Critical Path Methods

for project scheduling and resource allocation.

Consider a project (like building the new Gates Building or the expansion of the Triphammer Bridge) involving numerous tasks or activities. Each activity requires resources (e.g., people, equipment) and time to complete. The more resources allocated to any activity, the shorter the time that may be needed to complete it.

- How long will the entire project take to complete?
- Which activities determine total project time?
- Which activity times should be shortened, if possible, or in other words, how many resources should be allocated to each activity?

Questions such as these are addressed using deterministic Critical Path Methods (CPM) or probabilistic Program Evaluation and Review Techniques (PERT).

Scheduling of Activities

Some activities cannot begin until other activities are completed. For example one cannot begin building a new Commons in downtown Ithaca until the old one is removed. The old Commons cannot be removed until the construction area and its underground piping and other infrastructure is secured, etc. If all the activities in a project had to be done in sequence, then the time it would take to complete the entire project would be the sum of the activity durations. This is not the case when some activities can be accomplished simultaneously. The sequencing of activities can be shown by drawing a network of links and nodes.

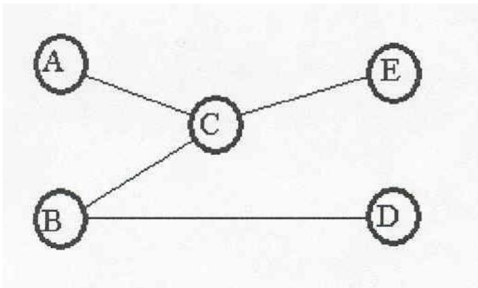
CPM Networks

There are two kinds of networks:

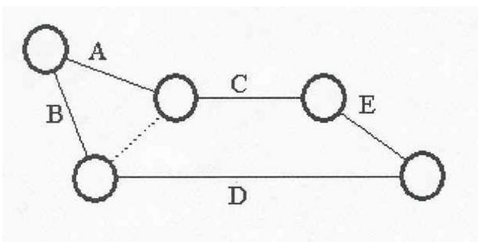
- Activity-on-Node (AON), and
- Activity-on-Link (AOL) or Activity on Arc (AOA).

Consider five activities: A, B, C, D, and E. Assume C cannot begin until A and B are completed, and D must begin after B is completed, and E must follow C.

If activities are represented by nodes, then the **AON network** is:



If activities are represented by links, then the **AOL network** is:



The dashed line is a "dummy link" required to show proper sequencing.

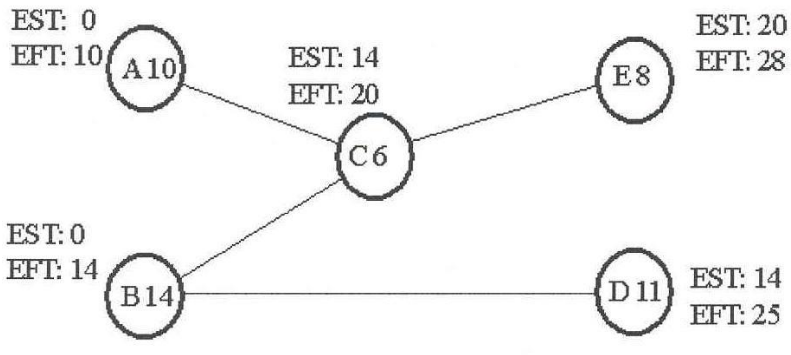
Project Duration and Activity Start and Finish Times

The project duration can now be computed for any set of estimated activity durations, assuming particular 'normal' resource allocations to each activity. For this example, assume:

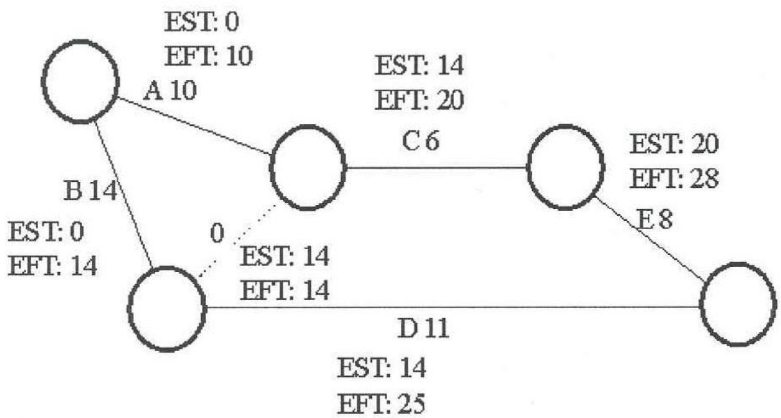
Activity	Normal Duration
A	10
B	14
C	6
D	11
E	8

To compute the project duration (time) and the sequence of activities that determine that time, the first step in is to compute **earliest start times** (EST) and **earliest finish times** (EFT) of each activity. Insuring that no activity can begin until those it must follow are completed, each activity's EST and EFT values can be written on the network next to the activity. The numbers in the nodes or on the links are the activity durations. Working from left to right, i.e., from the beginning to the end, the EST and EFT values are:

