ENERGY, FLOWS, & SPANNING TREES

MATH 7710
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$\boxed{\text{Ex}}$ $\pentagon = \ker(\operatorname{div} : \ell^2_-(\vec{E}) \to \ell^2(V))$

$\bigstar = \operatorname{im}(c\nabla : \ell^2(V) \to \ell^2_-(\vec{E}))$

DEF: A UNIT FLOW FROM a TO z IS ($a, z \in V$)
AN ELEMENT $\theta \in \ell^2_-(E)$ SUCH THAT

$$\operatorname{div} \theta = \delta_a - \delta_z = \begin{cases} 1 & \text{AT } a \\ 0 & \text{AT } x \neq a, z \\ -1 & \text{AT } z \end{cases}$$

EXAMPLE UNIT CURRENT FLOW i

$\boxed{\text{Ex}}$ $\mathcal{R}(a \leftrightarrow z) = v(a) - v(z)$ WHEN A UNIT CURRENT FLOWS FROM $a \to z$.

DEF: ENERGY $\mathcal{E}(\theta) = \|\theta\|_r^2 = \frac{1}{2} \sum_{e \in \vec{E}} \theta(e)^2 r(e)$

THEOREM: $\mathcal{E}(\overset{\text{UNIT CURRENT FLOW}}{i}) \leq \mathcal{E}(\theta)$ FOR ALL UNIT FLOWS $a \to z$. EQUALITY IF & ONLY IF $\theta = i$

PROOF: $\ell^2_-(\vec{E}) = \bigstar \oplus \pentagon$

WRITE $\theta = i + \underbrace{(\theta - i)}$

BY KIRCHOFF CYCLE LAW $\rightarrow$ $i \in \pentagon^{\perp}$ $\qquad$ $(\theta - i) \in \pentagon$ $\leftarrow$ SINCE $\operatorname{div}(\theta - i) = 0$

$$\mathcal{E}(\theta) = \|\theta\|_r^2 = \|i\|_r^2 + \|\theta - i\|_r^2$$
$$\geq \|i\|_r^2 = \mathcal{E}(i) \qquad \square$$

COR: $R(a \leftrightarrow z) = \mathcal{E}(i) = \min\left\{ \mathcal{E}(\theta) \;\middle|\; \theta \text{ IS A UNIT FLOW } a \to z \right\}$

PROOF: $\mathcal{E}(i) = (i, i)_r = (ri, i)$
$= (\nabla v, i)$
$= (v, \operatorname{div} i)$
$= v(a) - v(z)$
$= R(a \leftrightarrow z)$. $\square$

$\boxed{\text{Ex}}$ (DUAL TO THE COR)

WHAT FUNCTION $\boxed{h}$ MINIMIZES

$$\sum_{(x,y)\in\vec{E}} c(x,y)\big(h(x)-h(y)\big)^2$$ AMONG $\{h: V\to\mathbb{R} \mid h(a)=1,\ h(z)=0\}$?

RAYLEIGH MONOTONICITY: IF $r(e)\le r'(e)\ \forall e$ THEN $R_r(a\leftrightarrow z)\le R_{r'}(a\leftrightarrow z)$.

PROOF: $R_r(a\leftrightarrow z)=\mathcal{E}_r(i_r)\overset{\text{BY THEOREM}}{\le}\mathcal{E}_r(i_{r'})$

$\overset{\text{BY FORMULA FOR } \mathcal{E}}{\le}\mathcal{E}_{r'}(i_{r'})$

$=R_{r'}(a\leftrightarrow z)$. ∎

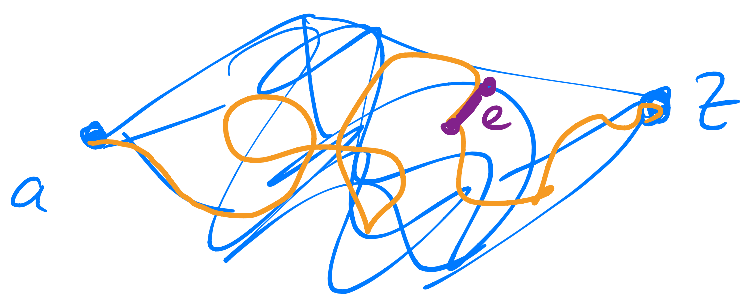
COR: ① DELETING AN EDGE (TAKING $r(e)\uparrow\infty$) WEAKLY INCREASES $R_r(a\leftrightarrow z)$

② CONTRACTING AN EDGE (TAKING $r(e)\downarrow 0$) WEAKLY DECREASES $R_r(a\leftrightarrow z)$.

$\boxed{\text{EX}}$: FIND AN EXAMPLE WHERE ① (OR ②) DOESN'T CHANGE $R(a \leftrightarrow z)$.

