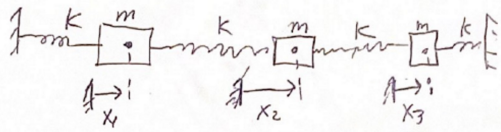
FINAL #13 Solution

13) (30 points) **Four springs, three masses.** Four equal springs k hold three equal masses m in place. The positions x_1, x_2 and x_3 are measured (positive to the right) relative to the equilibrium position.

a) Given that the system is released from rest with $x_1 = d$, $x_2 = 0$, and $x_3 = -d$, find $x_1(t)$, $x_2(t)$ and $x_3(t)$.

b) It so happens (really, it's true) that $[1, \sqrt{2}, 1]'$ is a normal mode. What is the frequency of oscillation for this mode? Answer in terms of k and m. [Note, for the motion $\sin(\lambda t)$, the frequency (sometimes called 'the angular frequency') is λ].



FBD:

$kx_1 \leftarrow \boxed{m_1} \rightarrow k(x_2 - x_1)$

$k(x_2 - x_1) \leftarrow \boxed{m_2} \rightarrow k(x_3 - x_2)$

$k(x_3 - x_2) \leftarrow \boxed{m_3} \rightarrow -kx_3$

LMBs:

m_1: $m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$

m_2: $m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = kx_1 - 2kx_2 + kx_3$

m_3: $m\ddot{x}_3 = -k(x_3 - x_2) - kx_3 = kx_2 - 2kx_3$

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}}_{\ddot{x}} = \underbrace{\begin{bmatrix} -2k & k & 0 \\ k & -2k & k \\ 0 & k & -2k \end{bmatrix}}_{-K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}$$

or $\ddot{x} = -M^{-1}Kx$

Normal Mode Requirements:

1) simple harmonic motion
2) masses exactly in/out of phase
3) oscillating at the same frequency

Normal mode!

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \cos\omega t$$

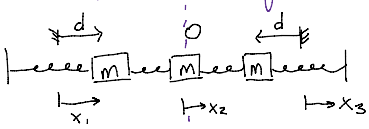
$\frac{d^2}{dt^2}$

$$\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix} = -\omega^2 \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \cos\omega t$$

a) Find $x_1(t)$, $x_2(t)$, $x_3(t)$ for given I.C.

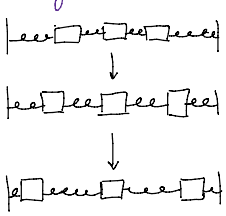
I.C.: $x_1(0) = d$, $x_2(0) = 0$, $x_3(0) = -d$

Does this correspond to a single normal mode?



Yes! I.C. results in a symmetric motion

Motion looks like this:



exactly out of phase

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ -d \end{bmatrix} \cos\omega t$$

Matches I.C.s and solves ODEs

What is ω? Find by isolating mass 1 (or 3)

$kx_1 \leftarrow \boxed{m_1} \rightarrow k(x_2 - x_1)$ $\quad m\ddot{x}_1 = -2kx_1 + k\cancel{x_2}$ always 0

b) Given $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$, find ω for this mode

$\ddot{x} = -M^{-1}Kx$ plug in normal mode equations

$$\cancel{-}\omega^2 \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \cancel{\cos\omega t} = \cancel{-}M^{-1}K \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \cancel{\cos\omega t}$$

$$\underbrace{\omega^2}_{\lambda} \underbrace{\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}}_{v} = \underbrace{M^{-1}K}_{A} \underbrace{\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}}_{v}$$

Eigenvalue problem!
Plug in A, v to find λ, or ω^2

identity!

$$M^{-1}K = \frac{1}{m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Given: $v = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

$$\lambda \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} 2-\sqrt{2} \\ 2\sqrt{2}-2 \\ 2-\sqrt{2} \end{bmatrix}$$

$$\omega^2 \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} 2-\sqrt{2} \\ 2\sqrt{2}-2 \\ 2-\sqrt{2} \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

$m\ddot{x}_1 = -2kx_1$ equivalent to simple harmonic oscillator with spring constant $2k$

$$\omega = \sqrt{\frac{2k}{m}}$$

$$\boxed{\begin{aligned} x_1 &= d\cos\sqrt{\tfrac{2k}{m}}\,t \\ x_2 &= 0 \\ x_3 &= -d\cos\sqrt{\tfrac{2k}{m}}\,t \end{aligned}}$$

Line 1: $\omega^2(1) = \frac{k}{m}(2-\sqrt{2}) \rightarrow \omega = \sqrt{\frac{k}{m}(2-\sqrt{2})}$

Line 2: $\omega^2(\sqrt{2}) = \frac{k}{m}(2\sqrt{2}-2) \rightarrow \omega = \sqrt{\frac{k}{m}(2-\sqrt{2})}$

ωs match! Normal mode confirmed

$$\boxed{\omega = \sqrt{\frac{k}{m}(2-\sqrt{2})}}$$

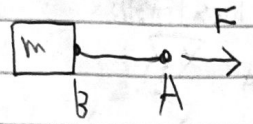
Alex Merrill sp'21

MAE 2030 Final Solutions

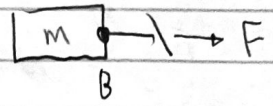
Problem 14

14. Given pulleys, walls and strings, A force F acting on point A to the right and a mass at B draw systems such that

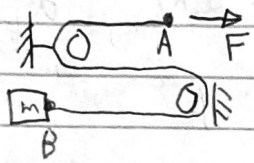
A. $a_B = F/m$ (Trivial Base Case.)