AON:

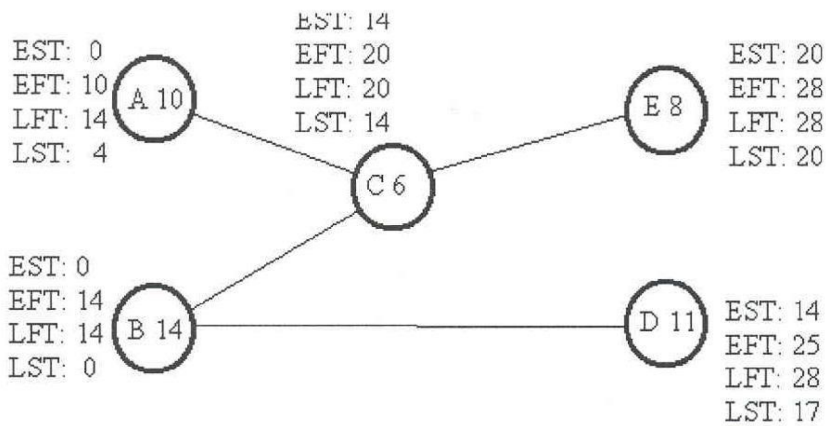


AOL:

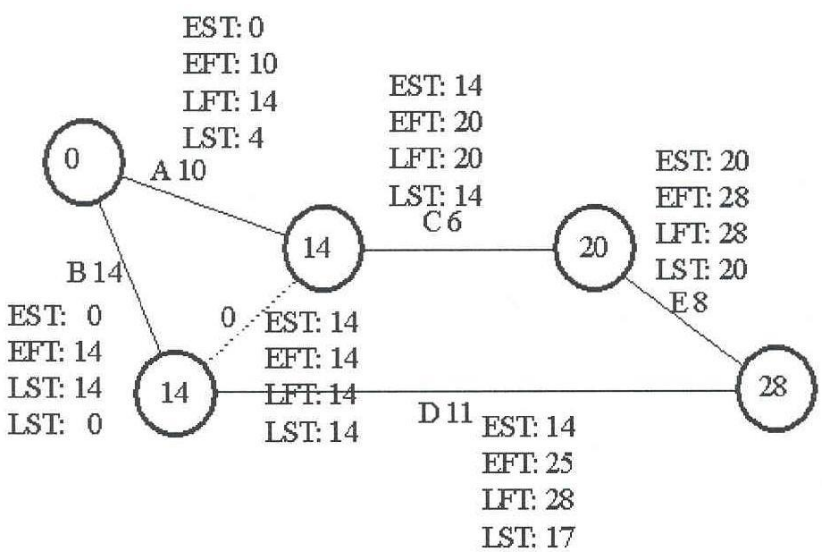


Next compute **earliest and latest finish times** (EFT, LFT) without extending the total project duration (of 28 in this example). Do this by working from right to left, i.e., from the end to the beginning.

AON:



AOL:

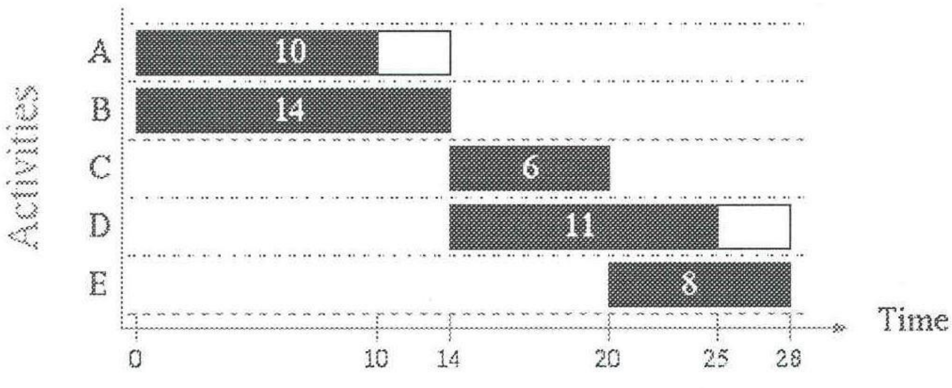


Now the **'critical path'** or sequence of activities that determines the total project time can be identified. The any durations of the activities on the critical path are increased, the time to complete the total project will also increase by the same amount(s). The critical path is defined by those activities that have identical early and late start times and hence identical early and late finish times. In this example it is B, C, and E. Activity A can start anytime between 0 and 4, and D can start anytime between 14 and 17 without delaying the total project completion time. Each of these activities have what is called **float**. The float in both cases is **free float**; it does not prevent any other activity from starting at its earliest start time. **Interfering float** will force a delay in the earliest start time of one or more other activities.

For example if activity Y follows X, and both have floats, then if X starts later than its earliest start time, it will cause Y to start after its earliest start time. Delaying the start of activity X interferes with the earliest start of activity Y.

