Electrical Networks

OHM'S LAW: $\nabla v = i r$

NODE LAW: $(\operatorname{div} i)(x) = 0 \quad \forall x \neq a, z.$

MATH 7710

2022-02-01

L. LEVINE

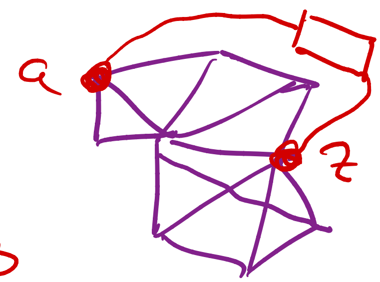
$G = (V, E)$

$c : E \to \mathbb{R}_{\geq 0}$

$c(e)$ CONDUCTANCE OF edge $e = (x,y)$

$c(x,y) = c(y,x)$

$r(e) = 1/c(e)$ RESISTANCE



VOLTAGE $v : V \to \mathbb{R}$

BATTERY $v(a) = 1, \quad v(z) = 0$

CURRENT $i : \vec{E} \to \mathbb{R}$

$i(x,y)$ = current flowing across edge (x,y)

$$i(x,y) = -i(y,x)$$

THE UNKNOWNS $(v(x))_{x \in V - \{a,z\}}$

$(i(x,y))_{(x,y) \in \vec{E}}$

ARE DERMINED BY

(1) OHM'S LAW: $v(x)-v(y) = i(x,y)\,r(x,y)$

$\forall (x,y) \in \vec{E}$

(2) KIRCHOFF'S NODE LAW:

$$\sum_{y\sim x} i(x,y) = 0 \qquad \forall x \neq a,z.$$

$$0 = \sum_{y\sim x} \big(v(x)-v(y)\big)\,c(x,y) = C_x v(x) - \sum_{y\sim x} c(x,y)\,v(y)$$

$$v(x) = \sum_{y\sim x} \boxed{\frac{c(x,y)}{C_x}}^{\,p(x,y)} v(y) \qquad \forall x \neq a,z$$

IN OTHER WORDS, v IS HARMONIC ON $V - \{a,z\}$ FOR THE MARKOV TRANSITION MATRIX $p(x,y)$

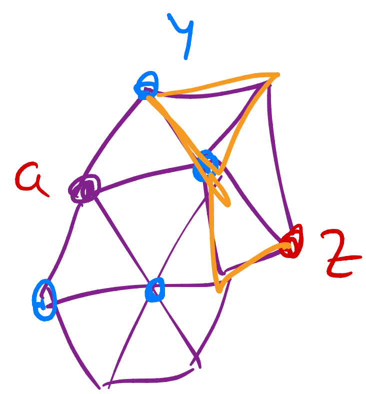
HENCE $\boxed{v(x) = \mathbb{P}_x(\tau_a < \tau_z)}$

(BOTH SIDES ARE 1 AT a, 0 AT z, HARMONIC AT $x \neq a,z$.)

DEF: THE EFFECTIVE CONDUCTANCE $C(a \leftrightarrow z)$ IS THE TOTAL CURRENT FLOWING $a \rightsquigarrow z$, NAMELY

$$\sum_{y \sim a} i(a,y) = \sum \left(v(a) - v(y)\right) c(a,y)$$

$$= \sum_{y \sim x} \left(1 - \mathbb{P}_y(\tau_a < \tau_z)\right) c(a,y)$$



$$= \sum_{y \sim x} \mathbb{P}_y(\tau_z < \tau_a)\, c(a,y)$$

$$= C_a \sum_{y \sim x} \mathbb{P}_y(\tau_z < \tau_a)\, p(a,y)$$

$$= C_a\, \mathbb{P}_a\left(\tau_z < \boxed{\tau_a^+}\right)$$

ESCAPE PROB.

DEF: $\inf\{n \geq 1 : X_n = a\}$

DEF: EFFECTIVE RESISTANCE

$$R(a \leftrightarrow z) = 1/C(a \leftrightarrow z).$$

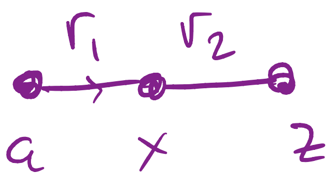
EX:

$$= \frac{1}{C\pi(a)} \mathbb{E}_a\{0 \leq n < \tau_z \mid X_n = a\}$$

STAT. DIST.