FBD of B

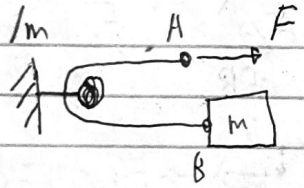


or

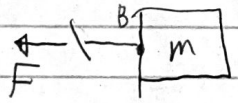


$a_B = F/m$

B. $a_B = -F/m$

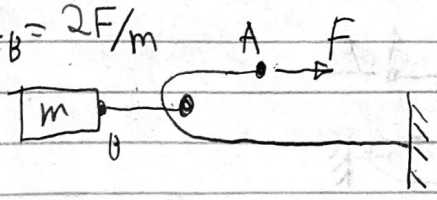


FBD of B

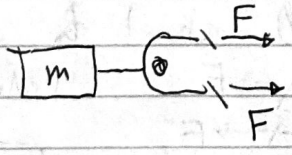


$a_B = -F/m$ ✓

C. $a_B = 2F/m$



FBD of B

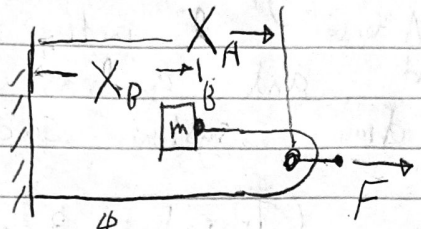


, $a_B = 2F/m$

2

D. $a_A = F/4m$

not B

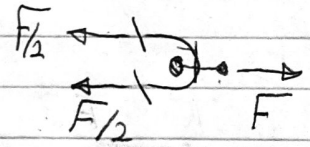


inextensible string length ℓ, $\dot{\ell} = 0$

$$\frac{d^2}{dt^2}\{\ell = 2 \cdot X_A - X_B\}$$

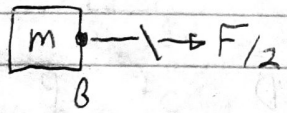
$$\ddot{X}_B = 2\ddot{X}_A, \quad a_B/2 = a_A \quad (1)$$

FBD of A



massless pulley $\Rightarrow F = 0$

FBD of B

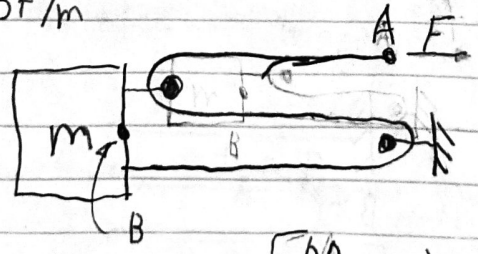


$a_B = F/2m \quad (2)$

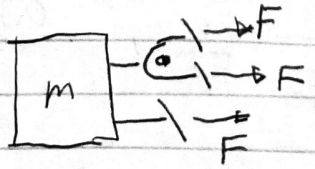
Combine 1, 2

$a_A = F/4m$ ✓

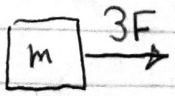
E $a_B = 3F/m$



FBD about B.



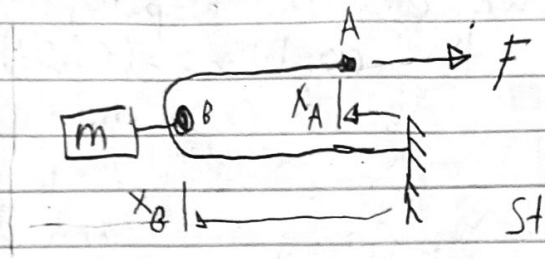
redraw



$a_B = 3F/m$ ✓

F $\quad a_A = 2F/m$ (Hard)

Try (This was a common answer)



(Same diagram as C)

String length l (ignore pully radius)
$\dot{l} = 0$

$\{l = 2x_B - x_A\}$
$d^2/dt^2 \{\}$

$\ddot{x}_A = 2\ddot{x}_B, \quad a_A = 2a_B$ ①

From C. $a_B = 2F/m$ ②
Combine 1,2 $a_a = 4F/m$:(this is NOT the desired result. Bad guess

---

Does this question have a solution?
Approach through logic not guessing

Say B is acted upon by force CF where C is the force multiplier as result of the pulley system.

