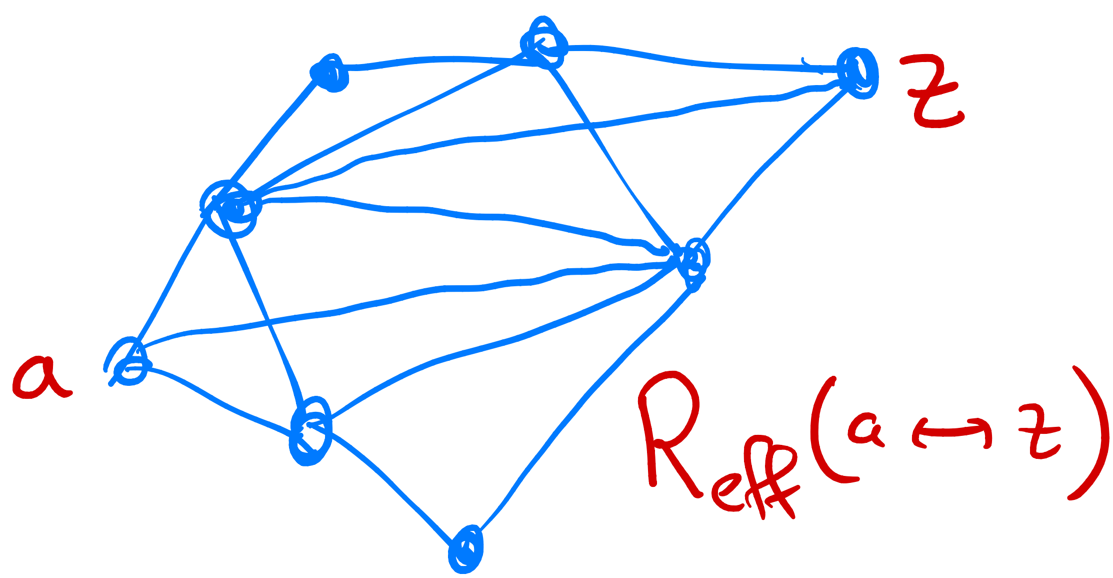
ELECTRICAL NETWORKS



MATH 7710
2022-01-27
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Reversible Markov Chains

$G=(V,E)$ connected finite graph

vertices — edges

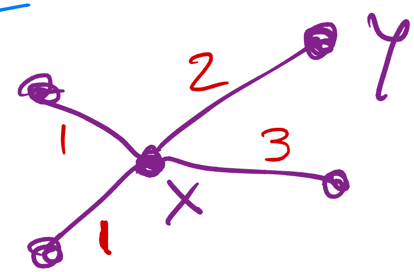
Weights $\quad c: E \to \mathbb{R}_{\geq 0}$

Convention: $c(x,y)=0$ for $(x,y)\notin E$.

$c(e)$ is the conductance of edge e

Markov chain: steps from $x \in V$ to $y \in V$ with prob. proportional to $\boxed{c(x,y)} = c(y,x)$ independent of the past

Ex: $p(x,y) = \dfrac{2}{2+3+1+1}$



STATIONARY DIST:

	a	b	c	d	e
C_x	1	2	2	2	1
$\pi(x)$:	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

$$\pi : V \to \mathbb{R}_{\geq 0}, \qquad \sum_{x \in V} \pi(x) = 1$$

(*)
$$\left.\begin{aligned} \pi(x) &= \mathbb{P}_\pi(X_1 = x) \\ &= \sum_{y \in V} \pi(y)\, p(y,x) \end{aligned}\right\} \quad \pi P = \pi$$

LEFT EIGENVECT OF P WITH EIGENVAL 1.

DEF REVERSIBLE

$$\boxed{c(x,y) = c(y,x)}$$

GUESS

$$\boxed{\pi(x) = \frac{C_x}{C}}$$

← LOCAL!