EXPECTED # RETURNS TO a BEFORE HITTING z.

RESISTORS IN SERIES



$1 = v(a)$ $\quad v(x)$ $\quad v(z) = 0$

$$v(a) - v(x) = i(a,x)\, r_1$$
$$v(x) - v(z) = i(x,z)\, r_2$$

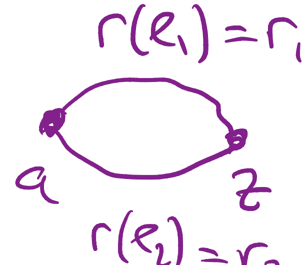
OHM'S LAW

NODE LAW: $i(a,x) = i(x,z)$

$$\Rightarrow 1 = v(a) - v(z) = i(a,x)\boxed{(r_1 + r_2)}$$

$$R(a \leftrightarrow z) = r_1 + r_2$$

RESISTORS IN PARALLEL



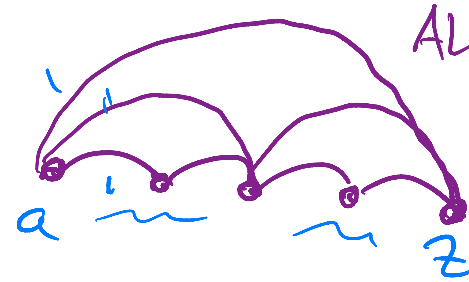
TOTAL CURRENT FLOW

$$i(e_1) + i(e_2) = (v(a) - v(z))\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$R(a \leftrightarrow z) = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2}}$$