$\boxed{m} \rightarrow CF$

Use Power, $P_A = P_B$ (Conservation of energy)

$P = F \cdot V \qquad P_A = V_A \cdot F, \quad P_B = V_B C F$

$(V_A \cdot F = V_B C F) \frac{d}{dt}$

$a_A \cdot F = a_B \cdot C \cdot F, \quad a_B = CF/m$
$a_A \cdot F = C^2 F^2/m \quad \boxed{a_A = C^2 F/m}$ ③

Using equation (3) and $a_A = 2F/m$

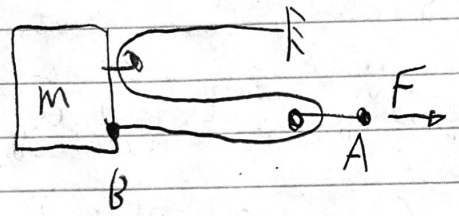
$C = \sqrt{2}$ ← irrational

Can we construct a pulley system that meets this condition ($C = \sqrt{2}$)

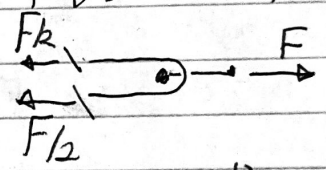
no we can only ~~but~~ add subtract multiply and divide

So lets get close

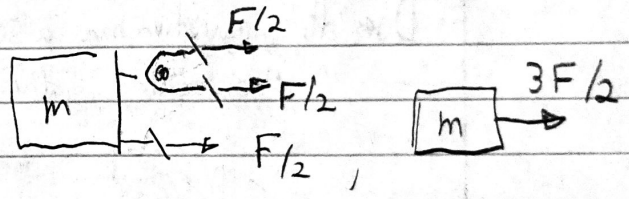
$3/2 = 1.5$ $\sqrt{2} = 1.41$ - close



FBD of A



FBD of B

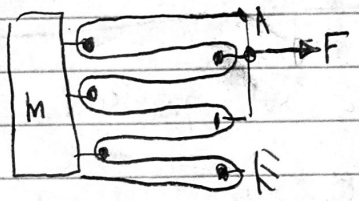


$C = 3/2$

plug in eqn (3) $a_A = 9/4\ F/m$

$a_A = 2.25\ F/m$

Or do $c = 7/5$ (1.4)



C_2^1 $\bar{D}_3^1$ Strands

C_4^2 Strands D_5^4 Force

C_6^3 block

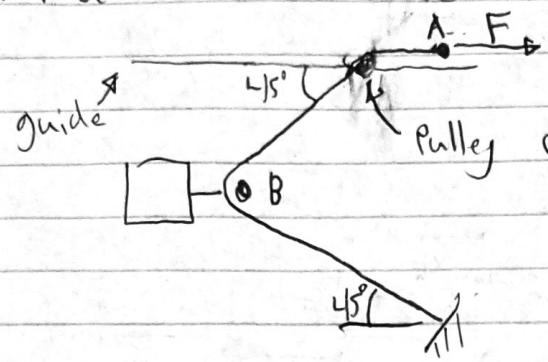
→

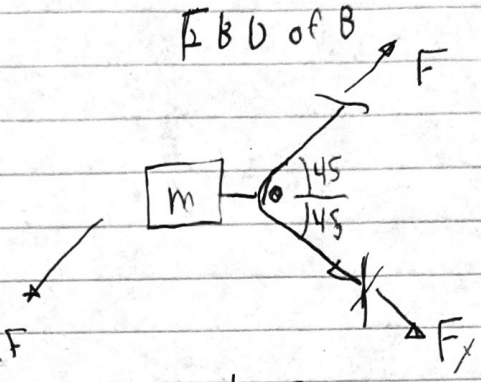
$C = 7/5$

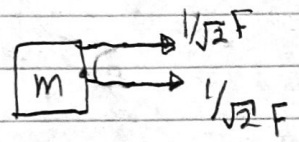
$(7/5)^2 = 1.96$

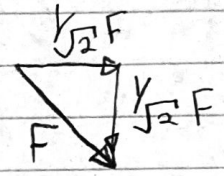
$a_A = 1.96\ F/m$

What if we did it in 2d









$2/\sqrt{2} = \sqrt{2}$

$C = \sqrt{2}$

$a_A = 2F/m$ Horray!