WHERE $C_x = \sum_y c(x,y)$

$C = \sum_x C_x$

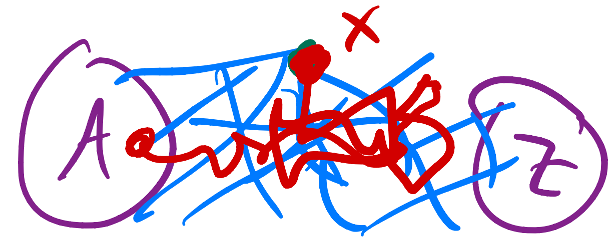
↳ CHECK (*) HOLDS!

BIRKHOFF ERGODIC THEOREM:

$$\frac{1}{n} \# \{ k \le n \mid X_k = x \} \longrightarrow \pi(x) \quad \text{a.s. and in } L^1$$

PROPORTION OF TIME SPENT IN STATE x.

HITTING PROBABILITIES



GIVEN $A, Z \subseteq V$ DISJOINT

DEFINE $F(x) := \boxed{\mathbb{P}_x(\tau_A < \tau_Z)}$

WHERE $\tau_S := \inf\{n \geq 0 \mid X_n \in S\}$

IS THE HITTING TIME OF $S \subseteq V$

$$F(x) = \begin{cases} 0 & \text{IF } x \in Z \\ 1 & \text{IF } x \in A \\ \displaystyle\sum_{y \sim x} \mathbb{P}_x(X_1 = y)\,\mathbb{P}_y(\tau_A < \tau_Z) & \text{IF } x \notin A \cup Z. \end{cases}$$

EX: PROVE THIS FORMALLY!

MEANS $(x,y) \in E$

$$\sum_{y \sim x} p(x,y) F(y)$$

$$= (PF)(x)$$

COMPARE WITH $\pi = \pi P$ FOR STAT. DIST.

DEF: A FUNCTION $h : V \to \mathbb{R}$ IS CALLED HARMONIC ON $W \subseteq V$ IF

$$h(x) = (Ph)(x) \quad \forall x \in W.$$

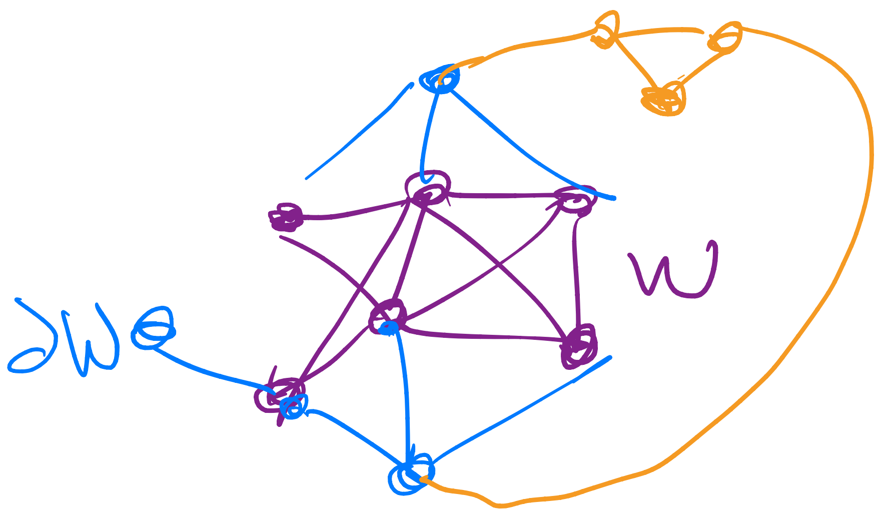
Ex: F above is harmonic on $V - (A \cup Z)$.

MAXIMUM PRINCIPLE

IF h IS HARMONIC ON W THEN

$$M := \max_{x \in W \cup \partial W} h(x) = \max_{z \in \partial W} h(z)$$

BOUNDARY $\partial W := \{ z \in V : z \notin W, \exists y \sim z, y \in W \}$



PROOF: LET $W_{max} := \{x \in W \cup \partial W \mid h(x) = M\}$

WANT TO SHOW $W_{max} \cap \partial W \neq \emptyset$

ENOUGH TO SHOW $W_{max} \cap W = \emptyset$

THEN DONE ✓

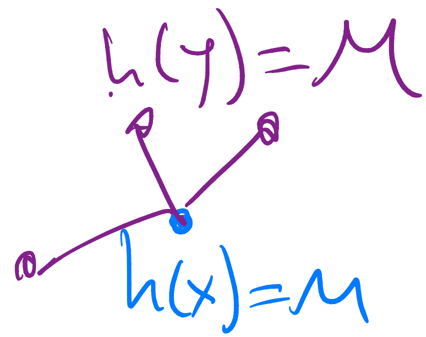
OTHERWISE THERE IS $x \in W_{max} \cap W$

