel{\dot{\omega}}^{0} \hat{k}$

$$\{\}\cdot\hat{k} \Rightarrow \quad M - Nr = 0 \Rightarrow M = Nr \quad (2)$$

$$\text{Power} \Rightarrow P = M\omega \underset{(1)\,\&\,(2)}{=} Nr\omega = 2m\dot{r}\omega^2 r$$

$$\text{Work} = \int P\,dt = 2m\omega^2\int_{t_1}^{t_2} r\,\overbrace{\dot{r}\,dt}^{dr} = \left[\int_{R_1}^{R_2} r\,dr\right]2m\omega^2$$

$$= m r^2\Big|_{R_1}^{R_2}\omega^2 = \boxed{m\omega^2(R_2^2 - R_1^2) = \text{Work}}$$

9) Block with springs. A uniform square plate (sides L, mass m, moment of inertia I^G) is suspended by two identical springs (spring constant = k). It is in equilibrium.

a) Then, spring CD is cut. Immediately after the cut, find the acceleration (a vector) of point H. Answer in terms of some or all of m, I^G, L, g, k and $\hat{\imath}$ & $\hat{\jmath}$.

b) Calculate I^G in terms of m and L.

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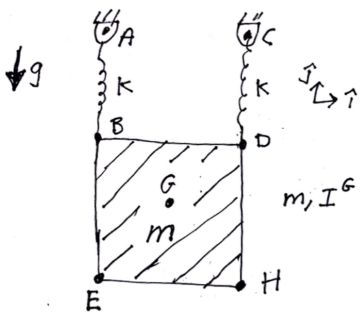
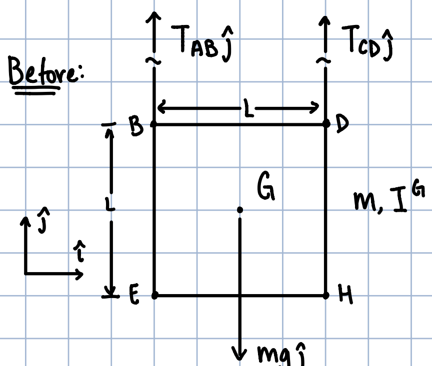


(a)

Before:



By symmetry and inspection $\Rightarrow T_{AB} = T_{CD} = \frac{mg}{2}$ *

$\Rightarrow$ Or, in detail:

LMB: $\Sigma F = m\vec{a}_G$

↳ Before Cutting: $\cancel{m\vec{a}_G}^{0} = T_{AB}\hat{\jmath} + T_{CD}\hat{\jmath} - mg\hat{\jmath}$

$0 = T_{AB}\hat{\jmath} + T_{CD}\hat{\jmath} - mg\hat{\jmath}$

$0 = T_{AB}\hat{\jmath} + T_{AB}\hat{\jmath} - mg\hat{\jmath}$

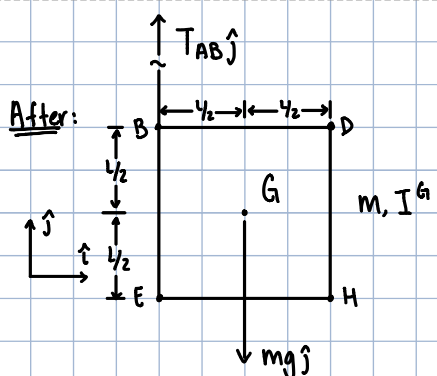
$0 = 2T_{AB}\hat{\jmath} - mg\hat{\jmath}$

$\{\}\cdot\hat{\jmath} \Rightarrow 0 = 2T_{AB} - mg$

$2T_{AB} = mg$

$T_{AB} = \frac{mg}{2}$ *

After:



↳ After cutting: $m\vec{a}_G = T_{AB}\hat{\jmath} - mg\hat{\jmath}$

$m\vec{a}_G = \left[\frac{mg}{2}\right]\hat{\jmath} - mg\hat{\jmath}$

Divide by m: $\vec{a}_G = \left[\frac{g}{2}\right]\hat{\jmath} - g\hat{\jmath}$

$\vec{a}_G = -\frac{g}{2}\hat{\jmath}$ *

AMB: $\Sigma \vec{M}_{/G} = \vec{r}_{G/G} \times m\vec{a}_G + I^G\alpha\hat{k}$

$-\frac{L}{2}\hat{\imath} \times T_{AB}\hat{\jmath} = \cancel{-0\hat{\imath} \times m(-\frac{g}{2}\hat{\jmath})}^{0} + I^G\dot{\omega}\hat{k}$

$-\frac{L}{2}\hat{\imath} \times \left(\frac{mg}{2}\right)\hat{\jmath} = I^G\dot{\omega}\hat{k}$

$-\frac{mgL}{4}\hat{k} = I^G\dot{\omega}\hat{k}$

$\{\}\cdot\hat{k} \Rightarrow -\frac{mgL}{4} = I^G\dot{\omega}$

$\dot{\omega} = -\frac{mgL}{4I^G}$ *

$\vec{a}_H = \vec{a}_G + \vec{a}_{H/G}$

$\vec{a}_H = \vec{a}_G + \alpha_H\hat{k} \times \vec{r}_{H/G} - \omega^2\vec{r}_{H/G}$

$\vec{a}_H = \vec{a}_G + \dot{\omega}\hat{k} \times \vec{r}_{H/G} - \cancel{(0)^2\vec{r}_{H/G}}^{0}$

$\vec{a}_H = -\frac{g}{2}\hat{\jmath} + \left(-\frac{mgL}{4I^G}\right)\hat{k} \times \left(\frac{L}{2}\hat{\imath} - \frac{L}{2}\hat{\jmath}\right)$

$\vec{a}_H = -\frac{g}{2}\hat{\jmath} + \left(-\frac{mgL^2}{8I^G}\hat{\jmath} - \frac{mgL^2}{8I^G}\hat{\imath}\right)$

$$\boxed{\vec{a}_H = -\frac{mgL^2}{8I^G}\hat{\imath} + \left(-\frac{g}{2} - \frac{mgL^2}{8I^G}\right)\hat{\jmath}}$$

(b) $I^G = \int r^2 dm$

$I^G = \int (x^2+y^2)\rho dA$

$I^G = \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} (x^2+y^2)\frac{m}{L^2}dxdy$ (same integral twice)

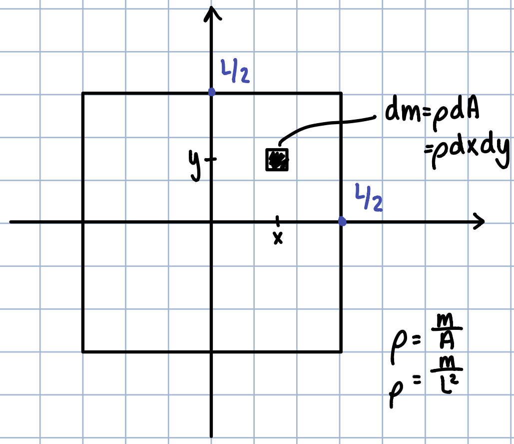
$I^G = (2)\frac{m}{L^2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} x^2 dxdy$

$I^G = \frac{2m}{L^2}\left(\frac{x^3}{3}\Big|_{-L/2}^{L/2}\right)(L)$ $\int_{-L/2}^{L/2} dy = L$

$I^G = \frac{2m}{L}\left[\frac{(L/2)^3}{3} - \frac{(-L/2)^3}{3}\right]$

$I^G = \frac{2m}{L}\left(\frac{L^3}{12}\right)$

$\boxed{I^G = \frac{mL^2}{6}}$



$\rho = \frac{m}{A}$

$\rho = \frac{m}{L^2}$