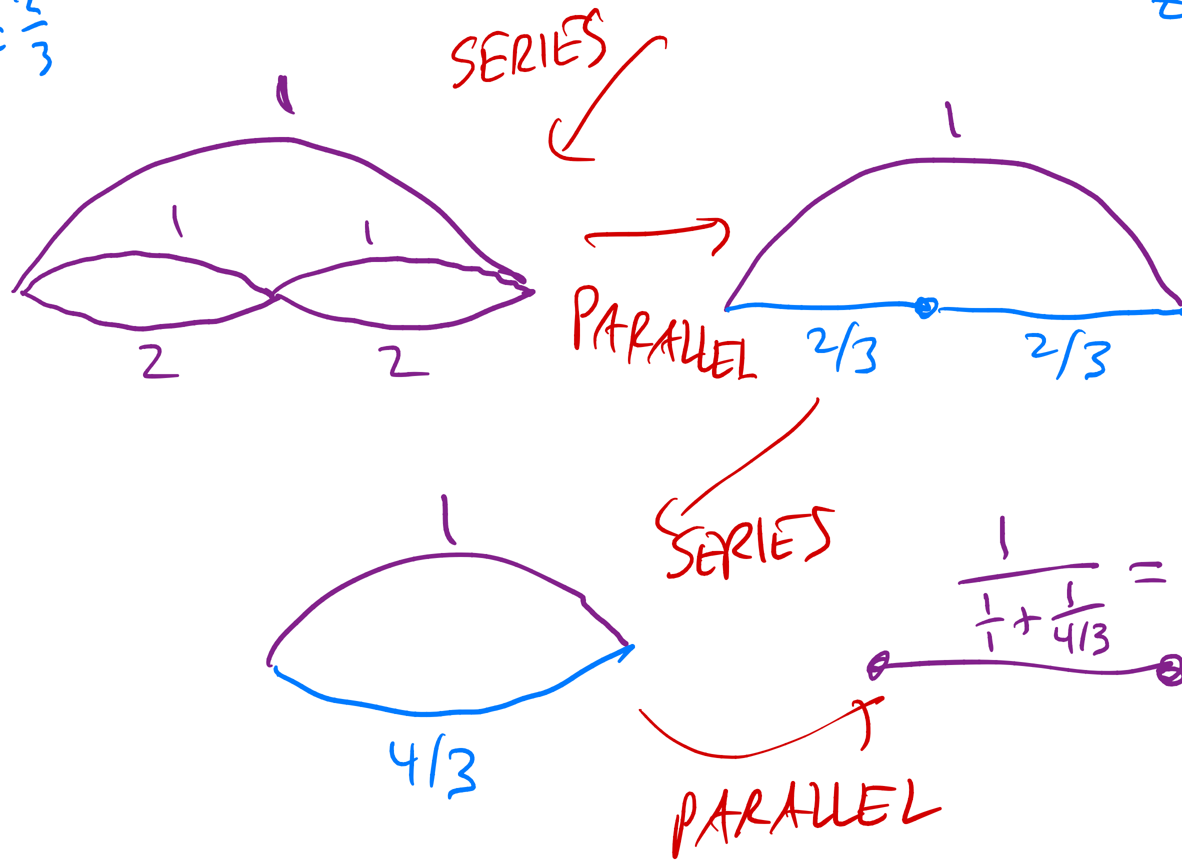
$C_a = 1 + 1 + 1 = 3$

ALL $r(e) = 1$



REDUCING NETWORKS

$$\frac{1}{\frac{1}{1} + \frac{1}{2}} = \frac{2}{3}$$



$$\frac{1}{\frac{1}{1} + \frac{1}{4/3}} = \frac{4}{7}$$

$$\frac{4}{7} = R(a \leftrightarrow z)$$

ESCAPE PROB $\quad \mathbb{P}(a \to z) := \mathbb{P}_a(\tau_z < \tau_a^+)$

$$= \frac{1}{C_a} \cdot C(a \leftrightarrow z)$$

$$= \frac{1}{3} \cdot \frac{7}{4} = \boxed{\frac{7}{12}}$$

FLOWS & ENERGY

$$\ell^2(V) = \{ f : V \to \mathbb{R} \} \qquad (f,g) = \sum_{x \in V} f(x) g(x)$$

$$\ell^2_-(\vec{E}) = \{ \theta : \vec{E} \to \mathbb{R} \mid \theta(e) = -\theta(-e) \ \forall e \in \vec{E} \}$$

SPACE OF "FLOWS"

$$(\theta_1, \theta_2) = \frac{1}{2} \sum_{e \in \vec{E}} \theta_1(e) \theta_2(e)$$

NOTATION
$E_{1/2}$ = A SET OF REPRESENTATIVES FOR THE EQUIV. CLASSES $\{e, -e\}$.

$$= \sum_{e \in E_{1/2}} \theta_1(e) \theta_2(e).$$

$$\begin{tikzcd} \ell^2(V) \arrow[r, bend left, "\nabla"] & \ell^2_-(\vec{E}) \arrow[l, bend left, "\mathrm{div}"] \end{tikzcd}$$

$$(\nabla f)(x,y) = f(x) - f(y)$$

$$(\mathrm{div}\, \theta)(x) = \sum_{y \sim x} \theta(x,y)$$

$$\boxed{(f, \mathrm{div}\, \theta)_V} = \sum_{x \in V} f(x) \sum_{y \sim x} \theta(x,y)$$

$$= \sum_{e \in E_{1/2}} \left(f(e^-)\theta(e) + f(e^+)\theta(e^-) \right)$$

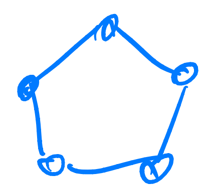
$$= \sum_{e \in E_{1/2}} \left(f(e^-) - f(e^+) \right) \theta(e)$$

$$= \boxed{(\nabla f, \theta)_E}$$

OHM'S LAW: $\nabla v = i r$

NODE LAW: $(\mathrm{div}\, i)(x) = 0 \quad \forall x \neq a, z.$

KIRCHOFF'S CYCLE LAW:



IF $x_1 \sim x_2 \sim \cdots \sim x_n \sim x_{n+1} = x_1$

THEN $$\sum_{j=1}^{n} i(x_j, x_{j+1})\, r(x_j, x_{j+1}) = 0$$

PROOF: SUM EQUALS $\sum \left(v(x_j) - v(x_{j+1}) \right) = 0.$

WEIGHTED INNER PROD.