$$M = h(x) = \sum_{y \sim x} p(x,y) h(y)$$

$$\leq \sum_{y \sim x} p(x,y) M = M$$

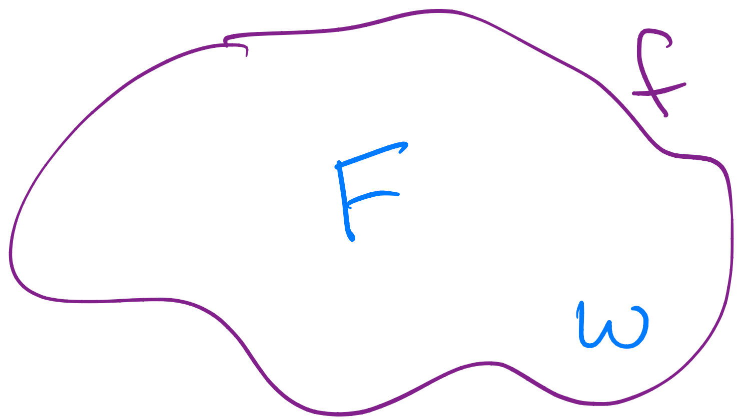
EQUALITY.

HENCE $h(y) = M$ FOR ALL $y \sim x$



SINCE G IS CONNECTED, $\exists$ PATH FROM $x \rightsquigarrow \partial W$. $\blacksquare$

DIRICHLET PROBLEM: GIVEN $W \subseteq V$ AND $f : V \setminus W \to \mathbb{R}$ THERE IS A UNIQUE $F : V \to \mathbb{R}$ SUCH THAT F IS HARMONIC ON W & $F = f$ ON $V - W$.



PROOF: UNIQUENESS: IF F_1, F_2 ARE BOTH SOLUTIONS,

THEN $F_1 - F_2 = f - f = 0$ ON $\boxed{\partial W}$

BY MAX PRINCIPLE, $F_1 - F_2 \leq 0$ ON W.

& $F_2 - F_1 \leq 0$ ON W

HENCE $F_1 = F_2$.

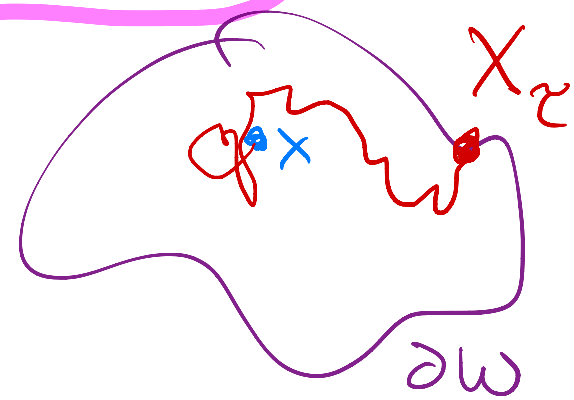
EXISTENCE: $\# W$ UNKNOWNS $F(x)$, $x \in W$

$\# W$ LINEAR EQ. $F(x) = PF(x)$

UNIQUENESS $\Rightarrow$ SOLUTION EXISTS!

EXPLICIT SOLUTION : $\boxed{F(x) = \mathbb{E}_x f(X_\tau)}$

WHERE $\tau = \tau_{\partial W}$



EX: PROVE THIS!

NOTE $\mathbb{E}_x f(X_\tau) = \sum_{z \in \partial W} \underbrace{\mathbb{P}_x(X_\tau = z)}_{\text{WEIGHTS SUMMING TO } 1} f(z)$

WEIGHTED AVERAGE OF VALUES $f(z)$.