But this is only at this instant as geometry will change

Problem 15:

15) (30 points) **Two bars, pin in slot.** Bar AD has a pin at D a distance L from A. Bar AE, with length H, has a long slot in which the pin D can slide. At the instant shown AE is vertical, AB is horizontal, and $d^2 + h^2 = L^2$. Both $\omega = \omega_{BD}$ and $\dot{\omega} = \dot{\omega}_{BD}$ are known. Answer in terms of some or all of $d, h, L, H, \omega, \dot{\omega}, \hat{\imath}$ and $\hat{\jmath}$.

a) (25/30) Find the velocity of point E.

b) (5/30) Find the acceleration of point E (Just set up clearly, no need to solve well defined equations)

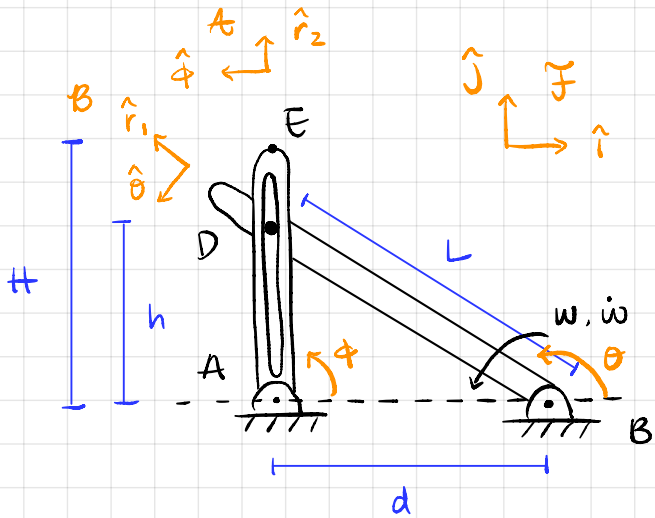
Givens:

$\omega = \omega_{BD}$

$\dot{\omega} = \dot{\omega}_{BD}$

$d^2 + h^2 = L^2$

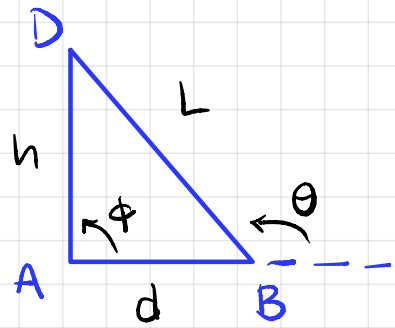
Answer: in terms of $d, h, L, H, \omega, \dot{\omega}, \hat{\imath}, \hat{\jmath}$



Frames:

script letters
- $\mathcal{F}$ - fixed frame
- $\mathcal{B}$ - body frame for bar BD
- $\mathcal{A}$ - body frame for bar AE

frames $\{ \mathcal{B} \neq B, \ \mathcal{A} \neq A \}$



$h = L\sin\theta$

$d = L\cos\theta$

a) Find: $\vec{v}_E$

$$\vec{v}_E = \cancel{\vec{v}_{E/A}}^{\,\vec{0}} + \vec{\omega}_{\mathcal{A}/\mathcal{F}} \times \vec{r}_{E/A}$$

find $\vec{\omega}_{\mathcal{A}/\mathcal{F}} = \dot{\vec{\phi}}$:

$$\vec{v}_D = \vec{v}_D$$

($\vec{v}_D$ is $\vec{v}_{D/\mathcal{F}}$: velocity of D with respect to fixed frame $\mathcal{F}$)

bar BD: bar AE:

$$\cancel{\vec{v}_{D/B}}^{\,\vec{0}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{D/B} = \vec{v}_{D/A} + \vec{\omega}_{\mathcal{A}/\mathcal{F}} \times \vec{r}_{D/A}$$

substitute: $\vec{\omega}_{\mathcal{B}/\mathcal{F}} = \omega\hat{k}$, $\vec{r}_{D/B} = L\hat{r}_1$, $\vec{\omega}_{\mathcal{A}/\mathcal{F}} = \dot{\phi}\hat{k}$, $\vec{r}_{D/A} = h\hat{r}_2$, $\vec{v}_{D/A} = \dot{h}\hat{r}_2$

$$\omega\hat{k} \times L\hat{r}_1 = \dot{h}\hat{r}_2 + \dot{\phi}\hat{k} \times h\hat{r}_2$$

substitute: $\hat{r}_1 = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ $\quad \hat{r}_2 = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$

$$\omega\hat{k} \times (L\cos\theta\,\hat{\imath} + L\sin\theta\,\hat{\jmath}) = \dot{h}(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) + \dot{\phi}\hat{k} \times h(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})$$

substitute : $h = L\sin\theta$ $\quad d = L\sin\theta$ $\quad \phi = 90^\circ$ (at instance shown)

$$\omega d\hat{j} - \omega h\hat{i} = (\dot{h}\overset{0}{\cancel{\cos\phi}}\,\hat{i} + \dot{h}\overset{1}{\cancel{\sin\phi}}\,\hat{j}) + (\dot{\phi}h\overset{0}{\cancel{\cos\phi}}\,\hat{j} - \dot{\phi}h\overset{1}{\cancel{\sin\phi}}\,\hat{i})$$

$$① \{\omega d\hat{j} - \omega h\hat{i} = \dot{h}\hat{j} - \dot{\phi}h\hat{i}\}$$

solve for $\dot{\phi}$:

$$\hat{i}\cdot\{①\} \rightarrow \qquad -h\omega = -\dot{\phi}h$$

$$\underline{\dot{\phi} = \omega}$$ wow, crazy!