The **total float** of an activity is the sum of its free and interfering floats.

Bar Charts:



Resource Allocations:

Using bar charts as above, together with the costs of shortening any activity duration, we can see how best to allocate resources in order to reduce total project costs. For example, if a special machine is needed for activities A and C, and the machine costs $2000 to bring to the site and $500 per time period to operate, then clearly activity A should start at 4. If the machine were needed for activities C and D, then one could compare and select the minimum cost of:

- renting two machines instead of one,
- beginning D at 20, causing a delay in project completion by 3 periods, or
- shortening activities C and/or D by a total of 3 periods so that they do not overlap.

In this case activity D will be on the critical path.

Note:

- There are no cost savings of shortening the duration of activities not on the critical path.
- As critical-path activity durations are shortened, the critical path itself may change.

PERT

Program Evaluation and Review Technique

This method extends CPM by considering the uncertainty in estimating activity duration in order to estimate the probability of finishing a project in a given time.

Three time estimates are required for each activity's duration:

- most likely (m)
- optimistic (a)
- pessimistic (b)

Assumptions:

- Expected value of activity duration e_i is a weighted combination of the most likely (mode of distribution), mi and the midpoint of the distribution, $(a_i + b_i)/2$ for each activity i :

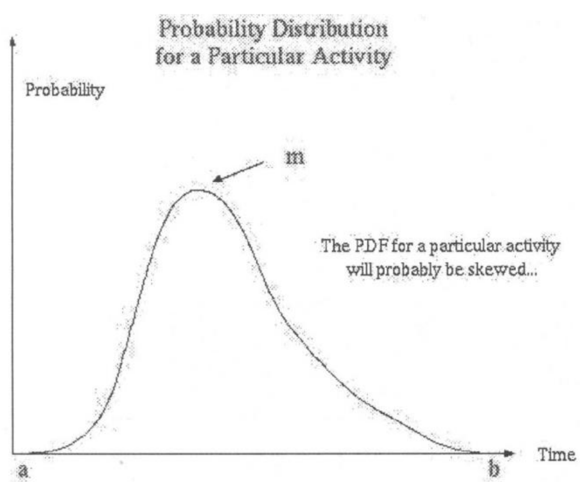
$$e_i = \frac{2m_i + \frac{a_i + b_i}{2}}{3} = [a_{\mathrm{i}} + 4m_{\mathrm{i}} + b_{\mathrm{i}}]/6$$

- A spread of about 6 standard deviations between the optimistic and pessimistic ranges of the duration distribution:

$$\sigma_i = \frac{(b_i - a_i)}{6}$$

- Activity durations are independent, and hence the sum of independent random expected values, e_i is normally distributed.

- Critical path activities (based on expected durations) will determine total expected project time.



Suppose a completion time is specified in the project contract. Call this given time ***t***. We are interested in estimating the probability that the actual completion time ***T*** is less than or equal to the specified completion time ***t***. ***T*** is a random variable. Thus, we want to know

$$\text{Prob}\{T \leq t\}$$

The probability distribution of the (random) actual completion time ***T*** is as shown below. It is a normal distribution having a mean of $\bar{e}$ and a standard deviation of σ_{proj} ; where expected project duration time is:

$$\bar{e} = \sum e_i$$

summed over all activities ***i*** on the critical path, and the project standard deviation σ_{proj} is

$$\sigma_{proj} = \sqrt{\sum \sigma_i^2}$$

over all activities ***i*** on the critical path.

Given any specified time "t", such as shown below, the probability Prob("T" less than or equal to "t") is the area under the distribution to the left of "t". These areas can be found in statistics books. But they can be estimated knowing that plus or minus 1 standard deviation σ from the mean is 68% of the area under the normal distribution curve, and plus or minus 2 standard deviations is 95% of the area. Thus if the project mean is 16 time periods and the standard deviation is 5 time units, the probability of completing the project by 11 time periods is 0.5 - 0.34 = 0.16 or 16 %.

Example: Expanding on the CPM example above,

Task	Durations			Expected	Std. Dev.	Variance
	Optimistic a(i)	Most Likely m(i)	Pessimistic b(i)	e(i)	σ(i)	$(\sigma(i))^2$
A	8	10	14	10.3	1	1
B	12	14	18	14.3	1	1
C	4	6	10	6.2	1	1
D	8	11	14	11.0	1	1
E	5	8	12	8.1	1.2	1.44

In this example the critical path remains the same as the CPM example, namely B, C, E for a total expected duration of 14.3 + 6.2 + 8.1 = 28.6.

The sum of the variances along the critical path is 3.44, the project time variance. Hence the project time standard deviation is √3.44 = 1.85.

The probability distribution of expected values is a normal distribution.

Hence referring to the normal distributions shown below

There is 50% chance that the project will end on or before 28.6 time periods.

There is a 84% chance the project will end by 28.6+1.85 = 30.45 time periods.

There is a 16% chance the project will end by 28.6-1.85 = 26.75 time periods.