THIS IS A (GLOBAL) "MEAN VALUE PROPERTY"

NOTE : BY DEF OF HARMONIC, $F(x) = \sum_{y \sim x} P_{xy} F(y)$

LOCAL M.V.P.

HARMONIC FUNCTIONS IN $\mathbb{R}^2$

GIVEN $W \subseteq \mathbb{R}^2$ BDD OPEN

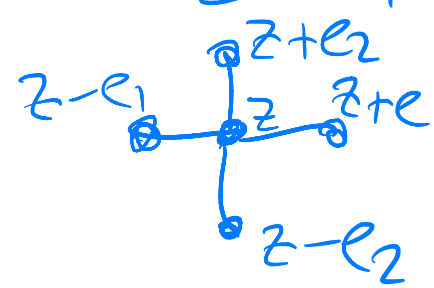
$F \in C^2(\bar{W})$ IS HARMONIC ON W

IF $\boxed{\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2} = 0.}$

HOW RELATED TO DISCRETE HARMONIC FUNCTIONS?

$F : \mathbb{Z}^2 \to \mathbb{R}$ HARMONIC AT $z \in \mathbb{Z}^2$ IF

$$F(z) = \frac{1}{4} \sum_{w \sim z} F(w)$$



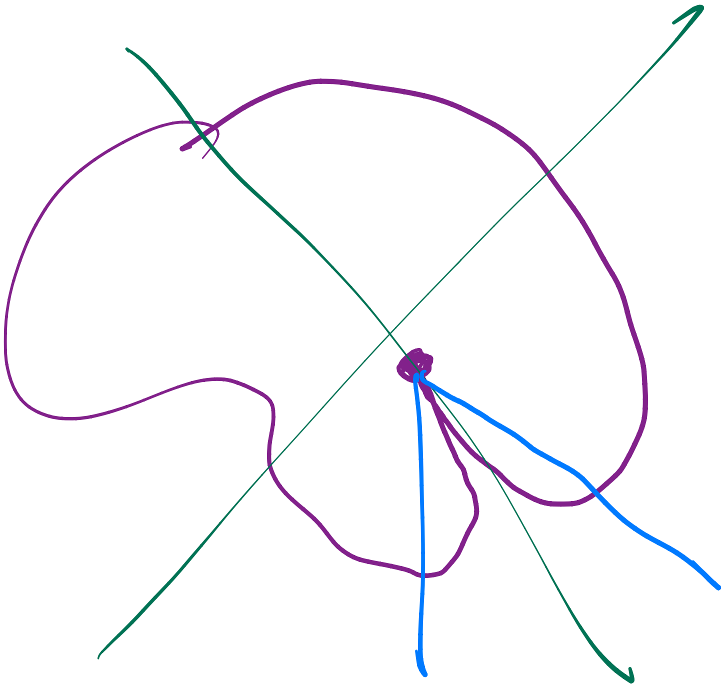
$$\iff 4F(z) = F(z+e_1) + F(z-e_1) + F(z+e_2) + F(z-e_2)$$

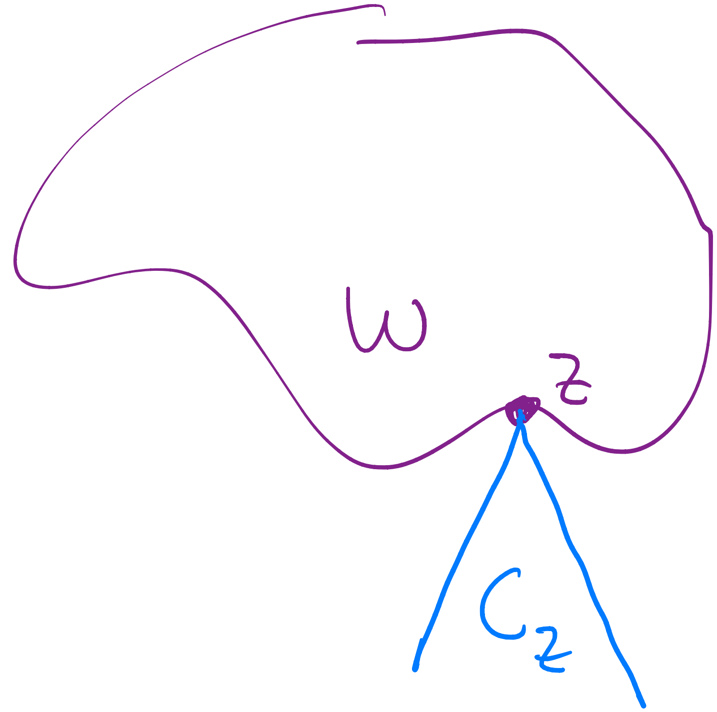
$$\iff F(z+e_1) - 2F(z) + F(z-e_1) + F(z+e_2) - 2F(z) + F(z-e_2) = 0$$