bar AE : rigid object $\rightarrow$ $\vec{\omega}_{A/F} = \dot{\phi}\hat{k} = \omega\hat{k}$

$$\vec{V}_E = \vec{\omega}_{A/F} \times \vec{r}_{E/A}$$

$$= \omega\hat{k} \times H\hat{j}$$

$$\boxed{\vec{V}_E = -\omega H\hat{i}}$$

b) Find : $\vec{a}_E$

$$\vec{a}_E = \cancelto{\vec{0}}{\vec{a}_{A/\mathcal{F}}} + \cancelto{\vec{0}}{\vec{a}_{E/A}} + \dot{\vec{\omega}}_A \times \vec{r}_{E/A} - \omega_A^2 \vec{r}_{E/A} + 2\vec{\omega}_A \times \cancelto{\vec{0}}{\vec{v}_{E/A}}$$

need to find $\dot{\vec{\omega}}_A$:

$$\vec{a}_D = \vec{a}_D \qquad (\vec{a}_D \text{ is } \vec{a}_{D/\mathcal{F}})$$

$$\cancelto{0}{\vec{a}_{B/\mathcal{F}}} + \cancelto{\vec{0}}{\vec{a}_{D/B}} + \dot{\vec{\omega}}_B \times \vec{r}_{D/B} - \omega_B^2 \vec{r}_{D/B} + 2\vec{\omega}_B \times \cancelto{\vec{0}}{\vec{v}_{D/B}} = \cancelto{\vec{0}}{\vec{a}_{A/\mathcal{F}}} + \cancelto{\vec{0}}{\vec{a}_{D/A}} - \omega_A^2 \vec{r}_{D/A} + \dot{\vec{\omega}}_A \times \vec{r}_{D/A} + 2\vec{\omega}_A \times \vec{v}_{D/A}$$

substitute: $\dot{\vec{\omega}}_B = \dot{\omega}\hat{k}$ $\qquad$ $\dot{\vec{\omega}}_A = \ddot{\phi}\hat{k}$

$\omega_B = \omega$ $\qquad$ $\omega_A = \dot{\phi} = \omega$

$\vec{r}_{D/B} = L\hat{r}$ $\qquad$ $\vec{r}_{D/A} = h\hat{r}_2$

$\vec{v}_{D/A} = \dot{h}\hat{r}_2$

$$\dot{\omega}\hat{k} \times L\hat{r}_1 - \omega^2 L\hat{r}_1 = -\omega^2 h\hat{r}_2 + \ddot{\phi}\hat{k} \times h\hat{r}_2 + 2\dot{\phi}\hat{k} \times \dot{h}\hat{r}_2$$

substitute: $\hat{r}_1 = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ $\qquad$ $\hat{r}_2 = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$

$$\dot{\omega}\hat{k} \times (L\cos\theta\,\hat{\imath} + L\sin\theta\,\hat{\jmath}) - \omega^2 L(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = -\omega^2 h(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})$$
$$+ \ddot{\phi}\hat{k} \times h(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) + 2\dot{\phi}\hat{k} \times \dot{h}(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath})$$

@ the instance shown : $\phi = 90°$

and recall $h = L\sin\theta$, $d = L\cos\theta$, $\dot{\phi} = \omega$

$$\dot{\omega}L\cos\theta\,\hat{\jmath} - \dot{\omega}L\sin\theta\,\hat{\imath} - \omega^2 L(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = (-\omega^2 h)(\cancelto{0}{\cos\phi}\,\hat{\imath} + \cancelto{1}{\sin\phi}\,\hat{\jmath})$$
$$+ (\ddot{\phi}h\cancelto{0}{\cos\phi}\,\hat{\jmath} - \ddot{\phi}h\cancelto{1}{\sin\phi}\,\hat{\imath}) + (2\dot{\phi}\dot{h}\cancelto{0}{\cos\phi}\,\hat{\jmath} - 2\dot{\phi}\dot{h}\cancelto{1}{\sin\phi}\,\hat{\imath})$$

② $\{(\dot{\omega}d - \omega^2 h)\hat{\jmath} - (\dot{\omega}h + \omega^2 d)\hat{\imath} = -\omega^2 h\hat{\jmath} + (-\ddot{\phi}h - 2\omega\dot{h})\hat{\imath}\}$

we need $\dot{h}$: (use eqn ① from part a)