ESCAPE PROBABILITIES: $R_r(a \leftrightarrow z) = \dfrac{1}{C_a \mathbb{P}_r(a \to z)}$

$\Rightarrow$ $\mathbb{P}_r(a \to z)$ WEAKLY DECREASES (INCREASES) WHEN WE DELETE (CONTRACT) ANY EDGE NOT INCIDENT TO a.



$\boxed{\text{EX}}$: SHOW THIS IS FALSE FOR EDGES INCIDENT TO a.

UNIFORM SPANNING TREES

Lyons-Peres
Ch. 4

$G = (V, E)$ connected finite graph

DEF A spanning subgraph of G is a graph of the form (V, A) for $A \subseteq E$.

all the vertices

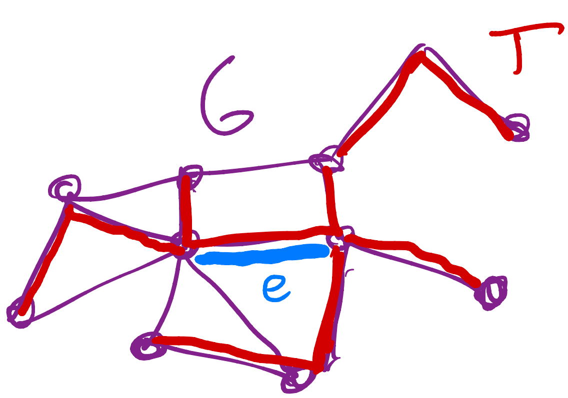
A spanning tree of G is a spanning subgraph T satisfying

(1) T is connected

(2) T has no cycles

(3) $\# E(T) = \# V - 1$

$\boxed{\text{Ex}}$ Any two of these implies the third!



THEOREM (KIRCHOFF 1847 (!))

$\mathbb{P}(U=T)=\frac{1}{N}$ $\forall$ SP. TREES T OF G

IF U IS A UNIFORM RANDOM SPANNING TREE OF G, AND $e \in E$ THEN

$$(*) \quad \mathbb{P}(e \in E(U)) = i^{\vec{e}}(\vec{e}) = \frac{R(e^- \leftrightarrow e^+)}{r(e)}$$

THE UNIT CURRENT FLOW FROM e^- TO e^+

(HERE WE TAKE $r(e)=1$ $\forall e$)

[MORE GENERALLY, $(*)$ STILL HOLDS IF WE REPLACE "UNIFORM" BY

$$\mathbb{P}(U=T) = \frac{\prod_{e \in E(T)} c(e)}{\sum_{\text{sp. trees } T'} \prod_{e \in E(T')} c(e)}$$
]

PROOF: WE'LL SHOW

CALL THIS Θ.

$$i^e(f) = \frac{1}{N} \sum_{T \in \mathcal{T}} \Theta_T(f)$$ WHERE

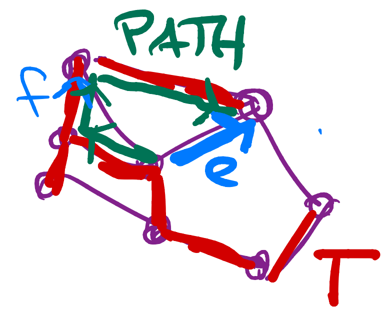
N: $\#\mathcal{T}$

$\mathcal{T}$: SET OF SPANNING TREES

$$\Theta_T = \chi_{(x_1, x_2)} + \cdots + \chi_{(x_n, x_{n+1})}$$ WHERE

$e^- = x_1 \sim \cdots \sim x_{n+1} = e^+$ IS THE PATH IN T

FROM e^- TO e^+



① EACH Θ_T IS A UNIT FLOW e^- TO e^+

SO Θ IS TOO.

$$\Rightarrow \quad P_{\star} \Theta = i^e$$

ORTHOG. PROJECTION $\ell^2_-(\vec{E}) \to \star$

IT REMAINS TO SHOW $\boxed{P_{\diamondsuit} \Theta = 0.}$

GIVEN A TWO-COMPONENT SPANNING FOREST

$$F = F_- \cup F_+$$

CALL F A CUT IF $e_- \in F_-$, $e_+ \in F_+$.

$T \setminus \{(x,y)\}$ IS A CUT

$\Longleftrightarrow$

(x,y) IS ON THE PATH IN T FROM e_- TO e_+

SO $$\sum_{\text{CUTS } F} \theta_F = \sum_{\text{TREES } T} \theta_T.$$

ANY CYCLE CROSSES THE CUT SAME # TIMES F_- TO F_+ AS F_+ TO F_-

$\Longrightarrow$ $\theta_F \perp$ ⬠. $\Longrightarrow$ $\theta \perp$ ⬠

$\Longrightarrow$ $P_{⬠}\,\theta = 0$. $\square$