$$\iff \boxed{D_x^2 F + D_y^2 F = 0}$$

DIRICHLET PROBLEM IN $\mathbb{R}^2$: SUPPOSE ∂W IS "NOT TOO BAD", IN THE SENSE THAT $\forall z \in \partial W$ $\exists$ CONE C_z APEX AT z

SUCH THAT $C_z \subseteq \mathbb{R}^2 \setminus W$





THEN GIVEN CONTINUOUS $f : \partial W \to \mathbb{R}$

(DEF: $\partial W = \overline{W} \setminus W$)

$\exists ! \, F : \overline{W} \to \mathbb{R}$

$F = f$ on ∂W

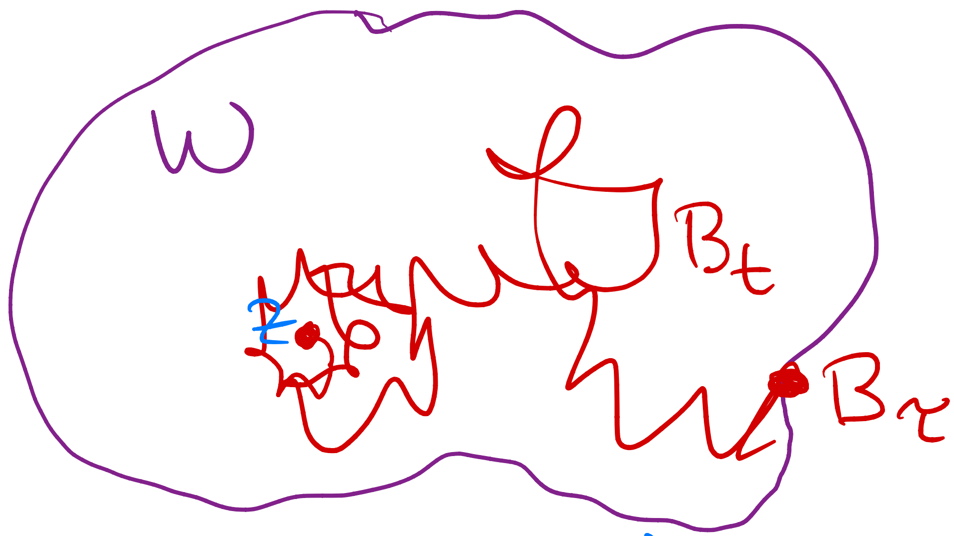
F harmonic on W.

PLANAR

BROWNIAN MOTION $(B_t)_{t \in [0,\infty)}$

$B_t = (B_t^1, B_t^2)$

INDEP STD B.M.