① $\{\omega d\,\hat{\jmath} - \omega h\,\hat{\imath} = \dot{h}\,\hat{\jmath} - \dot{\phi}h\,\hat{\imath}\}$

$$\hat{\jmath}\cdot\{①\} \rightarrow \omega d\,\hat{\jmath} = \dot{h}\,\hat{\jmath} \rightarrow \dot{h} = \omega d$$

solve for $\ddot{\phi}$:

$\hat{\imath} \cdot \{②\} \rightarrow$

$$-\dot{\omega} h - \omega^2 d = -\ddot{\phi} h - 2\omega \dot{h}$$

$$\dot{\omega} h + \omega^2 d = \ddot{\phi} h + 2\omega^2 d$$

$$\ddot{\phi} = \frac{\dot{\omega} h - \omega^2 d}{h}$$

substitute back int $\vec{v}_E$

$$\vec{a}_E = \dot{\vec{\omega}}_A \times \vec{r}_{E/A} - \omega_A^2 \vec{r}_{E/A}$$

$$\vec{a}_E = \frac{\dot{\omega} h - \omega^2 d}{h} \hat{k} \times H\hat{j} - \omega^2 \times H\hat{j}$$

$$\boxed{\vec{a}_E = -H\omega^2 \hat{j} - \frac{H}{h}(\dot{\omega} h - \omega^2 d)\hat{\imath}}$$

Alternate solution to (b) using polar coordinates $\longrightarrow$

b) Find $\vec{a}_E$

Using polar coordinates: $\vec{a}_E = (\cancel{\ddot{H}}^{0} - H\dot{\phi}^2)\,\hat{r}_2 + (H\ddot{\phi} + 2\cancel{\dot{H}}^{0}\dot{\phi})\,\hat{\phi}$

$$\vec{a}_E = -H\dot{\phi}^2\,\hat{r}_2 + H\ddot{\phi}\,\hat{\phi}$$

What is $\ddot{\phi}$?

↳ Use $\vec{a}_D = \vec{a}_D$

$$(\ddot{h} - h\dot{\phi}^2)\,\hat{r}_2 + (2\dot{h}\dot{\phi} + h\ddot{\phi})\,\hat{\phi} = (\cancel{\ddot{L}}^{0} - L\omega^2)\,\hat{r}_1 + (2\cancel{\dot{L}}^{0}\omega + L\dot{\omega})\,\hat{\theta}$$

$$\{(\ddot{h} - h\dot{\phi}^2)\,\hat{r}_2 + (2\dot{h}\dot{\phi} + h\ddot{\phi})\,\hat{\phi} = -L\omega^2\,\hat{r}_1 + L\dot{\omega}\,\hat{\theta}\}$$

$\{\}\cdot\hat{\phi}$: $2\dot{h}\dot{\phi} + h\ddot{\phi} = -L\omega^2\,\hat{r}_1\cdot\hat{\phi} + L\dot{\omega}\,\hat{\theta}\cdot\hat{\phi}$

$$2\dot{h}\dot{\phi} + h\ddot{\phi} = L\omega^2\cos\theta + L\dot{\omega}\sin\theta$$

$\hat{r}_1\cdot\hat{\phi} = (\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\cdot(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$

$= -\cos\theta\,\cancel{\sin\phi}^{1} + \sin\theta\,\cancel{\cos\phi}^{0}$

$= -\cos\theta$

$\hat{\theta}\cdot\hat{\phi} = (-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})\cdot(-\sin\phi + \cos\phi\,\hat{\jmath})$

$= \sin\theta\,\cancel{\sin\phi}^{1} + \cos\theta\,\cancel{\cos\phi}^{0}$

$= \sin\theta$

solve for $\ddot{\phi}$:

$$\ddot{\phi} = \frac{L\omega^2\cos\theta + L\dot{\omega}\sin\theta - 2\dot{h}\dot{\phi}}{h}$$

also $d = L\cos\theta$ $\quad h = L\sin\theta$ $\quad \dot{\phi} = \omega$

$$\ddot{\phi} = \frac{d\omega^2 + h\dot{\omega} - 2\dot{h}\,\omega}{h}$$

From part a) equation ①:

① $L\omega\cos\theta\,\hat{\jmath} - L\omega\sin\theta\,\hat{\imath} = \dot{h}\,\hat{\jmath} - \dot{\phi}h\,\hat{\imath}$

