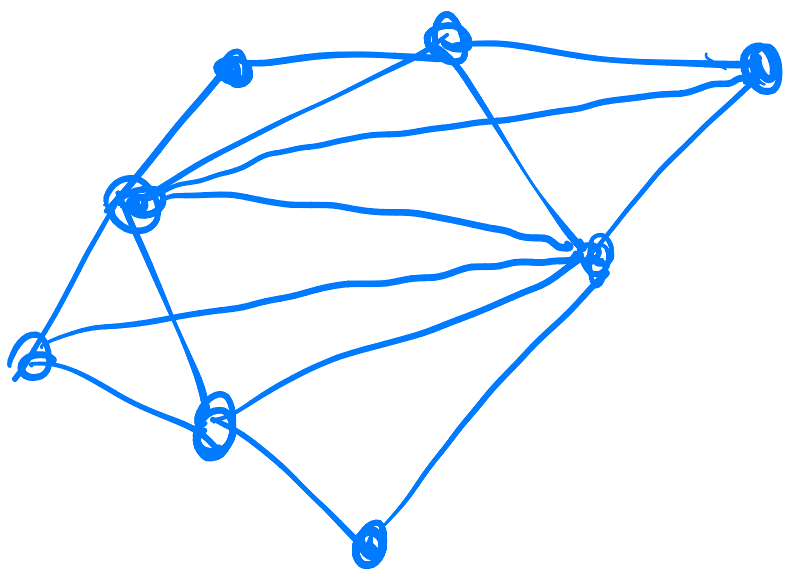
# Limits of Discrete Random Structures



Math 7710
2022-01-25
Lionel Levine

# REVERSIBLE MARKOV CHAINS

$G=(V,E)$ CONNECTED FINITE GRAPH

vertices ($V$), edges ($E$)

WEIGHTS $c: E \to \mathbb{R}_{\geq 0}$

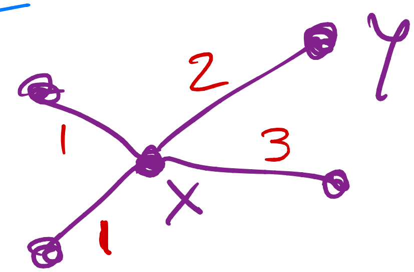
CONVENTION: $c(x,y)=0$ FOR $(x,y) \notin E$.

$c(e)$ IS THE CONDUCTANCE OF EDGE $e$

MARKOV CHAIN: STEPS FROM $x \in V$ TO $y \in V$ WITH PROB. PROPORTIONAL TO $c(x,y)$

INDEPENDENT OF THE PAST

EX: $p(x,y) = \dfrac{2}{2+3+1+1}$



FORMALLY: SEQ. OF RV's $(X_n)_{n\geq 0}$

$$X_n : \Omega \to V \quad \text{random vertex}$$

$$\Omega = \{(\omega_0, \omega_1, \omega_2, \ldots) \mid \omega_n \in V \ \forall n\} \qquad \boxed{X_n(\omega) = \omega_n}$$

$$\mathcal{F} = \sigma(\text{cylinder sets } \{\omega_0 = x_0, \ldots, \omega_n = x_n\} \mid x_0, \ldots, x_n \in V\}$$

PROB. MEASURE $\mathbb{P}_x$ ON $(\Omega, \mathcal{F})$ FOR $x \in V$.

$$\mathbb{P}_x(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n)$$

$$= \boxed{\mathbb{1}\{x_0 = x\}\, p(x_0, x_1)\, p(x_1, x_2) \cdots p(x_{n-1}, x_n)}$$

START AT x

WHERE $p(x,y) = \dfrac{c(x,y)}{C_x}$, $\quad C_x = \sum_{z \in V} c(x,z)$

Ex: $\mathbb{P}_x(X_n = y) = p^n(x,y)$ WHERE $P = (p(x,y))_{x,y \in V}$

nTH POWER OF MATRIX P.

$V \times V$ MATRIX CALLED THE TRANSITION MATRIX

ROW VECTORS

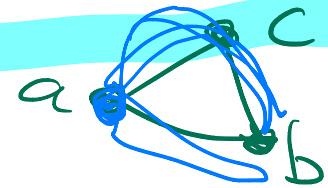
$\mu = [\quad\quad] : V \to \mathbb{R}_{\geq 0}$ MEASURES ON $V$

COL VECTORS $f = \begin{bmatrix} \\ \\ \end{bmatrix} : V \to \mathbb{R}$ FUNCTIONS ON $V$

Ex: $\underline{\mu} \underline{P}^n(x) = \mathbb{P}_{\underline{\mu}}(X_n = x)$

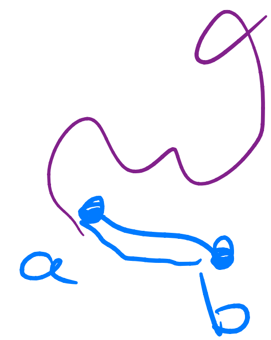
$p^n f(\underline{x}) = \mathbb{E}_x(f(X_n))$

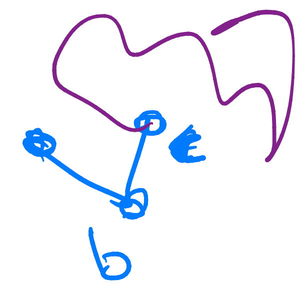
$\mathbb{P}_\mu = \sum_{z \in V} \mu(z) \mathbb{P}_z$




# EX OF CYLINDER SET

$$\{(\omega_0, \omega_1, \omega_2, \dots) \mid \omega_0 = a,\ \omega_1 = b,\ \omega_2 \in \{a, c\}\}$$



STATIONARY DIST:

| | a | b | c | d | e |
|---|---|---|---|---|---|
| $C_x$ | 1 | 2 | 2 | 2 | 1 |
| $\pi(x)$: | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |

$$\pi : V \to \mathbb{R}_{\geq 0}, \qquad \sum_{x \in V} \pi(x) = 1$$

(*) $$\pi(x) = \mathbb{P}_\pi(X_1 = x) = \sum_{y \in V} \pi(y) p(y,x)$$

$$\pi P = \pi$$

LEFT EIGENVECT OF $P$ WITH EIGENVAL 1.

DEF REVERSIBLE

$$\boxed{c(x,y) = c(y,x)}$$

GUESS $\pi(x) = \frac{C_x}{C}$ WHERE $C_x = \sum_y c(x,y)$ 1.

$$C = \sum_x C_x$$

$\Rightarrow$ CHECK (*) HOLDS!

BIRKHOFF ERGODIC THEOREM:

$$\frac{1}{n} \# \{ k \leq n \mid X_k = x \} \longrightarrow \pi(x) \quad \text{a.s. and in } L^1$$

PROPORTION OF TIME SPENT IN STATE $x$.