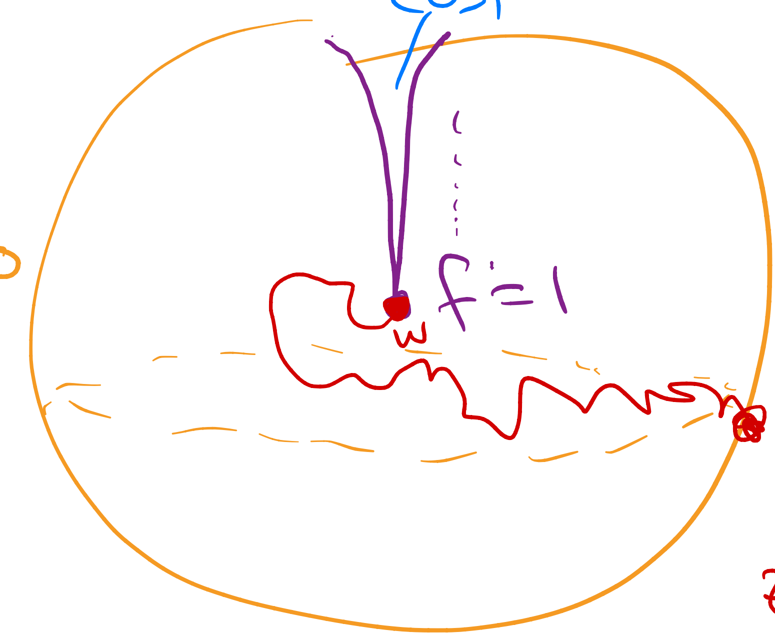


SOLUTION TO THE DIRICHLET PROBLEM

$$F(z) = E_z(f(B_\tau))$$

$$\tau = \tau_{\partial W} = \tau_{W^c} = \inf\{t : B_t \in \partial W\}$$

EX OF WHEN THIS CAN FAIL: (IN $\mathbb{R}^3$)



w = SPHERE − CUSP

$$\lim_{z \to w} \mathbb{E}_z f(B_\tau) < 1$$

MEAN VALUE PROPERTY OF HARMONIC $h : \mathbb{R}^d \to \mathbb{R}$

IF h IS HARMONIC ON A BALL $B(0,r) \subseteq \mathbb{R}^d$

THEN

$$(*) \quad \boxed{h(0) = \frac{1}{\operatorname{VOL}(B(0,r))} \int_{B(0,r)} h(x)\,dx}$$

LEB. MEAS OF BALL.

LEBESGUE MEAS. ($\mathbb{R}^d$)

DEF: h IS HARMONIC IF (#) HOLDS $\forall\, B(y,r) \subseteq W$ ON W

THM

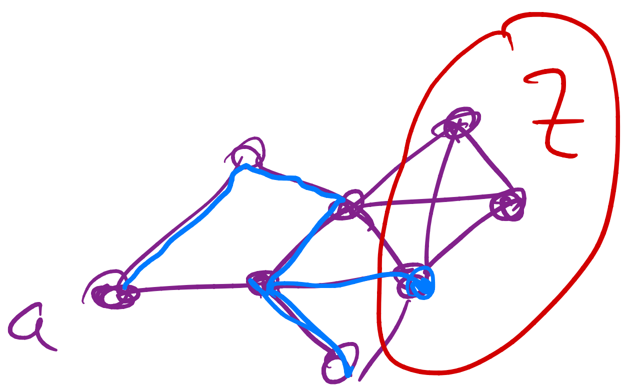
IF $h \in C^2(\bar{W})$ THEN THE TWO DEFS. OF HARMONIC ARE EQUIVALENT!

REV. MARKOV CHAIN ON $G=(V,E)$

$a \in V$, $Z \subseteq V$

DEF $\mathbb{P}(a \to Z) = \mathbb{P}_a(\tau_Z < \boxed{\tau_a^+})$ "ESCAPE PROBABILITY"

$\tau_a^+ = \inf\{n \geq 1 : X_n \in a\}$



ELECTRICAL NETWORK:

EACH EDGE $e \in E$ IS A WIRE
OF CONDUCTANCE $c(e)$
(RESISTANCE $1/c(e)$)

FOR $a, z \in V$ CONNECT TO BATTERY
WITH VOLTAGE $v(a) = 1$, $v(z) = 0$

THIS INDUCES VOLTAGES $v(x)$ FOR EACH $x \in V$.

⋮

WE'LL SHOW $v(x) = \mathbb{P}_x(\tau_a < \tau_z)$

Q: FIND A GRAPH $G = (V, E)$ AND $e \in E$
$a, z \in V$

SUCH THAT e NOT INCIDENT TO a

b

$$P_G(a \to z) < P_{G-e}(a \to z)$$

OR PROVE NO SUCH $G_{e, a, z}$ EXIST!