$①\cdot\hat{\jmath}$: $L\omega\cos\theta = \dot{h}$

$$\dot{h} = \omega d$$

Then $\ddot{\phi} = \dfrac{d\omega^2 + h\dot{\omega} - 2\omega^2 d}{h}$

$$= \frac{h\dot{\omega} - d\omega^2}{h}$$

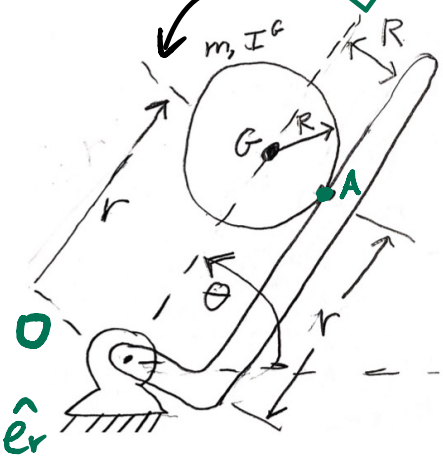
$$\vec{a}_E = -H\dot{\phi}^2\,\hat{r}_2 + H\ddot{\phi}\,\hat{\phi}$$

$$= -H\omega^2\,\hat{r}_2 + \frac{H}{h}(h\dot{\omega} - d\omega^2)\,\hat{\phi}$$

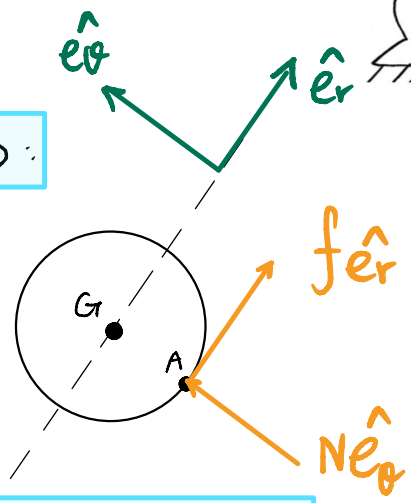
16) (30 points) **Disk rolls against rotating arm.** *Neglect gravity.* A motor powers an arm so as to rotate it at constant rate $\dot{\theta}$. A disk with radius R, mass m and moment of inertia I^G. The friction coefficient μ is big enough so the disk rolls without slip against the arm. Find $\ddot{r}$. Answer in terms of some or all of $r, \dot{r}, \theta, \dot{\theta}$ and parameters m, I^G, μ and R.

- Hint: this problem is not very hard.

$\vec{\omega} = \omega \hat{k}$ $\quad \hat{e}_\theta \quad \hat{e}_r$ $\quad \hat{k} = \hat{e}_r \times \hat{e}_\theta$



FBD:



acceleration of CoM of disk:

$$\vec{a}_G = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$\dot{\theta}$ is constant $\Rightarrow \ddot{\theta} = 0$

$$\Rightarrow \boxed{\vec{a}_G = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + 2\dot{r}\dot{\theta}\,\hat{e}_\theta}$$

AMB of disk w/ A

$$\sum \vec{M}_{/A} = \sum \dot{\vec{H}}_{/A} \Rightarrow \vec{0} = \vec{r}_{G/A} \times m\vec{a}_G + I^G \dot{\vec{\omega}} \qquad \vec{r}_{G/A} = R\hat{e}_\theta \qquad \dot{\vec{\omega}} = \dot{\omega}\hat{k}$$

$$\Rightarrow \vec{0} = (R\hat{e}_\theta) \times m[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + 2\dot{r}\dot{\theta}\hat{e}_\theta] + I^G\dot{\omega}\hat{k}$$

$$\{-Rm(\ddot{r} - r\dot{\theta}^2)\hat{k} + I^G\dot{\omega}\hat{k} = \vec{0}\} \cdot \hat{k}$$

$$\Rightarrow \boxed{-Rm(\ddot{r} - r\dot{\theta}^2) + I^G\dot{\omega} = 0}$$

$$\sim \frac{r\dot{\theta}^2}{1 + \frac{I^G}{mR^2}}$$

Rolling Constraint:

$$\text{Slipping Velocity} = \dot{r} + R\omega = 0 \Rightarrow \frac{d}{dt}(\dot{r} + R\omega) = \ddot{r} + R\dot{\omega} = 0 \Rightarrow \boxed{\dot{\omega} = -\frac{\ddot{r}}{R}}$$

CoM velocity in $\hat{e}_r$ direction (along the arm)

plug Rolling constraint back into AMB, eliminate $\dot{\omega}$

$$-Rm(\ddot{r} - r\dot{\theta}^2) + I^G\left(-\frac{\ddot{r}}{R}\right) = 0$$

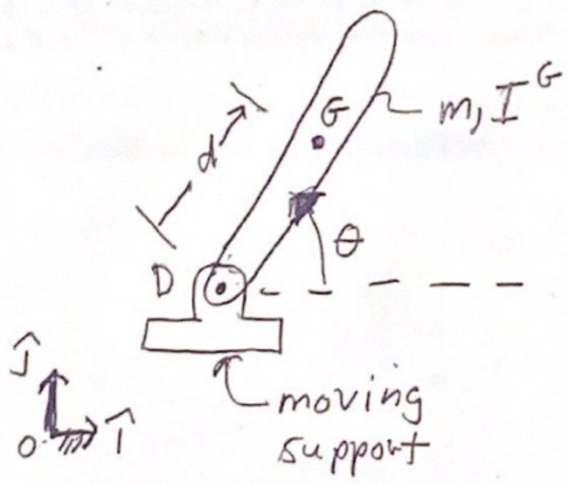
$$\Rightarrow \ddot{r} = \frac{mRr\dot{\theta}^2}{mR + \frac{I^G}{R}} = \boxed{\frac{r\dot{\theta}^2}{1 + \frac{I^G}{mR^2}}}$$