$$\boxed{(\Theta_1, \Theta_2)_r} = \frac{1}{2} \sum_{e \in \vec{E}} \Theta_1(e)\, \Theta_2(e)\, r(e)$$

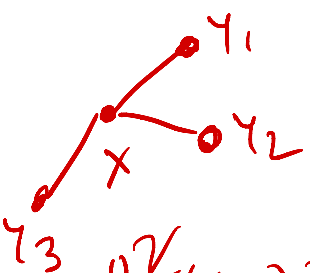
STAR SPACE

$$\bigstar = \operatorname{span}\Big\{ \underbrace{\boxed{\sum_{y \sim x} c(x,y)\, \chi_{(x,y)}}}_{} \Big\}, \qquad S_x := \sum_{y \sim x} c(x,y)\, \chi_{(x,y)}$$

TAKEN OVER $\cancel{x} \in V$

WHERE $\chi_{(x,y)} = \mathbb{1}_{(x,y)} - \mathbb{1}_{(y,x)}$

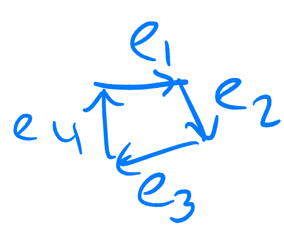
$\in \boxed{\ell^2_-(\vec{E})}$



$c(1)\chi_{(x,y_1)} + c(1)\chi_{(x,y_2)} + c(1)\chi_{(x,y_3)}$

CYCLE SPACE

$$\pentagon = \operatorname{span}\Big\{ \sum_{j=1}^{n} \chi_{e_j} \;\Big|\; e_1, \dots, e_n \text{ IS A DIRECTED CYCLE} \Big\}$$



LEMMA: $\ell^2_-(\vec{E}) = \bigstar \oplus \pentagon$

AND $(s,t)_r = 0 \quad \forall s \in \bigstar \ \forall t \in \pentagon$

PF: ① $\left(s_x, \text{cycle} \sum_{j=1}^{n} \chi_{e_j}\right)_r = 0$ SINCE

$$\xrightarrow{e_j} \bullet_x \xrightarrow{e_{j+1}}$$

$$(s_x, \chi_{e_j})_r = -1$$

$$(s_x, \chi_{e_{j+1}})_r = +1$$

SHOWS $\star \perp \pentagon$

② TO SHOW $\ell^2_-(\vec{E}) = \text{span}(\star, \pentagon)$, WE'LL SHOW THAT IF $\Theta \in \ell^2_-(\vec{E})$ IS ORTHOGONAL TO BOTH $\star$ AND $\pentagon$ THEN $\Theta \equiv 0$.

FIX $a \in V$, LET $f(x) = \sum_{j=1}^{n} \Theta(x_j, x_{j+1})\, r(x_j, x_{j+1})$

ALONG ANY PATH $a \rightsquigarrow x$.

IF $\Theta \perp \text{⬠}$ THEN f IS WELL-DEFINED:

$$\begin{tikzcd}[row sep=small]
 & x_1 \arrow[r] & x_2 \arrow[r, dashed] & \cdot \arrow[r] & x \\
a \arrow[ur] \arrow[dr] & & & & \\
 & y_1 \arrow[r] & y_2 \arrow[r, dashed] & \cdot \arrow[uur] &
\end{tikzcd}$$

cycle t

THEN $0 = (\Theta, t)_r = f_{\underline{x}} - f_{\underline{y}}$.

So $\boxed{\nabla f = r\Theta.}$ Now if $\Theta \perp \star$,

THEN

$$0 = (\Theta, s_x)_r = (r\Theta, s_x) = (\nabla f, s_x)_{\vec{E}} = (f, \operatorname{div}(s_x))_V$$

$$0 = \sum_{y \sim x} (f(x) - f(y))\, c(x,y)$$

$\Rightarrow$ f IS HARMONIC ON V.

$\Rightarrow$ f IS CONSTANT

$\Rightarrow$ $\nabla f = 0$ $\Rightarrow$ $\Theta = 0$. $\square$