Final

Wednesday, May 19, 2021 8:17 PM

17) (30 points) **Pendulum with moving support.** *Neglect gravity*. The pendulum support point D is connected to a machine, not shown, that causes D to move with known velocity $\vec{v}_D = v_{Dx}\hat{\imath} + v_{Dy}\hat{\jmath}$ and known acceleration $\vec{a}_D = a_{Dx}\hat{\imath} + a_{Dy}\hat{\jmath}$. At the instant of interest, θ and $\dot{\theta}$ are known. Find $\ddot{\theta}$.

- Answer in terms of some or all of $v_{Dx}, v_{Dy}, a_{Dx}, \hat{\imath}, a_{Dy}, \theta, \dot{\theta}$ as well as parameters d, m and I^G.
- Full credit (30/30) if this is reduced to a clear set of equations from which it is clear that one can solve for $\ddot{\theta}$ (It is more important to make your engineering work clear than it is to do algebra that could be done by a mathematician).



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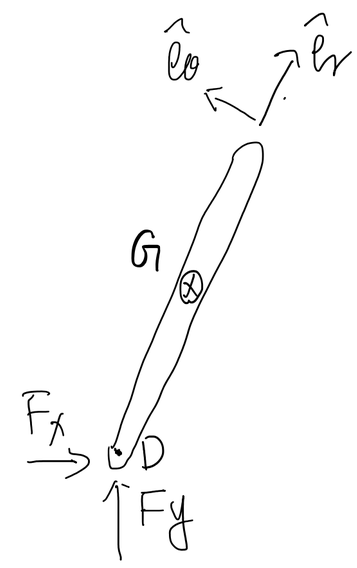

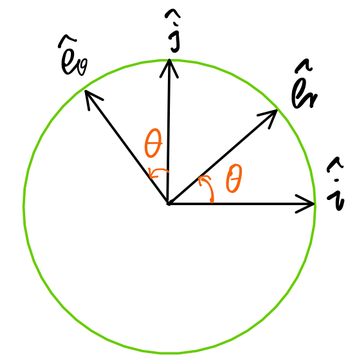

$$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

AMB about point D: $\Sigma \vec{M}_D = \vec{r}_{G/D} \times m\vec{a}_{G/\mathcal{F}} + I^G\ddot{\theta}\hat{k}$

$$\vec{0} = \vec{r}_{G/D} \times m\vec{a}_{G/\mathcal{F}} + I^G\ddot{\theta}\hat{k} \quad (*)$$

$\Rightarrow$ $\boxed{\text{Find } \vec{a}_{G/\mathcal{F}}}$ $\quad \vec{a}_{G/\mathcal{F}} = \underset{①}{\vec{a}_{D/\mathcal{F}}} + \underset{②}{\vec{a}_{G/D}}$

① is given: $\vec{a}_{D/\mathcal{F}} = a_{Dx}\hat{\imath} + a_{Dy}\hat{\jmath}$

② use polar coordinates: G goes in circle relative to D

$$\vec{r}_{G/D} = d\hat{e}_r$$

$$\vec{a}_{G/D} = -d\dot{\theta}^2\hat{e}_r + d\ddot{\theta}\hat{e}_\theta$$

$$\Rightarrow \vec{a}_{G/\mathcal{F}} = a_{Dx}\hat{\imath} + a_{Dy}\hat{\jmath} - d\dot{\theta}^2\hat{e}_r + d\ddot{\theta}\hat{e}_\theta \quad (**)$$

subsititute (**) into (*): $\vec{0} = d\hat{e}_r \times m(a_{Dx}\hat{\imath} + a_{Dy}\hat{\jmath} - d\dot{\theta}^2\hat{e}_r + d\ddot{\theta}\hat{e}_\theta) + I^G\ddot{\theta}\hat{k}$

$$\{\vec{0} = md(-a_{Dx}\sin\theta + a_{Dy}\cos\theta + d\ddot{\theta})\hat{k} + I^G\ddot{\theta}\hat{k}\}$$

$$\{\ \}\cdot\hat{k} \Rightarrow 0 = (I^G + md^2)\ddot{\theta} + md(-a_{Dx}\sin\theta + a_{Dy}\cos\theta)$$

$$\Rightarrow \ddot{\theta} = \frac{md(a_{Dx}\sin\theta - a_{Dy}\cos\theta)}{I^G + md^2} = \frac{\frac{a_{Dx}}{d}\sin\theta - \frac{a_{Dy}}{d}\cos\theta}{1 + \frac{I^G}{md^2}}$$